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STRUCTURES OF HYPERHOLOMORPHIC FUNCTIONS ON DUAL QUATERNION NUMBERS

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Abstract. We research properties of a corresponding Cauchy theorem of hyperholomorphic functions in an open set of product complex spaces, in the sense of complex Clifford analysis.

1. Introduction

Theory of holomorphic functions of several complex variables was developed by a constructive method of complex analysis. Weil [14] generalized the Cauchy integral formula to polynomial polyhedra in $\mathbb{C}^n$. Oka [6-13] solved the so-called fundamental problems, that is, Cousin problem and Levi problem et al. Likewise in the theory of holomorphic functions of several complex variables, investigations of hyperholomorphic functions, regular functions, monogenic functions, biregular functions and their natural domains of existence are rooted in abstract versions of complex Clifford analysis and their applications. In practice, we require n solutions of polynomial equations having the form

$$c_0z^n + c_1z^{n-1} + c_2z^{n-2} + \cdots + c_{n-1}z + c_n = 0,$$

where $c_0 \neq 0, c_1, \cdots, c_n$ are given complex numbers and n is a positive integer called the degree of the equation. The fundamental theorem of algebra does not hold in the case of complex Clifford analysis.

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2. Preliminaries

For the real space $\mathbb{R}^4$, the non-commutative extension of the complex numbers was discovered by Hamilton [3], and was called the quaternion in Clifford analysis. Futer [1] gave a definition of regular quaternion function in $\mathbb{R}^4$ and developed theory of quaternion functions in 1935. Recently, theory of quaternion analysis have been applied in physics (see [2]).

The field $\mathcal{T}$ of quaternions

$$z = \sum_{j=o}^{3} e_j x_j \ (x_0, x_1, x_2, x_3 \in \mathbb{R})$$

is a four dimensional non-commutative $\mathbb{R}$-field of real numbers such that its four base elements $e_0 = id., e_1, e_2$ and e_3 satisfy the followings:

$$\begin{aligned} e_j^2 &= -1 \ (j = 1, 2, 3), \quad & e_1 e_2 &= e_3 = -e_2 e_1, \\ e_2 e_3 &= e_1 = -e_3 e_2, \quad & e_3 e_1 &= e_2 = -e_1 e_3. \end{aligned}$$

Identifying the element e_1 with the imaginary unit $\sqrt{-1}$ in the $\mathbb{C}$-field of complex numbers, we have a quaternion that z is regarded as

$$z = z_1 + z_2 e_2 \in \mathcal{T},$$

where $z_1 = x_0 + e_1 x_1, \ z_2 = x_2 + e_1 x_3 \in \mathbb{C}$. Thus, we identify $\mathcal{T}$ with $\mathbb{C}^2$. We define the quaternion multiplication of two quaternion numbers $z = z_1 + z_2 e_2$ and $w = w_1 + w_2 e_2 \in \mathcal{T}$, by where $\bar{z_1} = x_1 - e_1 x_2, \ \bar{z_2} = x_2 - e_1 x_3 \in \mathbb{C}$. For $z = z_1 + z_2 e_2$, the quaternion conjugate z^*, the absolute value $|z|$ of z and the inverse z^{-1} of z in $\mathcal{T}$ are defined by the followings:

$$z^* = \bar{z_1} - z_2 e_2, \ |z| = \sqrt{|z_1|^2 + |z_2|^2}, \ z^{-1} = \frac{z^*}{|z|^2} \ (z \neq 0).$$

We use the following quaternion differential operators:

$$D_z := \frac{\partial}{\partial z_1} - e_2 \frac{\partial}{\partial \bar{z_2}} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3}\right),$$

$$D_z^* = \frac{\partial}{\partial \bar{z_1}} + e_2 \frac{\partial}{\partial \bar{z_2}} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} - e_3 \frac{\partial}{\partial x_3}\right),$$

where $\frac{\partial}{\partial z_1}, \frac{\partial}{\partial \bar{z_1}}, \frac{\partial}{\partial z_2}$ and $\frac{\partial}{\partial \bar{z_2}}$ are usual differential operators. And, we have

$$\begin{aligned}\frac{\partial}{\partial z_1}e_2 &= \frac{1}{2}\left(\frac{\partial}{\partial x_0} - e_1\frac{\partial}{\partial x_1}\right)e_2 = \frac{1}{2}\left(e_2\frac{\partial}{\partial x_0} - e_1e_2\frac{\partial}{\partial x_1}\right)\\ &= \frac{1}{2}\left(e_2\frac{\partial}{\partial x_0} + e_2e_1\frac{\partial}{\partial x_1}\right) = e_2\frac{\partial}{\partial \bar{z_1}},\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial \bar{z_1}}e_2 &= \frac{1}{2}\left(\frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1}\right)e_2 = \frac{1}{2}\left(e_2\frac{\partial}{\partial x_0} + e_1e_2\frac{\partial}{\partial x_1}\right)\\ &= \frac{1}{2}\left(e_2\frac{\partial}{\partial x_0} - e_2e_1\frac{\partial}{\partial x_1}\right) = e_2\frac{\partial}{\partial z_1}.\end{aligned}$$

The operator

$$\begin{aligned}D_zD_z^* &= \left(\frac{\partial}{\partial z_1} - e_2\frac{\partial}{\partial \bar{z_2}}\right)\left(\frac{\partial}{\partial \bar{z_1}} + e_2\frac{\partial}{\partial \bar{z_2}}\right)\\ &= \frac{1}{4}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2}\right) = \Delta_x.\end{aligned}$$

Let D be an open set in $\mathbb{C}^2$ and $f(z) = f_1(z) + f_2e_2$ be a function defined on D with values in $\mathcal{T}$, where $z = (z_1, z_2)$ and $f_1(z)$ and $f_2(z)$ are complex-valued functions on D.

Definition 1. Let D be an open set in $\mathbb{C}^2$. A function $f(z) = f_1(z) + f_2(z)e_2$ is said to be $L(R)$-hyperholomorphic on D, if
(1) $f_1(z)$ and $f_2(z)$ are continuously differentiable on D,
(2) $\frac{\partial}{\partial z^*}f = 0$ $(f\frac{\partial}{\partial z^*} = 0)$ on D.

The above equations (2) of the definition 1 operate to f as follows:

$$D_z^*f = \left(\frac{\partial}{\partial \bar{z_1}} + e_2\frac{\partial}{\partial \bar{z_2}}\right)(f_1 + f_2e_2) = \left(\frac{\partial f_1}{\partial \bar{z_1}} - \frac{\partial \bar{f_2}}{\partial z_2}\right) + \left(\frac{\partial f_2}{\partial \bar{z_1}} + \frac{\partial \bar{f_1}}{\partial z_2}\right)e_2,$$

$$fD_z^* = (f_1 + f_2e_2)\left(\frac{\partial}{\partial \bar{z_1}} + e_2\frac{\partial}{\partial \bar{z_2}}\right) = \left(\frac{\partial f_1}{\partial \bar{z_1}} - \frac{\partial f_2}{\partial \bar{z_2}}\right) + \left(\frac{\partial f_2}{\partial z_1} + \frac{\partial f_1}{\partial z_2}\right)e_2.$$

The above equation (2) of the definition 1 for L-hyperholomorphic function $f(z)$ is equivalent to the following corresponding Cauchy-Riemann system in $\mathcal{T}$:

$$\frac{\partial f_1}{\partial \bar{z_1}} = \frac{\partial \bar{f_2}}{\partial z_2},\ \frac{\partial f_2}{\partial \bar{z_1}} = -\frac{\partial \bar{f_1}}{\partial z_2}.$$

3. Dual quaternion

The dual numbers extended the real numbers by adjoining one new element ε with the property $\varepsilon^2 = 0$. Every dual number has the form $z = x + \varepsilon y$ with x and y uniquely determined real numbers. And the conjugate dual number z^* of z is defined by $z^* = x - \varepsilon y$ and we obtain $|z|^2 = x^2$. If we use matrices, dual numbers can be represented as

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ z = x + \varepsilon y = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}.$$

For the real polynomial, we have

$$\begin{aligned} P(x) &= p_n x^n + p_{n-1} x^{n-1} + \cdots + p_1 x + p_0, \\ P(x + \varepsilon y) &= p_n (x + \varepsilon y)^n + p_{n-1} (x + \varepsilon y)^{n-1} + \cdots + p_1 (x + \varepsilon y) + p_0 \\ &= p_n x^n + \varepsilon n p_n x^{n-1} y + p_{n-1} x^{n-1} + \varepsilon (n-1) p_{n-1} x^{n-2} y \\ &\quad + \cdots + p_1 x + \varepsilon p_1 y + p_0 \\ &= P(x) + \varepsilon y P'(x). \end{aligned}$$

For a Taylor series of a holomorphic function $f(a + \varepsilon b)$, we have

$$f(x + \varepsilon y) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} (\varepsilon y)^n = f(x) + \varepsilon y f'(x).$$

Hence, we have

$$\exp(\varepsilon y) = \sum_{n=0}^{\infty} \frac{(\varepsilon y)^n}{n!} = 1 + \varepsilon y.$$

Thus,

$$(x + \varepsilon y)^{c + \varepsilon d} = x^c + \varepsilon [y(c x^{c-1}) + d(x^c \ln x)].$$

Therefore, the dual numbers are elements of the 2-dimensional real algebra

$$D = \{z = x + \varepsilon y \mid x, y \in \mathbb{R}, \varepsilon^2 = 0\}$$

generated by 1 and ε. The dual quaternion $z = \sum_{j=0}^{3} e_j x_j + \varepsilon \sum_{j=0}^{3} e_j y_j$ is written as

$$z = a + \varepsilon b.$$

The conjugation number z^*, the absolute value $|z|$ and the inverse z^{-1} of $z = a + \varepsilon b$ are given by the followings:

$$z^* = x_0 - e_1 x_1 - e_2 x_2 - e_3 x_3 + \varepsilon (y_0 - e_1 y_1 - e_2 y_2 - e_3 y_3) = a^* + \varepsilon b^*,$$

$$|z|^2 = z z^* = \sum_{j=0}^{3} {x_j}^2 + 2\varepsilon \sum_{j=0}^{3} x_j y_j = \sum_{j=0}^{3} {\zeta_j}^2, \ z^{-1} = \frac{z^*}{|z|^2} \ (z \neq 0),$$

where $\zeta_j = x_j + \varepsilon y_j$ $(j = 0, 1, 2, 3)$, a^* and b^* are conjugate numbers of a and b, respectively.

We consider the following differential operators:

$$D := \frac{\partial}{\partial x_0} - e_1\frac{\partial}{\partial x_1} - e_2\frac{\partial}{\partial x_2} - e_3\frac{\partial}{\partial x_3} + \varepsilon\left(\frac{\partial}{\partial y_0} - e_1\frac{\partial}{\partial y_1} - e_2\frac{\partial}{\partial y_2} - e_3\frac{\partial}{\partial y_3}\right),$$
$$D^* = \frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} + e_3\frac{\partial}{\partial x_3} + \varepsilon\left(\frac{\partial}{\partial y_0} + e_1\frac{\partial}{\partial y_1} + e_2\frac{\partial}{\partial y_2} + e_3\frac{\partial}{\partial y_3}\right).$$

Then, we have the following for dual quaternion operators:

$$DD^* = D^*D = \sum_{j=0}^{3}\frac{\partial^2}{\partial {x_j}^2} + 2\varepsilon\left(\sum_{j=0}^{3}\frac{\partial^2}{\partial x_j \partial y_j}\right) = \sum_{j=0}^{3}\frac{\partial^2}{\partial {\tau_j}^2} = \Delta_z,$$

where

$$\frac{\partial}{\partial \tau_j} = \frac{\partial}{\partial x_j} + \varepsilon\frac{\partial}{\partial y_j}\ (j = 0, 1, 2, 3).$$

Definition 2. Let Ω be an open subset of $\mathbb{C}^2 \times \mathbb{C}^2$. A function $F(z) = f(a) + \varepsilon g(b) = \sum_{j=0}^{3} e_j u_j(a) + \varepsilon(\sum_{j=0}^{3} e_j v_j(b))$ is said to be hyperholomorphic on Ω if
(1) $u_j(a)$ and $v_j(b)$ $(j = 0, 1, 2, 3)$ are continuously differentiable on Ω,
(2) $D^*F(z) = 0$ on Ω.

Equation (2) of Definition 2 operates to $F(z)$ as follows:

$$\begin{aligned}
D^*F &= \left(\sum_{j=0}^{3} e_j\frac{\partial}{\partial x_j} + \varepsilon\sum_{j=0}^{3} e_j\frac{\partial}{\partial y_j}\right)\left(\sum_{j=0}^{3} e_j u_j + \varepsilon\sum_{j=0}^{3} e_j v_j\right)\\
&= \left(\frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} + e_3\frac{\partial}{\partial x_3} + \varepsilon\left(\frac{\partial}{\partial y_0} + e_1\frac{\partial}{\partial y_1} + e_2\frac{\partial}{\partial y_2}\right.\right.\\
&\quad \left.\left.+ e_3\frac{\partial}{\partial x_3}\right)\right)\cdot(u_0 + e_1u_1 + e_2u_2 + e_3u_3 + \varepsilon(v_0 + e_1v_1 + e_2v_2 + e_3v_3))\\
&= \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3} + e_1\left(\frac{\partial u_1}{\partial x_0} + \frac{\partial u_0}{\partial x_1} + \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)\\
&\quad + e_2\left(\frac{\partial u_2}{\partial x_0} - \frac{\partial u_3}{\partial x_1} + \frac{\partial u_0}{\partial x_2} + \frac{\partial u_1}{\partial x_3}\right) + e_3\left(\frac{\partial u_3}{\partial x_0} + \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} + \frac{\partial u_0}{\partial x_3}\right)\\
&\quad + \varepsilon\left(\frac{\partial v_0}{\partial x_0} - \frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2} - \frac{\partial v_3}{\partial x_3} + \frac{\partial u_0}{\partial y_0} - \frac{\partial u_1}{\partial y_1} - \frac{\partial u_2}{\partial y_2} - \frac{\partial u_3}{\partial y_3}\right.\\
&\quad + e_1\left(\frac{\partial v_1}{\partial x_0} + \frac{\partial v_0}{\partial x_1} + \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} + \frac{\partial u_1}{\partial y_0} + \frac{\partial u_0}{\partial y_1} + \frac{\partial u_3}{\partial y_2} - \frac{\partial u_2}{\partial y_3}\right)
\end{aligned}$$

$$+ e_2 \left(\frac{\partial v_2}{\partial x_0} - \frac{\partial v_3}{\partial x_1} + \frac{\partial v_0}{\partial x_2} + \frac{\partial v_1}{\partial x_3} + \frac{\partial u_2}{\partial y_0} - \frac{\partial u_3}{\partial y_1} + \frac{\partial u_0}{\partial y_2} + \frac{\partial u_1}{\partial y_3} \right)$$
$$+ e_3 \left(\frac{\partial v_3}{\partial x_0} + \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} + \frac{\partial v_0}{\partial x_3} + \frac{\partial u_3}{\partial y_0} + \frac{\partial u_2}{\partial y_1} - \frac{\partial u_1}{\partial y_2} + \frac{\partial u_0}{\partial y_3} \right) \Bigg).$$

Therefore, the Equation (2) of Definition 2 for $F(z)$ is equivalent to the following system of equations:

$$\frac{\partial u_0}{\partial x_0} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}, \ \frac{\partial u_2}{\partial x_3} = \frac{\partial u_1}{\partial x_0} + \frac{\partial u_0}{\partial x_1} + \frac{\partial u_3}{\partial x_2},$$
$$\frac{\partial u_3}{\partial x_1} = \frac{\partial u_2}{\partial x_0} + \frac{\partial u_0}{\partial x_2} + \frac{\partial u_1}{\partial x_3}, \ \frac{\partial u_1}{\partial x_2} = \frac{\partial u_3}{\partial x_0} + \frac{\partial u_2}{\partial x_1} + \frac{\partial u_0}{\partial x_3},$$
$$\frac{\partial v_0}{\partial x_0} + \frac{\partial u_0}{\partial y_0} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} + \frac{\partial u_1}{\partial y_1} + \frac{\partial u_2}{\partial y_2} + \frac{\partial u_3}{\partial y_3},$$
$$\frac{\partial v_2}{\partial x_3} + \frac{\partial u_2}{\partial y_3} = \frac{\partial v_1}{\partial x_0} + \frac{\partial v_0}{\partial x_1} + \frac{\partial v_3}{\partial x_2} + \frac{\partial u_1}{\partial y_0} + \frac{\partial u_0}{\partial y_1} + \frac{\partial u_3}{\partial y_2},$$
$$\frac{\partial u_3}{\partial y_1} + \frac{\partial v_3}{\partial x_1} = \frac{\partial v_2}{\partial x_0} + \frac{\partial v_0}{\partial x_2} + \frac{\partial v_1}{\partial x_3} + \frac{\partial u_2}{\partial y_0} + \frac{\partial u_0}{\partial y_2} + \frac{\partial u_1}{\partial y_3},$$
$$\frac{\partial v_1}{\partial x_2} + \frac{\partial u_1}{\partial y_2} = \frac{\partial v_3}{\partial x_0} + \frac{\partial v_2}{\partial x_1} + \frac{\partial v_0}{\partial x_3} + \frac{\partial u_3}{\partial y_0} + \frac{\partial u_2}{\partial y_1} + \frac{\partial u_0}{\partial y_3}. \tag{3.1}$$

Now, we add the following condition of integrability:

$$\frac{\partial v_0}{\partial x_0} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}, \ \frac{\partial v_2}{\partial x_3} = \frac{\partial v_1}{\partial x_0} + \frac{\partial v_0}{\partial x_1} + \frac{\partial v_3}{\partial x_2},$$
$$\frac{\partial v_3}{\partial x_1} = \frac{\partial v_2}{\partial x_0} + \frac{\partial v_0}{\partial x_2} + \frac{\partial v_1}{\partial x_3}, \ \frac{\partial v_1}{\partial x_2} = \frac{\partial v_3}{\partial x_0} + \frac{\partial v_2}{\partial x_1} + \frac{\partial v_0}{\partial x_3}. \tag{3.2}$$

We let

$$dxy = dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dy_0 \wedge dy_1 \wedge dy_2 \wedge dy_3.$$

Theorem 3.1. *Under the condition of integrability* (3.2), *let $F(z)$ be a hyperholomorphic function in an open set Ω of $\mathbb{C}^2 \times \mathbb{C}^2$ and*

$$\kappa = \hat{dy_0} - e_1\hat{dy_1} + e_2\hat{dy_2} - e_3\hat{dy_3} + \varepsilon(\hat{dx_0} - e_1\hat{dx_1} + e_2\hat{dx_2} - e_3\hat{dx_3}),$$

where $\hat{dx_j}$ is the dx_j-removed form on dxy, and $\hat{dy_j}$ is the dy_j-removed form on dxy $(j = 0, 1, 2, 3)$. Then for any domain $G \subset \Omega$ with smooth distinguished boundary bG of $\mathbb{C}^2 \times \mathbb{C}^2$,

$$\int_{bG} \kappa F(z) = 0,$$

where $\kappa F(z)$ is the product of quaternion numbers of the form κ on the function $F(z)$.

Proof. By the rule of the multiplication of quaternion numbers, we have

$$\begin{aligned}\kappa F(z) =&[\hat{dy_0} - e_1\hat{dy_1} + e_2\hat{dy_2} - e_3\hat{dy_3} + \varepsilon(\hat{dx_0} - e_1\hat{dx_1} + e_2\hat{dx_2} - e_3\hat{dx_3})] \\ &\cdot \left(\sum_{j=0}^{3} e_j u_j + \varepsilon \sum_{j=0}^{3} e_j v_j\right) \\ =&u_o\hat{dy_0} + e_1u_1\hat{dy_0} + e_2u_2\hat{dy_0} + e_3u_3\hat{dy_0} - e_1u_0\hat{dy_1} + u_1\hat{dy_1} \\ &- e_3u_2\hat{dy_1} + e_2u_3\hat{dy_1} + e_2u_0\hat{dy_2} - e_3u_1\hat{dy_2} - u_2\hat{dy_2} + e_1u_3\hat{dy_2} \\ &- e_3u_0\hat{dy_3} - e_2u_1\hat{dy_3} + e_1u_2\hat{dy_3} + u_3\hat{dy_3} + \varepsilon[v_0\hat{dy_0} + e_1v_1\hat{dy_0} \\ &+ e_2v_2\hat{dy_0} + e_3v_3\hat{dy_0} - e_1v_0\hat{dy_1} + v_1\hat{dy_1} - e_3v_2\hat{dy_1} + e_2v_3\hat{dy_1} \\ &+ e_2v_0\hat{dy_2} - e_3v_1\hat{dy_2} - v_2\hat{dy_2} + e_1v_3\hat{dy_2} - e_3v_0\hat{dy_3} - e_2v_1\hat{dy_3} \\ &+ e_1v_2\hat{dy_3} + v_3\hat{dy_3} + u_0\hat{dx_0} + e_1u_1\hat{dx_0} + e_2u_2\hat{dx_0} + e_3u_3\hat{dx_0} \\ &- e_1u_0\hat{dx_1} + u_1\hat{dx_1} - e_3u_2\hat{dx_1} + e_2u_3\hat{dx_1} + e_2u_0\hat{dx_2} - e_3u_1\hat{dx_2} \\ &- u_2\hat{dx_2} + e_1u_3\hat{dx_2} - e_3u_0\hat{dx_3} - e_2u_1\hat{dx_3} + e_1u_2\hat{dx_3} + u_3\hat{dx_3}].\end{aligned}$$

Therefore,

$$\begin{aligned}d(\kappa F) =& \left(\sum_{j=o}^{3} \frac{\partial}{\partial x_j}dx_j + \varepsilon \sum_{j=o}^{3} \frac{\partial}{\partial y_j}dy_j\right)(\kappa F) \\ =& \varepsilon\left[\left(\frac{\partial u_0}{\partial y_0} - \frac{\partial u_1}{\partial y_1} - \frac{\partial u_2}{\partial y_2} - \frac{\partial u_3}{\partial y_3} + \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3}\right)dxy\right. \\ &+ e_1\left(\frac{\partial u_1}{\partial y_0} + \frac{\partial u_0}{\partial y_1} + \frac{\partial u_3}{\partial y_2} - \frac{\partial u_2}{\partial y_3} + \frac{\partial u_1}{\partial x_0} + \frac{\partial u_0}{\partial x_1} + \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)dxy \\ &+ e_2\left(\frac{\partial u_2}{\partial y_0} - \frac{\partial u_3}{\partial y_1} + \frac{\partial u_0}{\partial y_2} + \frac{\partial u_1}{\partial y_3} + \frac{\partial u_2}{\partial x_0} - \frac{\partial u_3}{\partial x_1} + \frac{\partial u_0}{\partial x_2} + \frac{\partial u_1}{\partial x_3}\right)dxy \\ &\left.+e_3\left(\frac{\partial u_3}{\partial y_0} + \frac{\partial u_2}{\partial y_1} - \frac{\partial u_1}{\partial y_2} + \frac{\partial u_0}{\partial y_3} + \frac{\partial u_3}{\partial x_0} + \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} + \frac{\partial u_0}{\partial x_3}\right)dxy\right].\end{aligned}$$

By Equations (3.1) and (3.2), we have $d(\kappa F) = 0$. By Stokes theorem, we have

$$\int_{bG} \kappa F = \int_G d(\kappa F) = 0. \qquad \square$$

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