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A HOMOMORPHISM OF POINTED MINIMAL SETS AND ELLIS GROUPS

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Abstract. In this paper we give some results on homomorphisms of pointed minimal sets. In particular, we investigate some characterizations on Ellis groups.

1. Introduction

Universal minimal sets were studied by R. Ellis in [2]. In [3], S. Glasner introduced the Ellis group which is a certain group of the universal minimal set. Given a homomorphism of pointed minimal sets $\pi : (X, x_0) \to (Y, y_0)$, we can define the Ellis groups $\mathcal{G}(X, x_0)$ and $\mathcal{G}(Y, y_0)$ and give a relationship between the homomorphism and the Ellis groups.

The purpose of this paper is to study some characterizations on Ellis groups and investigate the equivalent conditions for the homomorphism to be proximal. Also we give some results on homomorphisms of pointed minimal sets.

2. Preliminaries

A *transformation group*, or *flow*, (X, T), will consist of a jointly continuous action of the topological group T on the compact Hausdorff space X. The group T, with identity e, is assumed to be topologically discrete

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and remain fixed throughout this paper, so we may write X instead of (X, T).

A *homomorphism* of flows is a continuous, equivariant map. A homomorphism whose range is minimal is always onto, and a homomorphism whose domain is point transitive is determined by its value at a single point.

A *point transitive flow*, (X, x_0) consists of a flow X with a distinguished point x_0 which has dense orbit. Espectially, (Z, z_0) is a *universal point transitive flow* if (a) (Z, z_0) is point transitive and (b) if (X, x_0) is point transitive, then there exists a homomorphism from (Z, z_0) onto (X, x_0).

A flow is said to be *minimal* if every point has dense orbit. Minimal flows are also referred to as minimal sets. Espectially, (Z, T) is a *universal minimal set* if (a) (Z, T) is minimal and (b) if (X, T) is minimal, then there exists a homomorphism from (Z, T) onto (X, T).

The compact Hausdorff space X carries a natural uniformity whose indices are the neighborhoods of the diagonal in $X \times X$. Two points $x, x' \in X$ are said to be *proximal* if, given any index α, there exists $t \in T$ such that $(xt, x't) \in \alpha$. The proximal relation in X, denoted by $P(X, T)$, is the set of all proximal pairs in X. X is said to be *distal* if $P(X, T) = \triangle$, the diagonal of $X \times X$ and is said to be *proximal* if $P(X, T) = X \times X$. Given $x \in X$, we define $P(x) = \{x' \in X \mid (x, x') \in P(X, T)\}$.

A homomorphism $\pi : X \to Y$ is said to be *proximal* (resp. *distal*) if whenever $x, x' \in \pi^{-1}(y)$ then x and x' are proximal (resp. distal).

Given a flow (X, T), we may regard T as a set of self-homeomorphisms of X. The *enveloping semigroup* of X, denoted by $E(X)$, is the closure of T in X^X taken with the product topology. $E(X)$ is at once a transformation group and a sub-semigroup of X^X. The minimal right ideals of $E(X)$, considered as a semigroup, coincide with the minimal sets of $E(X)$.

If E is some enveloping semigroup, and there exists a homomorphism $\theta : (E, e) \to (E(X), e)$ we say that E is an *enveloping semigroup for* X. If such a homomorphism exists, it must be unique, and, given $x \in X$ and $p \in E$ we may write xp to mean $x\theta(p)$ unambiguously.

LEMMA 2.1. [4] *If (X, x) and (Y, y) are point transitive flows, and E is an enveloping semigroup for X and Y, there exists a unique homomorphism $\pi : (X, x) \to (Y, y)$ if and only if $xp = xq$ for $p, q \in E$ implies $yp = yq$.*

LEMMA 2.2. [4] *Let E be an enveloping semigroup for X and let I be a minimal right ideal in E. The following are true :*

(1) *The set $J(I)$ of idempotent elements in I is non-empty.*
(2) *$up = p$ whenever $p \in I$ and $u \in J(I)$.*
(3) *Iu is a group with identity u for each $u \in J(I)$.*
(4) *Given $x \in X$, the following conditions are equivalent :*
  (a) *x is an almost periodic point.*
  (b) *$\overline{xT} = xI$.*
  (c) *$x = xu$ for some $u \in J(I)$.*

LEMMA 2.3. [4] *Let E be an enveloping semigroup for X. Then for any points $x, x' \in X$, (1) and (2) are equivalent :*

(1) *$(x, x') \in P(X, T)$.*
(2) *There exists a minimal right ideal I in E such that $xp = x'p$ for every $p \in I$.*

3. Some results on homomorphisms of pointed minimal sets

Let βT denote the Stone-Cěch compactification of T. Then $(\beta T, e)$ is a universal point transitive flow. It is also clear that βT is an enveloping semigroup for X, whenever X is a flow with acting group T.

Let M be a fixed minimal right ideal in βT and let $J = J(M)$. We choose a distinguished idempotent $u \in J$ and let G denote the group Mu. Given a minimal set X, we choose a point $x_0 \in Xu = \{xu \mid x \in X\}$. Under the canonical map $(\beta T, e) \to (X, x_0)$, M is mapped onto X and u onto x_0. Thus (M, u) is a universal minimal pointed set. Given a pointed minimal set (X, x_0), we define the Ellis group of (X, x_0) to be

$$\mathcal{G}(X, x_0) = \{\alpha \in G \mid x_0\alpha = x_0\}.$$

Clearly $\mathcal{G}(X, x_0)$ is a subgroup of G.

Let $\pi : (X, x_0) \to (Y, y_0)$ be a homomorphism of pointed minimal sets. Then there exist homomorphisms $\gamma : (M, u) \to (X, x_0)$ and $\delta : (M, u) \to (Y, y_0)$ such that $\pi \circ \gamma = \delta$.

LEMMA 3.1. [3] *Let $\pi : (X, x_0) \to (Y, y_0)$ be a homomorphism of pointed minimal sets. Then the following statements are true :*

(1) *$\mathcal{G}(X, x_0) \subset \mathcal{G}(Y, y_0)$.*
(2) *$\mathcal{G}(X, x_0) = \mathcal{G}(Y, y_0)$ if and only if π is proximal.*
(3) *π is distal if and only if for every $y \in Y$ and $p \in M$ such that $y_0p = y$, $\pi^{-1}(y) = x_0\mathcal{G}(Y, y_0)p$.*

LEMMA 3.2. *Let $\pi : (X, x_0) \to (Y, y_0)$ be a homomorphism of pointed minimal sets. Then π is proximal if and only if $\pi^{-1}(y) \subset xJ$ for any $x \in \pi^{-1}(y)$.*

Proof. This follows immediately from [3, Proposition 4.1] and the fact that if $x, x' \in \pi^{-1}(y)$, then there exists $u \in J$ such that $x' = xu$. □

THEOREM 3.3. *Let $\pi : (X, x_0) \to (Y, y_0)$ be a homomorphism of pointed minimal sets. Then the following conditions are equivalent :*

(1) *π is proximal.*
(2) *Suppose $y \in Y$. Then $\pi^{-1}(y) \subset xJ$ for any $x \in \pi^{-1}(y)$.*
(3) *Suppose $y \in Y$ and that $v \in J$ with $yv = y$. Then $\pi^{-1}(y)v$ is a singleton.*
(4) *$\mathcal{G}(X, x_0) = \mathcal{G}(Y, y_0)$.*
(5) *For any two points $x, x' \in X$ with (x, x') almost periodic and $\pi(x) = \pi(x')$, we have $x = x'$.*
(6) *Given any pair of homomorphisms $\gamma : M \to X$ and $\delta : M \to X$ with $\pi\delta = \pi\gamma$, we have $\delta = \gamma$.*

Proof. That (1) and (2), (1) and (3), (1) and (4) are equivalent follows from Lemma 3.2, [4, Lemma 2.5.8] and Lemma 3.1 respectively.

(1) implies (5). This follows from the fact that if a pair of points is both proximal and almost periodic, the two points are identical.

(5) implies (6). Given homomorphisms $\gamma : M \to X$ and $\delta : M \to X$ with $\pi\delta = \pi\gamma$ we let $\gamma(u) = x'$ and $\delta(u) = x$. Then $(x', x)u = (x', x)$ so (x', x) is almost periodic. Now $\pi(x') = \pi\gamma(u) = \pi\delta(u) = \pi(x)$. Thus we get $x' = x$, so that $\gamma(u) = \delta(u)$. Since M is minimal, it follows that $\gamma = \delta$.

(6) implies (1). Suppose $x, x' \in X$ and $\pi(x) = \pi(x')$. Since X is minimal, there exists $v \in J$ with $xv = x$. Let $x'' = x'v$. Then $x''v = (x'v)v = x'v$. Now we define homomorphisms $\gamma : M \to X$ and $\delta : M \to X$ by $\gamma(v) = x''$ and $\delta(v) = x$. Then $\pi\gamma(v) = \pi(x'') = \pi(x'v) = \pi(x')v = \pi(x)v = \pi(x) = \pi\delta(v)$ so $\pi\gamma = \pi\delta$. By hypothesis, we get $\gamma = \delta$. Thus $x'' = \gamma(v) = \delta(v) = x$. Hence $(x, x') \in P(X, T)$. □

REMARK 3.4. *Notice that if X is proximal and minimal, then the only endomorphism of X is the identity. Suppose that $\pi : X \to X$ is a homomorphism and that X is proximal and minimal. By Theorem 3.3 (3), we get $\pi^{-1}(x)v$ is a singleton whenever $x \in X$ and $v \in J$ with $xv = x$. Now let $\pi^{-1}(x)v = \{x_1\}$. Then there exists $x' \in \pi^{-1}(x)$ with $x_1 = x'v$, and so $x_1v = x'v = x_1$. Hence $(x, x_1)v = (x, x_1)$ so (x, x_1) is* *almost periodic. Since* $(x, x_1) \in P(X, T)$*, it follows that* $x_1 = x$*. This means that* $\pi^{-1}(x) = \{x\}$*. That* π *is an identity automorphism follows from the fact that* X *is minimal.*

PROPOSITION 3.5. *Let* $\pi : (X, x_0) \to (Y, y_0)$ *be a homomorphism of pointed minimal sets, and let* $y \in Y$ *and* $p \in M$ *with* $y_0 p = y$*. If* $x_0\mathcal{G}(X, x_0)p \subset x_0 J$*, then* $P(y_0) = Y$*.*

Proof. Suppose that $y \in Y$ and that $p \in M$ with $y_0 p = y$. Since $x_0\mathcal{G}(X, x_0)p \subset x_0 J$, it follows that for any $\alpha \in \mathcal{G}(X, x_0)$ there exists $v \in J$ such that $x_0 \alpha p = x_0 p = x_0 v$. Then $y = y_0 p = \pi(x_0 p) = \pi(x_0 v) = y_0 v$ whence $(y, y_0) \in P(Y, T)$. Thus $P(y_0) = Y$. □

The proof of the following result are similar to that of Proposition 3.5.

PROPOSITION 3.6. *Let* $\pi : (X, x_0) \to (Y, y_0)$ *be a homomorphism of pointed minimal sets, and let* $x \in X$ *and* $p \in M$ *with* $x_0 p = x$*. If* $x_0\mathcal{G}(X, x_0)p \subset x_0 J$*, then* $P(x_0) = X$*.*

PROPOSITION 3.7. *Let* $\pi : (X, x_0) \to (Y, y_0)$ *be a homomorphism of pointed minimal sets, and let* $x \in X$ *and* $p \in M$ *with* $x_0 p = x$*. If* $y_0\mathcal{G}(X, x_0)p \subset y_0 J$ *and* π *is proximal, then* $P(x_0) = X$*.*

Proof. Suppose that $x \in X$, $p \in M$ with $x_0 p = x$. Given $\alpha \in \mathcal{G}(X, x_0)$ we can pick $v \in J$ such that $y_0 \alpha p = y_0 v$. Then $\pi(x) = \pi(x_0 \alpha p) = y_0 \alpha p = y_0 v = \pi(x_0 v)$. But since π is proximal, it follows that $(x, x_0 v) \in P(X, T)$. Hence there exists $q \in M$ with $xq = x_0 vq = x_0 q$. This means that $(x, x_0) \in P(X, T)$. Thus $P(x_0) = X$. □

REMARK 3.8. *Notice that if in addition to the assumptions of Proposition 3.6 and Proposition 3.7,* $E(X)$ *contains the unique minimal right ideal then* X *is proximal.*

THEOREM 3.9. *Let* $\pi : (X, x_0) \to (Y, y_0)$ *be a homomorphism of pointed minimal sets, and let* $y \in Y$ *and* $v \in J$ *with* $yv = y$*. If* $\pi^{-1}(y)v \subset x_0 J$*, then the following statements are true :*

(1) $x \in P(x_0)$ *for all* $x \in \pi^{-1}(y)v$*.*
(2) $y \in P(y_0)$*.*
(3) $\pi^{-1}(y)v$ *is a singleton.*

Proof. Suppose that $y \in Y$ and that $v \in J$ with $yv = y$. Since $\pi^{-1}(y)v \subset x_0 J$, it follows that for any $x, x' \in \pi^{-1}(y)v$, there exist

$x_1, x_2 \in \pi^{-1}(y)$ and $w_1, w_2 \in J$ such that $x = x_1 v = x_0 w_1$ and $x' = x_2 v = x_0 w_2$, whence $x \in P(x_0)$ by Lemma 2.3. That $y \in P(y_0)$ follows from the fact that $y = yv = \pi(x_1)v = \pi(x) = \pi(x_0 w_1) = y_0 w_1$. Also $(x, x')v = (x_1 v, x_2 v)v = (x_1 v, x_2 v) = (x, x')$ so (x, x') is an almost periodic point. But $xw_2 = (x_0 w_1)w_2 = x_0(w_1 w_2) = x_0 w_2 = x'$ implies that $(x, x') \in P(X, T)$. Hence $x = x'$ whence $\pi^{-1}(y)v$ is a singleton. □

PROPOSITION 3.10. *Let $\pi : (X, x_0) \to (Y, y_0)$ be a homomorphism of pointed minimal sets and $y \in Y$. Then $\pi^{-1}(y) = \bigcup\{\pi^{-1}(y)v \mid v \in J \text{ and } yv = y\}$.*

Proof. Suppose that $v \in J$ with $yv = y$ and that $x \in \pi^{-1}(y)v$. Pick $x_1 \in \pi^{-1}(y)$ with $x = x_1 v$. Then $\pi(x) = \pi(x_1)v = yv = y$ so $x \in \pi^{-1}(y)$.

Conversely, suppose $x \in \pi^{-1}(y)$. Since X is minimal, we have $v \in J$ with $x = xv$. Then $y = \pi(x) = \pi(x)v = yv$ and so $x = xv \in \pi^{-1}(y)v$. □

The proof of the following corollary is immediate from Theorem 3.9 and Proposition 3.10.

COROLLARY 3.11. *Let $\pi : (X, x_0) \to (Y, y_0)$ be a homomorphism of pointed minimal sets, and let $y \in Y$ and $\pi^{-1}(y) \subset x_0 J$. Then the following statements are true :*

(1) $\pi^{-1}(y) \subset P(x_0)$.

(2) $y \in P(y_0)$.

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