

NOTE ON AVERAGE OF CLASS NUMBERS OF CUBIC FUNCTION FIELDS

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ABSTRACT. Let $k = \mathbb{F}_q(T)$ be the rational function field over a finite field \mathbb{F}_q , where $q \equiv 1 \pmod{3}$. In this paper, we determine asymptotic values of average of ideal class numbers of some family of cubic Kummer extensions of k .

1. Introduction and statement of result

Let $k = \mathbb{F}_q(T)$ be the rational function field over a finite field \mathbb{F}_q and $\mathbb{A} = \mathbb{F}_q[T]$. Average values of L -functions associated to orders in quadratic extensions of k are obtained by Hoffstein and Rosen [3] when q is odd and by Chen [2] when q is even. Rosen [6] generalized some results of [3] to general Kummer extensions of k of degree ℓ , where ℓ is a prime divisor of $q - 1$, and determined average values of ideal class numbers of Kummer extensions of k of degree 3. Prime [5] obtained an L -function average over imaginary quadratic extensions of k with prime discriminants. Recently, Bae, Jung and Kang [1] extended the result of Chen to general Artin-Schreier extensions of k and determined average values of ideal class numbers of Artin-Schreier extensions of k of degree 3. They also extended the result of Prime to general Kummer extensions of k and obtained a similar formulas of average values of prime L -functions

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of Artin-Schreier extensions of k of degree 2. The aim of this paper is to determine asymptotic values of average of ideal class numbers of some family of cubic Kummer extensions of k .

To state our main result, we introduce some notations. Assume that $q \equiv 1 \pmod{3}$. A monic irreducible polynomial in \mathbb{A} will be called a *prime* polynomial. Any (geometric) cubic Kummer extension K of k can be written as $K = k(\sqrt[3]{D})$ for some cubic power free polynomial $D \in \mathbb{A}$. We say that K/k is ramified imaginary, real or inert imaginary according as the infinite prime ∞_k of k ramifies, splits completely or is inert in K . Let \mathcal{O}_K be the integral closure of \mathbb{A} in K . Write $h(\mathcal{O}_K)$, $d(\mathcal{O}_K)$ and $R(\mathcal{O}_K)$ for the ideal class number, the discriminant and the regulator of \mathcal{O}_K , respectively. For $K = k(\sqrt[3]{D})$, we have $d(\mathcal{O}_K) = \text{rad}(D)^2$ (see (2.1)), where $\text{rad}(D)$ is the product of distinct prime divisors of D , and $R(\mathcal{O}_K) = 1$ if K/k is imaginary. Let \mathcal{A} be the family of all ramified imaginary cubic Kummer extensions K of k such that $d(\mathcal{O}_K)$ is a square of a prime polynomial and $\mathcal{A}_n = \{K \in \mathcal{A} : \deg d(\mathcal{O}_K) = 2n\}$ for each positive integer n with $3 \nmid n$. Similarly, replacing “ramified imaginary” by “real” or “inert imaginary” and “ $3 \nmid n$ ” by “ $3 \mid n$ ”, we define \mathcal{B} , \mathcal{B}_n or \mathcal{C} , \mathcal{C}_n , respectively. We determine asymptotic values of average of ideal class numbers $h(\mathcal{O}_K)$ (or times regulator $R(\mathcal{O}_K)$) as K varies over \mathcal{A} , \mathcal{B} , or \mathcal{C} . Our main result is the following theorem.

THEOREM 1.1. *Assume that $q \equiv 1 \pmod{3}$. Then we have*

1. as $n \rightarrow \infty$ with $3 \nmid n$,

$$\frac{1}{\#\mathcal{A}_n} \sum_{K \in \mathcal{A}_n} h(\mathcal{O}_K) = \frac{\zeta_{\mathbb{A}}(2)\zeta_{\mathbb{A}}(3)^2}{\zeta_{\mathbb{A}}(6)} q^{n-1} + O(4^n q^{\frac{n}{2}}),$$

2. as $n \rightarrow \infty$ with $3 \mid n$,

$$\frac{1}{\#\mathcal{B}_n} \sum_{K \in \mathcal{B}_n} h(\mathcal{O}_K)R(\mathcal{O}_K) = \frac{\zeta_{\mathbb{A}}(2)^3 \zeta_{\mathbb{A}}(3)^2}{\zeta_{\mathbb{A}}(6)} q^{n-2} + O(4^n q^{\frac{n}{2}}),$$

3. as $n \rightarrow \infty$ with $3 \mid n$,

$$\frac{1}{\#\mathcal{C}_n} \sum_{K \in \mathcal{C}_n} h(\mathcal{O}_K) = \frac{3\zeta_{\mathbb{A}}(3)^2 \zeta_{\mathbb{A}}(4)}{\zeta_{\mathbb{A}}(6)} q^{n-2} + O(4^n q^{\frac{n}{2}}),$$

where $\zeta_{\mathbb{A}}(s) = \frac{1}{1-q^{1-s}}$ is the zeta function of \mathbb{A} .

2. L -functions of Kummer extensions and Class number formula

In this section we recall some basic results on Kummer extensions of k . For more details, we refer to [6, §1]. Let \mathbb{A}^+ be the subset of \mathbb{A} consisting of all monic polynomials and $\mathcal{P}(\mathbb{A})$ be the set of prime polynomials in \mathbb{A} . Let $\mathbb{A}_n^+ = \{N \in \mathbb{A}^+ : \deg N = n\}$ and $\mathcal{P}_n(\mathbb{A}) = \mathcal{P}(\mathbb{A}) \cap \mathbb{A}_n^+$ for $n \geq 1$. Assume that q is a power of an odd prime. Let ℓ be a prime divisor of $q - 1$. For an ℓ -th power free polynomial $D \in \mathbb{A}$, let \mathcal{O}_K be the integral closure of \mathbb{A} in $K = k(\sqrt[\ell]{D})$. Let $h(\mathcal{O}_K)$, $d(\mathcal{O}_K)$ and $R(\mathcal{O}_K)$ be the ideal class number, the discriminant and the regulator of \mathcal{O}_K , respectively. By [6, Theorem 1.2], we have

$$(2.1) \quad d(\mathcal{O}_K) = \text{rad}(D)^{\ell-1},$$

where $\text{rad}(D)$ is the product of distinct prime divisors of D . The decomposition of the infinite prime ∞_k of k in $K = k(\sqrt[\ell]{D})$ is determined as follows:

- ∞_k ramifies in K if and only if $\ell \nmid \deg D$. In this case K/k is called a ramified imaginary extension.
- ∞_k splits completely in K if and only if $\ell \mid \deg D$ with $\text{sgn}(D) \in \mathbb{F}_q^{*\ell}$. In this case K/k is called a real extension.
- ∞_k is inert in K if and only if $\ell \mid \deg P$ with $\text{sgn}(D) \notin \mathbb{F}_q^{*\ell}$. In this case K/k is called an inert imaginary extension.

We note that $R_{\mathcal{O}_K} = 1$ if K/k is imaginary. To define the character χ_D , we first need to fix an isomorphism ω between the group of ℓ -th roots of unity in \mathbb{C} and the group of ℓ -th roots of unity in \mathbb{F}_q . For a prime polynomial $P \in \mathcal{P}(\mathbb{A})$, define $\chi_D(P) = 0$ if $P \mid D$, and if $P \nmid D$, $\chi_D(P) \in \mathbb{C}^*$ is defined by

$$D^{\frac{|P|-1}{\ell}} \equiv \omega(\chi_D(P)) \pmod{P}.$$

Now we extend the definition to all of \mathbb{A}^+ by multiplicativity. Then the L -function $L(s, \chi_D^i)$ associated to χ_D^i ($0 \leq i \leq \ell - 1$) is defined by

$$L(s, \chi_D^i) = \sum_{N \in \mathbb{A}^+} \frac{\chi_D^i(N)}{|N|^s}.$$

It is well known ([6, Lemma 2.1]) that $L(s, \chi_D^i)$ is a polynomial in q^{-s} of degree at most $\deg D - 1$ for $1 \leq i \leq \ell - 1$. Thus, we can write

$$L(s, \chi_D^i) = \sum_{n=0}^{\deg D-1} \sum_{N \in \mathbb{A}_n^+} \chi_D^i(N) q^{-ns}.$$

For $K = k(\sqrt[3]{D})$, we have the following class number formula (see [6, Theorem 1.3]):

(2.2)

$$\prod_{i=1}^{\ell-1} L(1, \chi_D^i) = \begin{cases} q^{\frac{\ell-1}{2}} \frac{h(\mathcal{O}_K)}{\sqrt{|d(\mathcal{O}_K)|}} & \text{if } K/k \text{ is ramified imaginary,} \\ (q-1)^{\ell-1} \frac{h(\mathcal{O}_K)R(\mathcal{O}_K)}{\sqrt{|d(\mathcal{O}_K)|}} & \text{if } K/k \text{ is real,} \\ \frac{q^{\ell-1}}{\ell(q-1)} \frac{h(\mathcal{O}_K)}{\sqrt{|d(\mathcal{O}_K)|}} & \text{if } K/k \text{ is inert imaginary.} \end{cases}$$

3. Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. Assume that $q \equiv 1 \pmod 3$. Let γ be a generator of \mathbb{F}_q^* and $\gamma_j = \gamma^j$ for $j \geq 0$. Then $\{\gamma_0 = 1, \gamma_1, \gamma_2\}$ forms a complete set of representatives of $\mathbb{F}_q^*/\mathbb{F}_q^{*3}$. By Kummer theory, it is easy to see that any cubic Kummer extension K of k such that the discriminant $d(\mathcal{O}_K)$ is a square of a prime polynomial can be written uniquely as $K = k(\sqrt[3]{\gamma_j P})$ with $P \in \mathcal{P}(\mathbb{A})$ and $0 \leq j \leq 2$. For $K = k(\sqrt[3]{\gamma_j P})$, by (2.1), we have $d(\mathcal{O}_K) = P^2$ and K/k is ramified imaginary, splits completely or inert imaginary according as $3 \nmid \deg P$, $3 \mid \deg P$ with $j = 0$ or $3 \mid \deg P$ with $j = 1, 2$, respectively. Moreover, by (2.2), we have

(3.1)

$$\prod_{i=1}^2 L(1, \chi_{\gamma_j P}^i) = \begin{cases} q^{1-\deg P} h(\mathcal{O}_K) & \text{if } 3 \nmid \deg P, \\ (q-1)^2 q^{-\deg P} h(\mathcal{O}_K) R(\mathcal{O}_K) & \text{if } 3 \mid \deg P \text{ and } j = 0, \\ \frac{q^{-\deg P} (q^3-1)}{3(q-1)} h(\mathcal{O}_K) & \text{if } 3 \mid \deg P \text{ and } j = 1, 2. \end{cases}$$

Hence, for a positive integer n with $3 \nmid n$, we have

$$\mathcal{A}_n = \{k(\sqrt[3]{\gamma_j P}) : P \in \mathcal{P}_n(\mathbb{A}), 0 \leq j \leq 2\}$$

and

$$(3.2) \quad \frac{1}{\#\mathcal{A}_n} \sum_{K \in \mathcal{A}_n} h(\mathcal{O}_K) = \frac{1}{3\#\mathcal{P}_n(\mathbb{A})} \sum_{P \in \mathcal{P}_n(\mathbb{A})} \sum_{j=0}^2 h(\mathcal{O}_{k(\sqrt[3]{\gamma_j P})}).$$

For a positive integer n with $3 \mid n$, we have

$$(3.3) \quad \mathcal{B}_n = \{k(\sqrt[3]{P}) : P \in \mathcal{P}_n(\mathbb{A})\}, \quad \mathcal{C}_n = \{k(\sqrt[3]{\gamma_j P}) : P \in \mathcal{P}_n(\mathbb{A}), j = 1, 2\},$$

$$\frac{1}{\#\mathcal{B}_n} \sum_{K \in \mathcal{B}_n} h(\mathcal{O}_K)R(\mathcal{O}_K) = \frac{1}{\#\mathcal{P}_n(\mathbb{A})} \sum_{P \in \mathcal{P}_n(\mathbb{A})} h(\mathcal{O}_{k(\sqrt[3]{P})})R(\mathcal{O}_{k(\sqrt[3]{P})}),$$

and

$$(3.4) \quad \frac{1}{\#\mathcal{C}_n} \sum_{K \in \mathcal{C}_n} h(\mathcal{O}_K) = \frac{1}{2\#\mathcal{P}_n(\mathbb{A})} \sum_{P \in \mathcal{P}_n(\mathbb{A})} \sum_{j=1}^2 h(\mathcal{O}_{k(\sqrt[3]{\gamma_j P})}).$$

We need a lemma which will be used in the proof of Proposition .

LEMMA 3.1. *For any positive integer n , $D \in \mathbb{A}^+$ not cubic power and $i = 1, 2$, we have*

$$(3.5) \quad \left| \sum_{P \in \mathcal{P}_n(\mathbb{A})} \chi_{\gamma_j P}^i(D) \right| \leq \frac{(\deg D + 1)}{n} q^{\frac{n}{2}}.$$

In particular, if $\deg D < n$, then

$$(3.6) \quad \left| \sum_{P \in \mathcal{P}_n(\mathbb{A})} \chi_{\gamma_j P}^i(D) \right| \leq 2q^{\frac{n}{2}}.$$

Proof. (3.5) follows from Theorem 2.1 in [4] and cubic power reciprocity law. For (3.6), by (3.5), we have

$$\left| \sum_{P \in \mathcal{P}_n(\mathbb{A})} \chi_{\gamma_j P}^i(D) \right| \leq \left(1 + \frac{1}{n}\right) q^{\frac{n}{2}} \leq 2q^{\frac{n}{2}}.$$

□

For a positive integer n , let

$$\mathcal{Z}_{n,j}(s) = \frac{1}{\#\mathcal{P}_n(\mathbb{A})} \sum_{P \in \mathcal{P}_n(\mathbb{A})} L(s, \chi_{\gamma_j P})L(s, \chi_{\gamma_j P}^2) \quad (0 \leq j \leq 2).$$

PROPOSITION 3.2. We have
(3.7)

$$\mathcal{Z}_{n,j}(1) = \frac{1}{\#\mathcal{P}_n(\mathbb{A})} \sum_{P \in \mathcal{P}_n(\mathbb{A})} \prod_{i=1}^2 L(1, \chi_{\gamma_j^i P}) = \frac{\zeta_{\mathbb{A}}(2)\zeta_{\mathbb{A}}(3)^2}{\zeta_{\mathbb{A}}(6)} + O(4^n q^{-\frac{n}{2}}).$$

Proof. Since $L(s, \chi_{\gamma_j P})$ and $L(s, \chi_{\gamma_j^2 P})$ are polynomials in q^{-s} of degree $\leq n - 1$, we can write

$$\prod_{i=1}^2 L(s, \chi_{\gamma_j^i P}) = \sum_{m=0}^{2n-2} \sum_{\substack{m_1+m_2=m \\ (M_1, M_2) \in \mathbb{A}_{m_1}^+ \times \mathbb{A}_{m_2}^+}} \chi_{\gamma_j P}(M_1 M_2^2) q^{-ms}.$$

Thus,

$$\mathcal{Z}_{n,j}(s) = \frac{1}{\#\mathcal{P}_n(\mathbb{A})} \sum_{m=0}^{2n-2} \sum_{P \in \mathcal{P}_n(\mathbb{A})} a_m(\gamma_j P) q^{-ms}$$

with

$$a_m(\gamma_j P) = \sum_{\substack{m_1+m_2=m \\ (M_1, M_2) \in \mathbb{A}_{m_1}^+ \times \mathbb{A}_{m_2}^+}} \chi_{\gamma_j P}(M_1 M_2^2).$$

By [6, Lemma 2.3], we have

$$|a_m(\gamma_j P)| \leq \binom{2n-2}{m} q^{\frac{m}{2}},$$

so

$$\left| \sum_{m=n}^{2n-2} a_m(\gamma_j P) q^{-ms} \right| \leq 2^{2n-2} (q^{\frac{1}{2}-\sigma})^n (1 - q^{\frac{1}{2}-\sigma})^{-1}$$

for $s \in \mathbb{C}$ with $\sigma = \text{Re}(s) > 1/2$. Hence, we have

$$(3.8) \quad \left| \frac{1}{\#\mathcal{P}_n(\mathbb{A})} \sum_{P \in \mathcal{P}_n(\mathbb{A})} \sum_{m=n}^{2n-2} a_m(\gamma_j P) q^{-ms} \right| \leq 2^{2n-2} (q^{\frac{1}{2}-\sigma})^n (1 - q^{\frac{1}{2}-\sigma})^{-1}.$$

Now, we consider

$$\begin{aligned} \mathcal{Z}'_{n,j}(s) &= \frac{1}{\#\mathcal{P}_n(\mathbb{A})} \sum_{m=0}^{n-1} \sum_{P \in \mathcal{P}_n(\mathbb{A})} a_m(\gamma_j P) q^{-ms} \\ &= \frac{1}{\#\mathcal{P}_n(\mathbb{A})} \sum_{m=0}^{n-1} \sum_{\substack{m_1+m_2=m \\ (M_1, M_2) \in \mathbb{A}_{m_1}^+ \times \mathbb{A}_{m_2}^+}} \sum_{P \in \mathcal{P}_n(\mathbb{A})} \chi_{\gamma_j P}(M_1 M_2^2) q^{-ms}. \end{aligned}$$

Write $\mathcal{Z}'_{n,j}(s) = \alpha_{n,j}(s) + \beta_{n,j}(s)$ with

$$\alpha_{n,j}(s) = \frac{1}{\#\mathcal{P}_n(\mathbb{A})} \sum_{m=0}^{n-1} \sum_{\substack{m_1+m_2=m \\ (M_1, M_2) \in \mathbb{A}_{m_1}^+ \times \mathbb{A}_{m_2}^+ \\ M_1 M_2^2: \text{ not cube}}} \sum_{P \in \mathcal{P}_n(\mathbb{A})} \chi_{\gamma_j P}(M_1 M_2^2) q^{-ms}$$

and

$$\beta_{n,j}(s) = \frac{1}{\#\mathcal{P}_n(\mathbb{A})} \sum_{m=0}^{n-1} \sum_{\substack{m_1+m_2=m \\ (M_1, M_2) \in \mathbb{A}_{m_1}^+ \times \mathbb{A}_{m_2}^+ \\ M_1 M_2^2: \text{ cube}}} \sum_{P \in \mathcal{P}_n(\mathbb{A})} \chi_{\gamma_j P}(M_1 M_2^2) q^{-ms}.$$

If $M_1 M_2^2$ is not cube, by (3.6), we have

$$(3.9) \quad \left| \sum_{P \in \mathcal{P}_n(\mathbb{A})} \chi_{\gamma_j P}(M_1 M_2^2) \right| \leq 2q^{\frac{n}{2}}.$$

For $s \in \mathbb{C}$ with $\sigma = \text{Re}(s) > \frac{1}{2}$, using (3.9) and the fact that $\#\mathcal{P}_n(\mathbb{A}) > \frac{q^n}{2n}$, we have

$$\begin{aligned} |\alpha_{n,j}(s)| &\leq \frac{1}{\#\mathcal{P}_n(\mathbb{A})} \sum_{m=0}^{n-1} \sum_{\substack{m_1+m_2=m \\ (M_1, M_2) \in \mathbb{A}_{m_1}^+ \times \mathbb{A}_{m_2}^+ \\ M_1 M_2^2: \text{ not cube}}} \left| \sum_{P \in \mathcal{P}_n(\mathbb{A})} \chi_{\gamma_j P}(M_1 M_2^2) \right| q^{-ms} \\ &< 4nq^{-\frac{n}{2}} \sum_{m=0}^{n-1} \sum_{\substack{m_1+m_2=m \\ (M_1, M_2) \in \mathbb{A}_{m_1}^+ \times \mathbb{A}_{m_2}^+}} q^{-m\sigma} \\ (3.10) \quad &< 4nq^{-\frac{n}{2}} \left(\frac{1 - q^{n(1-\sigma)}(1 + n - nq^{(1-\sigma)})}{(1 - q^{(1-\sigma)})^2} \right) \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Now, we consider $\beta_{n,j}(s)$. Since $P \nmid M_1M_2$ and $M_1M_2^2$ is cube, we have $\chi_{\gamma_j P}(M_1M_2^2) = 1$. Hence, we have

$$\beta_{n,j}(s) = \sum_{\substack{m_1+m_2=n \\ (M_1, M_2) \in \mathbb{A}_{m_1}^+ \times \mathbb{A}_{m_2}^+ \\ M_1M_2^2 : \text{cube}}} |M_1|^{-s} |M_2|^{-s}.$$

Put

$$L(s) = \sum_{\substack{(M_1, M_2) \in \mathbb{A}^+ \times \mathbb{A}^+ \\ P \nmid M_1M_2, M_1M_2^2 : \text{cube}}} |M_1|^{-s} |M_2|^{-s}.$$

Then, as in [6, §2], we have

$$L(s) = \frac{\zeta_{\mathbb{A}}(3s)^2 \zeta_{\mathbb{A}}(2s)}{\zeta_{\mathbb{A}}(6s)}$$

and

$$(3.11) \quad |\beta_{n,j}(s) - L(s)| \leq Cn^2 q^{\frac{n}{3}(1-3\sigma)}$$

for $\sigma = \text{Re}(s) > \frac{1}{3}$ and some constant C which depends on s but is independent of n . By (3.10) and (3.11), we have that for $\sigma = \text{Re}(s) > \frac{1}{3}$,

$$(3.12) \quad \mathcal{Z}'_{n,j}(s) = \frac{\zeta_{\mathbb{A}}(3s)^2 \zeta_{\mathbb{A}}(2s)}{\zeta_{\mathbb{A}}(6s)} + O(Cn^2 q^{\frac{n}{3}(1-3\sigma)}).$$

Since $Cn^2 q^{-\frac{2n}{3}} = o(4^n q^{-\frac{n}{2}})$, by (3.8) and (3.12), we have

$$\mathcal{Z}_{n,j}(1) = \frac{1}{\#\mathcal{P}_n(\mathbb{A})} \sum_{P \in \mathcal{P}_n(\mathbb{A})} \prod_{i=1}^2 L(1, \chi_{\gamma_j^i P}) = \frac{\zeta_{\mathbb{A}}(2)\zeta_{\mathbb{A}}(3)^2}{\zeta_{\mathbb{A}}(6)} + O(4^n q^{-\frac{n}{2}}).$$

□

Let $P \in \mathcal{P}_n(\mathbb{A})$. By (3.1), we have that if $3 \nmid n$,

$$(3.13) \quad \prod_{i=1}^2 L(1, \chi_{\gamma_j^i P}) = q^{1-n} h(\mathcal{O}_{k(\sqrt[3]{\gamma_j P})}) \quad (0 \leq j \leq 2)$$

and, if $3 \mid n$,

$$(3.14) \quad \begin{aligned} \prod_{i=1}^2 L(1, \chi_P^i) &= q^{-n} (q-1)^2 h(\mathcal{O}_{k(\sqrt[3]{P})}) R(\mathcal{O}_{k(\sqrt[3]{P})}) \\ &= \frac{q^{-n+2}}{\zeta_{\mathbb{A}}(2)^2} h(\mathcal{O}_{k(\sqrt[3]{P})}) R(\mathcal{O}_{k(\sqrt[3]{P})}) \end{aligned}$$

$$\begin{aligned}
 (3.15) \quad \prod_{i=1}^2 L(1, \chi_{\gamma_j P}^i) &= \frac{q^{-n}(q^3 - 1)}{3(q - 1)} h(\mathcal{O}_{k(\sqrt[3]{\gamma_j P})}) \\
 &= \frac{\zeta_{\mathbb{A}}(2)q^{-n+2}}{3\zeta_{\mathbb{A}}(4)} h(\mathcal{O}_{k(\sqrt[3]{\gamma_j P})}) \quad (j = 1, 2).
 \end{aligned}$$

As $n \rightarrow \infty$ with $3 \nmid n$, by (3.2), (3.7) and (3.13), we have

$$\frac{1}{\#\mathcal{A}_n} \sum_{K \in \mathcal{A}_n} h(\mathcal{O}_K) = \frac{\zeta_{\mathbb{A}}(2)\zeta_{\mathbb{A}}(3)^2}{\zeta_{\mathbb{A}}(6)} q^{n-1} + O(4^n q^{\frac{n}{2}}),$$

and $n \rightarrow \infty$ with $3 \mid n$, by (3.3), (3.4), (3.7), (3.14) and (3.15), we have

$$\begin{aligned}
 \frac{1}{\#\mathcal{B}_n} \sum_{K \in \mathcal{B}_n} h(\mathcal{O}_K) R(\mathcal{O}_K) &= \frac{\zeta_{\mathbb{A}}(2)^3 \zeta_{\mathbb{A}}(3)^2}{\zeta_{\mathbb{A}}(6)} q^{n-2} + O(4^n q^{\frac{n}{2}}), \\
 \frac{1}{\#\mathcal{C}_n} \sum_{K \in \mathcal{C}_n} h(\mathcal{O}_K) &= \frac{3\zeta_{\mathbb{A}}(3)^2 \zeta_{\mathbb{A}}(4)}{\zeta_{\mathbb{A}}(6)} q^{n-2} + O(4^n q^{\frac{n}{2}}).
 \end{aligned}$$

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