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# A NOTE ON THE WEIGHTED $q$-GENOCCHI NUMBERS AND POLYNOMIALS WITH THEIR INTERPOLATION FUNCTION

Serkan Araci, Mehmet Açikgöz and Jong Jin Seo

**Abstract.** Recently, T. Kim has introduced and analysed the $q$-Bernoulli numbers and polynomials with weight $\alpha$ cf.[7]. By the same motivaton, we also give some interesting properties of the $q$-Genocchi numbers and polynomials with weight $\alpha$. Also, we derive the $q$-extensions of zeta type functions with weight $\alpha$ from the Mellin transformation of this generating function which interpolates the $q$-Genocchi polynomials with weight $\alpha$ at negative integers.

## 1. Introduction, Definitions and Notations

Let $p$ be a fixed odd prime number. Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of $p$-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ . The $p$-adic absolute value is defined by $|p|_p = \frac{1}{p}$. In this paper we assume $|q-1|_p < 1$ as an indeterminate. In [12-15], the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by *Kim* as follows:

$$(1.1) \qquad I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x$$

where $[x]_q$ is a $q$-extension of $x$ which is defined by

$$[x]_q = \frac{1-q^x}{1-q}, \quad \text{see [1-15]}$$

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Note that $\lim_{q\to 1}[x]_q = x$.

The $q$-Genocchi numbers are defined as follows:

$$G_{0,q}=0, \text{ and } q\left(qG_q+1\right)^n+G_{n,q}=\begin{cases}[2]_q, & n=1;\\ 0, & n>1\end{cases} \tag{1.2}$$

with the usual convention of replacing $(G_q)^n$ by $G_{n,q}$ (see [1]).

The $(h,q)$-Genocchi numbers are defined as follows:

$$G_{0,q}^{(h)}=0, \text{ and } q^{h-2}\left(qG_q^{(h)}+1\right)^n+G_{n,q}^{(h)}=\begin{cases}[2]_q, & n=1;\\ 0, & n>1,\end{cases}$$

with usual the convention about replacing $\left(G_q^{(h)}\right)^n$ by $G_{n,q}^{(h)}$ (see [2]).

In [7], the $q$-Bernoulli numbers and polynomials with weight $\alpha$ has been investigated some interesting properties by *Kim*. By using $p$-adic $q$-integral on $\mathbb{Z}_p$, we also investigate some interesting identities of the $q$-Genocchi numbers and polynomials with weight $\alpha$. Furthermore, we derive the $q$-extensions of zeta type functions with weight $\alpha$ from the Mellin transformation of this generating function which interpolates the $q$-Genocchi polynomials with weight $\alpha$ at negative integers.

## 2. On the weighted $q$-Genocchi numbers and polynomials

Let $f_n(x)=f(x+n)$. By using definition (1.1), we easily get

$$\begin{aligned}-qI_{-q}(f_1) &= \lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{x=0}^{p^N-1}f(x+1)(-q)^{x+1}\\ &= \lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{x=0}^{p^N-1}f(x)(-q)^x-(1+q)\\ &\quad\cdot\lim_{N\to\infty}\frac{f\left(p^N\right)q^{p^N}+f(0)}{1+q^{p^N}}\\ &= I_{-q}(f)-[2]_q f(0) \qquad (2.1)\end{aligned}$$

and

$$\begin{aligned} q^2 I_{-q}(f_2) &= q^2 \int_{\mathbb{Z}_p} f(x+2)\, d\mu_{-q}(x) \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x+2)(-q)^{x+2} \\ &= -qI_{-q}(f_1) + q(1+q) \lim_{N\to\infty} \frac{f\left(p^N+1\right) q^{p^N} + f(1)}{1+q^{p^N}} \\ &= I_{-q}(f) - [2]_q f(0) + [2]_q f(1). \end{aligned}$$

Thus we have

$$-I_{-q}(f) + q^2 I_{-q}(f_2) = [2]_q \sum_{l=0}^{1} (-1)^l q^{1-l} f(l).$$

Continuing this process, we obtain the following Lemma

**Lemma 1.** *For $n \in \mathbb{N}^*$ ,we obtain*

$$(-1)^{n-1} I_{-q}(f) + q^n I_{-q}(f_n) = [2]_q \sum_{l=0}^{n-1} (-1)^l q^{n-l-1} f(l), \tag{2.2}$$

**Definition 1.** *Let $\alpha, n \in \mathbb{N}^*$. The q-Genocchi numbers with weight $\alpha$ are defined as follows:*

$$\frac{\widetilde{G}^{(\alpha)}_{n+1,q}}{n+1} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m [m]^n_{q^\alpha}. \tag{2.3}$$

From (2.3) we obtain,

$$\begin{aligned} &\frac{\widetilde{G}^{(\alpha)}_{n+1,q}}{n+1} \\ &= \frac{[2]_q}{(1-q^\alpha)^n} \sum_{m=0}^{\infty} (-1)^m q^m (1-q^{m\alpha})^n \\ &= \frac{[2]_q}{(1-q^\alpha)^n} \sum_{m=0}^{\infty} (-1)^m q^m \sum_{l=0}^{n} \binom{n}{l} (-1)^l (q^{m\alpha})^l \\ &= \frac{[2]_q}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \sum_{m=0}^{\infty} (-1)^m q^{m\alpha l+m} \\ &= \frac{[2]_q}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{\alpha l+1}}. \end{aligned}$$

Therefore we obtain the following theorem:

**Theorem 1.** *Let* $\alpha, n \in \mathbb{N}^*$ *and we have*

$$\frac{\widetilde{G}_{n+1,q}^{(\alpha)}}{n+1} = \frac{[2]_q}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{\alpha l+1}}. \tag{2.4}$$

In (1.1), replace $f(x)$ by $[x]_{q^\alpha}^n$ we have,

$$\begin{aligned}
\int_{\mathbb{Z}_p} [x]_{q^\alpha}^n \, d\mu_{-q}(x) &= \frac{1}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \int_{\mathbb{Z}_p} q^{\alpha l x} d\mu_{-q}(x) \\
&= \frac{1}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} \left(-q^{\alpha l+1}\right)^x \\
&= \frac{1}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{(1+q)}{1+q^{\alpha l+1}} \lim_{N\to\infty} \frac{1+\left(q^{\alpha l+1}\right)^{p^N}}{1+q^{p^N}} \qquad (2.5) \\
&= \frac{[2]_q}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{\alpha l+1}} \\
&= \frac{\widetilde{G}_{n+1,q}^{(\alpha)}}{n+1}.
\end{aligned}$$

From (2.4) and (2.5) we obtain $q$-Genocchi numbers with weight $\alpha$ witt's type formula the following theorem:

**Theorem 2.** *For* $\alpha, n \in \mathbb{N}^*$ *and we have*

$$\frac{\widetilde{G}_{n+1,q}^{(\alpha)}}{n+1} = \int_{\mathbb{Z}_p} [x]_{q^\alpha}^n \, d\mu_{-q}(x). \tag{2.6}$$

From (2.3) we easily get,

$$\int_{\mathbb{Z}_p} e^{t[x]_{q^\alpha}} d\mu_{-q}(x) = [2]_q \, t \sum_{m=0}^{\infty} (-1)^m q^m e^{t[m]_{q^\alpha}}. \tag{2.7}$$

By (2.7) we have

$$\sum_{n=0}^{\infty} \widetilde{G}_{n,q}^{(\alpha)} \frac{t^n}{n!} = [2]_q \, t \sum_{m=0}^{\infty} (-1)^m q^m e^{t[m]_{q^\alpha}}.$$

Therefore we obtain the following corollary:

**Corollary 1.** *Let* $D_q^{(\alpha)}(t)=\sum_{n=0}^{\infty}\widetilde{G}_{n,q}^{(\alpha)}\frac{t^n}{n!}$. *Then we have*

$$D_q^{(\alpha)}(t)=[2]_q\, t\sum_{m=0}^{\infty}(-1)^m q^m e^{t[m]_{q^\alpha}}.$$

Now, we consider the $q$-Genocchi polynomials with weight $\alpha$ as follows:

$$\frac{\widetilde{G}_{n+1,q}^{(\alpha)}(x)}{n+1}=\int_{\mathbb{Z}_p}[x+y]_{q^\alpha}^n\, d\mu_{-q}(y)\,,\quad n\in\mathbb{N}\text{ and }\alpha\in\mathbb{N}^*. \tag{2.8}$$

From (2.8) we see that

$$\begin{aligned}\frac{\widetilde{G}_{n+1,q}^{(\alpha)}(x)}{n+1} &= \frac{[2]_q}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}\frac{1}{1+q^{\alpha l+1}}\\ &= [2]_q\sum_{m=0}^{\infty}(-1)^m q^m [m+x]_{q^\alpha}^n\,. \qquad (2.9)\end{aligned}$$

Let $D_q^{(\alpha)}(t,x)=\sum_{n=0}^{\infty}\widetilde{G}_{n,q}^{(\alpha)}(x)\frac{t^n}{n!}$. Then we have

$$\begin{aligned}D_q^{(\alpha)}(t,x) &= [2]_q\, t\sum_{m=0}^{\infty}(-1)^m q^m e^{t[m+x]_{q^\alpha}}\\ &= \sum_{n=0}^{\infty}\widetilde{G}_{n,q}^{(\alpha)}(x)\frac{t^n}{n!}. \qquad (2.10)\end{aligned}$$

By *Lemma* 1, we see that

$$(-1)^{n-1}\frac{\widetilde{G}_{m+1,q}^{(\alpha)}}{m+1}+q^n\frac{\widetilde{G}_{m+1,q}^{(\alpha)}(n)}{m+1}=[2]_q\sum_{l=0}^{n-1}(-1)^l q^{n-l-1}[l]_{q^\alpha}^m\,.$$

Therefore we obtain the following theorem:

**Theorem 3.** *For* $m\in\mathbb{N}$, *and* $\alpha,n\in\mathbb{N}^*$, *one has*

$$(-1)^{n-1}\frac{\widetilde{G}_{m+1,q}^{(\alpha)}}{m+1}+q^n\frac{\widetilde{G}_{m+1,q}^{(\alpha)}(n)}{m+1}=[2]_q\sum_{l=0}^{n-1}(-1)^l q^{n-l-1}[l]_{q^\alpha}^m\,.$$

In (2.1) it is known that

$$qI_{-q}(f_1)+I_{-q}(f)=[2]_q f(0)\,.$$

If we take $f(x)=e^{t[x]_{q^\alpha}}$, then we have

$$\begin{aligned} [2]_q &= q\int_{\mathbb{Z}_p} e^{t[x+1]_{q^\alpha}} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} e^{t[x]_{q^\alpha}} d\mu_{-q}(x) \\ &= \sum_{n=0}^{\infty}\left(q\widetilde{G}_{n,q}^{(\alpha)}(1)+\widetilde{G}_{n,q}^{(\alpha)}\right)\frac{t^{n-1}}{n!}. \end{aligned} \tag{2.11}$$

Therefore, by (2.11), we obtain the following theorem:

**Theorem 4.** *For $\alpha\in\mathbb{N}^*$ and $n\in\mathbb{N}$, we get*

$$\widetilde{G}_{0,q}^{(\alpha)}=0,\ \ and\ \ q\widetilde{G}_{n,q}^{(\alpha)}(1)+\widetilde{G}_{n,q}^{(\alpha)}=\begin{cases}[2]_q, & if\ n=1;\\ 0, & if\ n\neq 1.\end{cases}$$

From (2.8), we can easily derive

$$\begin{aligned} \int_{\mathbb{Z}_p}[x+y]_{q^\alpha}^n d\mu_{-q}(y) &= \frac{[d]_{q^\alpha}^n}{[d]_{-q}}\sum_{a=0}^{d-1}(-1)^a q^a\int_{\mathbb{Z}_p}\left[\frac{x+a}{d}+y\right]_{q^{d\alpha}}^n d\mu_{(-q)^d}(y) \\ &= \frac{[d]_{q^\alpha}^n}{[d]_{-q}}\sum_{a=0}^{d-1}(-1)^a q^a\frac{\widetilde{G}_{n+1,q^d}^{(\alpha)}\left(\frac{x+a}{d}\right)}{n+1}. \end{aligned} \tag{2.12}$$

Therefore, by (2.12), we obtain the following theorem:

**Theorem 5.** *For $d\equiv 1\ (mod\ 2)$, $n\in\mathbb{N}^*$ and $\alpha\in\mathbb{N}$, we get*

$$\widetilde{G}_{n,q}^{(\alpha)}(x)=\frac{[d]_{q^\alpha}^{n-1}}{[d]_{-q}}\sum_{a=0}^{d-1}(-1)^a q^a\widetilde{G}_{n,q^d}^{(\alpha)}\left(\frac{x+a}{d}\right).$$

## 3. Interpolation function of the polynomials $\widetilde{G}_{n,q}^{(\alpha)}(x)$

In this section, we give interpolation function of the generating functions of $q$-Genocchi polynomials with weight $\alpha$. For $s\in\mathbb{C}$, by applying the Mellin transformation to (2.10), we obtain

$$\begin{aligned} \xi_q^{(\alpha)}(s,x) &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}\left\{-D_q^{(\alpha)}(-t,x)\right\}dt \\ &= [2]_q\sum_{m=0}^{\infty}(-1)^m q^m\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-t[m+x]_{q^\alpha}}dt \\ &= [2]_q\sum_{m=0}^{\infty}\frac{(-1)^m q^m}{[m+x]_{q^\alpha}^s}, \end{aligned}$$

where $\Gamma(s)$ is Euler-gamma function.

Thus, we define $q-$extension zeta type function as follows:

**Definition 2.** *For $s \in \mathbb{C}$ and $\alpha \in \mathbb{N}^*$ we have*

$$\xi_q^{(\alpha)}(s,x) = [2]_q \sum_{m=0}^{\infty} \frac{(-1)^m q^m}{[m+x]_{q^\alpha}^s} \tag{3.1}$$

$\xi_q^{(\alpha)}(s,x)$ can be continued analytically to an entire function.
By subsituting $s=-n$ into (3.1) we easily get

$$\xi_q^{(\alpha)}(-n,x) = \frac{\widetilde{G}_{n+1,q}^{(\alpha)}(x)}{n+1}.$$

Therefore, we obtain the following theorem:

**Theorem 6.** *Let $q,s \in \mathbb{C}$ with $|q|<1$ and $0<x\leq 1$. Then we define*

$$\xi_q^{(\alpha)}(-n,x) = \frac{\widetilde{G}_{n+1,q}^{(\alpha)}(x)}{n+1}.$$

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Serkan Aracı
Department of Mathematics, Faculty of Science and Arts, University of Gaziantep,
Gaziantep 27310, Turkey.
E-mail: mtsrkn@hotmail.com

Mehmet Açıkgöz
Department of Mathematics, Faculty of Science and Arts, University of Gaziantep,
Gaziantep 27310, Turkey.
E-mail: acikgoz@gantep.edu.tr

Jong Jin Seo
Department of Applied Mathematics, Pukyong National University,
Busan 608-737, Korea.
E-mail: seo2011@pknu.ac.kr