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ORDER SYSTEMS, IDEALS AND RIGHT FIXED MAPS OF SUBTRACTION ALGEBRAS

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Abstract. Conditions for an ideal to be irreducible are provided. The notion of an order system in a subtraction algebra is introduced, and related properties are investigated. Relations between ideals and order systems are given. The concept of a fixed map in a subtraction algebra is discussed, and related properties are investigated.

1. Introduction

B. M. Schein [14] considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [2]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [15] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun et al. [10] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [6], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results. Y. B. Jun and K. H. Kim [11] introduced the notion of prime and irreducible ideals of a subtraction algebra, and gave a characterization of a prime ideal. They also provided a condition for an ideal to be a prime/irreducible ideal. In this paper, we give conditions for an ideal to be irreducible. We introduce the notion of an order system in a subtraction algebra, and investigate related properties. We provide relations between ideals and order systems. We deal with the concept of a fixed map in a subtraction algebra, and investigate related properties.

2. Preliminaries

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

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(S1) $x - (y - x) = x$;
(S2) $x - (x - y) = y - (y - x)$;
(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [2], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true (see [10, 11]):

(a1) $(x - y) - y = x - y$.
(a2) $x - 0 = x$ and $0 - x = 0$.
(a3) $(x - y) - x = 0$.
(a4) $x - (x - y) \leq y$.
(a5) $(x - y) - (y - x) = x - y$.
(a6) $x - (x - (x - y)) = x - y$.
(a7) $(x - y) - (z - y) \leq x - z$.
(a8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.
(a9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.
(a10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.
(a11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.
(a12) $(x - y) - z = (x - z) - (y - z)$.

As a weak form of a subtraction algebra, Jun et al. discussed the weak subtraction algebras as follows:

Definition 2.1 ([8])**.** By a *weak subtraction algebra* (*WS-algebra*), we mean a triplet $(W, -, 0)$, where W is a nonempty set, $-$ is a binary operation on W and $0 \in W$ is a nullary operation, called *zero element*, such that

(S3) $(\forall x, y, z \in W)$ $((x - y) - z = (x - z) - y)$,
(S4) $(\forall x \in W)$ $(x - 0 = x,\ x - x = 0)$,
(a12) $(\forall x, y, z \in W)$ $((x - y) - z = (x - z) - (y - z))$.

Note that every subtraction algebra is a WS-algebra, but the converse is not true in general (see [8]).

3. Order systems and ideals in WS-algebras

In what follows, let X denote a WS-algebra unless otherwise specified.

Definition 3.1. A nonempty subset A of X is called an *ideal* of X if it satisfies

(b1) $0 \in A$

(b2) $(\forall x \in X)\ (\forall y \in A)\ (x - y \in A \Rightarrow x \in A)$.

The set of all ideals of X will be denoted by $Id(X)$.

Lemma 3.2. *An ideal A of a subtraction algebra X has the following property:*

$$(\forall x \in X)(\forall y \in A)(x \leq y \Rightarrow x \in A).$$

Proof. Straightforward. □

Theorem 3.3. *Let A be a nonempty subset of X. Then the set*

$$K := \left\{ x \in X \;\middle|\; \begin{array}{l} (\cdots((x - a_1) - a_2) - \cdots) - a_n = 0 \\ \text{for some } a_1, a_2, \ldots, a_n \in A \end{array} \right\}$$

is a minimal ideal of X containing A.

Proof. It is similar to the proof of Theorem 3.2 in [6]. □

We say that the ideal K is the *ideal generated* by A, and is denoted by $\langle A \rangle$.

Definition 3.4. An ideal A of X is said to be *irreducible* if for any ideals C and D of X, $A = C \cap D$ implies $A = C$ or $A = D$.

Theorem 3.5. *If $A \in Id(X)$ satisfies the following assertion:*

$$(\forall x, y \in X \setminus A)\,(\exists z \in X \setminus A)\,(z - x \in A,\ z - y \in A), \tag{1}$$

then A is an irreducible ideal of X.

Proof. Assume that $A \in Id(X)$ satisfies (1). Let $C, D \in Id(X)$ be such that $A = C \cap D$, $A \neq C$ and $A \neq D$. Then there exist $x \in C \setminus A \subset X \setminus A$ and $y \in D \setminus A \subset X \setminus A$. It follows from (1) that there exists $z \in X \setminus A$ such that $z - x \in A$ and $z - y \in A$. Since $x \in C$ and $z - x \in A = C \cap D \subset C$, we have $z \in C$ because C is an ideal of X. Also, $y \in D$ and $z - y \in D$, which imply $z \in D$. Hence $z \in C \cap D = A$, which is a contradiction. Hence A is an irreducible ideal of X. □

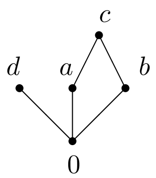
Corollary 3.6 ([11])**.** *Let $A \in Id(X)$. Assume that for any $x, y \in X \setminus A$, there exists $z \in X \setminus A$ such that $z \leq x$ and $z \leq y$. Then A is an irreducible ideal of X.*

Definition 3.7. Let X be a poset. A nonempty subset I of X is called an *order system* of X if it satisfies:

(b3) $(\forall x \in X)\ (\forall y \in I)\ (x \leq y \Rightarrow x \in I)$,
(b4) $(\forall x, y \in I)\ (\exists z \in I)\ (x \leq z,\ y \leq z)$.

The set of all order systems of a poset X will be denoted by $O_s(X)$. Note that if X is a poset with the bottom element $\perp$, then every order system of X contains the bottom element $\perp$.

Example 3.8. Let $X = \{0, a, b, c, d\}$ be a poset with the following Hasse diagram:



Then $I_1 := \{0, a\} \in O_s(X)$, $I_2 := \{0, a, b, c\} \in O_s(X)$, but $J_1 := \{0, b, c\} \notin O_s(X)$ and $J_2 := \{0, a, d\} \notin O_s(X)$.

Theorem 3.9. *For every WS-algebra X, we have $O_s(X) \subset Id(X)$.*

Proof. Let $I \in O_s(X)$. Since I is nonempty, obviously $0 \in I$. Now let $x, y \in X$ satisfy $x - y \in I$ and $y \in I$. Then there exists $z \in I$ such that $x - y \leq z$ and $y \leq z$ by (b4). It follows from (a2) and (a12) that

$$x - z = (x - z) - 0 = (x - z) - (y - z) = (x - y) - z = 0 \in I$$

so from (b2) that $x \in I$. Therefore $I \in Id(X)$, and so $O_s(X) \subset Id(X)$. □

The following example shows that an ideal is not an order system in general.

Example 3.10. (1) Let $X = \{0, a, b, c, d\}$ be a set with the following Cayley table:

$-$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	c
d	d	d	d	d	0

Then $(X, -)$ is a subtraction algebra, and hence a WS-algebra. It is easy to verify that $Q_1 := \{0, a, d\} \in Id(X)$, but $Q_1 := \{0, a, d\} \notin O_s(X)$.

(2) Let $X = \{0, a, b, c, d\}$ be a set with the following Cayley table:

$-$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	0
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

Then $(X, -)$ is a WS-algebra, which is not a subtraction algebra. It is easy to verify that $Q_2 := \{0, a, c\} \in Id(X)$, but $Q_2 := \{0, a, c\} \notin O_s(X)$.

To make an ideal to be an order system, we need more strong condition.

Definition 3.11 ([9])**.** A subtraction algebra X is said to be *complicated* if for any $a, b \in X$ the set

$$\mathscr{G}(a, b) := \{x \in X \mid x - a \leq b\}$$

has the greatest element.

The greatest element of $\mathscr{G}(a,b)$ is denoted by $a+b$.

Lemma 3.12 ([9]). *If X is a complicated subtraction algebra, then*

$$(\forall a,b\in X)\,(a\le a+b,\; b\le a+b).$$

Theorem 3.13. *In a complicated subtraction algebra X, every ideal is an order system.*

Proof. Let Q be an ideal of a complicated subtraction algebra X. The condition (b3) follows from Lemma 3.2. Now let $x,y\in Q$. Since $(x+y)-x\le y$, it follows from Lemma 3.2 and (b2) that $x+y\in Q$ so from Lemma 3.12 that (b4) is valid. Hence Q is an order system of X. □

Corollary 3.14 ([9]). *Let Q be a nonempty subset of a complicated subtraction algebra X. Then Q is an ideal of X if and only if Q is an order system of X.*

Theorem 3.15. *Let $Q\in O_s(X)$. If Q is irreducible as an ideal of X, then*

$$(\forall a,b\in X\setminus Q)\,(\exists c\in X\setminus Q)\,(c\le a,\; c\le b).$$

Proof. Assume that

$$(\exists a,b\in X\setminus Q)(\forall c\in X)(c\le a,\; c\le b\;\Rightarrow\; c\in Q). \tag{2}$$

Let $Q(a)$ and $Q(b)$ be the ideals of X generated by $Q\cup\{a\}$ and $Q\cup\{b\}$ respectively. Then $Q\subset Q(a)\cap Q(b)$. Let $x\in Q(a)\cap Q(b)$. Then $x\in Q(a)$ and $x\in Q(b)$. Thus

$$(\cdots(((x-a)-c_1)-c_2)-\cdots)-c_m=0$$

and

$$(\cdots(((x-b)-d_1)-d_2)-\cdots)-d_n=0$$

for some $c_1,c_2,\dots,c_m,d_1,d_2,\dots,d_n\in Q$. Since Q is an ideal of X, it follows from (b1) and (b2) that $x-a\in Q$ and $x-b\in Q$ so from (b4) that there exists $z\in Q$ such that $x-a\le z$ and $x-b\le z$. Hence

$$(x-z)-a=(x-a)-z=0 \text{ and } (x-z)-b=(x-b)-z=0,$$

and so $x-z\in Q$ by (2). But $Q\in Id(X)$ and $z\in Q$, and thus $x\in Q$ by (b2). Thus $Q(a)\cap Q(b)\subset Q$, and consequently $Q=Q(a)\cap Q(b)$ which is a contradiction. □

4. Right fixed maps

Definition 4.1. A *right fixed map* α of X is defined to be a self map $\alpha: X\to X$ satisfying $\alpha(x-y)=\alpha(x)-y$ for all $x,y\in X$.

Example 4.2. (1) Let $X=\{0,a,b\}$ be a set with the following Cayley table:

$-$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Then $(X, -)$ is a subtraction algebra, and hence a WS-algebra. It can be easily verify that the self map α of X defined by $\alpha(0) = 0$, $\alpha(a) = 0$, and $\alpha(b) = b$ is a right fixed map.

(2) Consider a subtraction algebra, and hence a WS algebra, $X = \{0, a, b, c\}$ with the following Cayley table:

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

Let $\beta : X \to X$ be defined by $\beta(0) = 0$, $\beta(a) = 0$, $\beta(b) = c$, and $\beta(c) = c$. Then β is not a right fixed map since $\beta(b - c) \neq \beta(b) - c$.

(3) Let $X = \{0, a, b, c, d\}$ be a set with the following Cayley table:

$-$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

Then $(X, -)$ is a WS-algebra, which is not a subtraction algebra. Let γ be a self map of X defined by $\gamma(0) = \gamma(a) = \gamma(b) = 0$, $\gamma(c) = c$ and $\gamma(d) = d$. Then γ is a right fixed map.

(4) Let $X = \{0, a, b, c, d\}$ be a set with the following Cayley table:

$-$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	b
c	c	c	c	0	c
d	d	d	d	d	0

Then $(X, -)$ is a WS-algebra, which is not a subtraction algebra. Let α be a self map of X defined by

$$\alpha(x) = \begin{cases} 0 & \text{if } \ x \in \{0, a\}, \\ x & \text{otherwise.} \end{cases}$$

Then α is a right fixed map of X.

Proposition 4.3. *If α is a right fixed map of X, then*

(i) $\alpha(0) = 0$,
(ii) $(\forall x \in X)$ $(\alpha(0 - x) = 0)$,
(iii) $(\forall x \in X)$ $(\alpha(x) \leq x)$,
(iv) $(\forall x, y \in X)$ $(x \leq y \Rightarrow \alpha(x) \leq y)$.

Proof. (i) For every $x, y \in X$, we have

$$\alpha(0) = \alpha(0 - \alpha(0)) = \alpha(0) - \alpha(0) = 0.$$

(ii) For every $x \in X$, we have $\alpha(0 - x) = \alpha(0) = 0$.

(iii) For any $x \in X$, we get $0 = \alpha(0) = \alpha(x - x) = \alpha(x) - x$, and so $\alpha(x) \leq x$.

(iv) Assume that $x \leq y$ for every $x, y \in X$. Then $0 = \alpha(0) = \alpha(x - y) = \alpha(x) - y$, and so $\alpha(x) \leq y$. □

Definition 4.4. For a right fixed map α of X, the *kernel* of α, denoted by $\ker(\alpha)$, is defined to be the set

$$\ker(\alpha) = \{x \in X \mid \alpha(x) = 0\}.$$

Obviously $\ker(\alpha) \neq \emptyset$ since $0 \in \ker(\alpha)$.

Theorem 4.5. *Let α be a right fixed map of X. Then α is one-to-one if and only if* $\ker(\alpha) = 0$.

Proof. Assume that α is one-to-one and let $x \in \ker(\alpha)$. Then $\alpha(x) = 0 = \alpha(0)$, and thus $x = 0$, i.e., $\ker(\alpha) = \{0\}$. Conversely suppose that $\ker(\alpha) = \{0\}$. Let $x, y \in X$ satisfy $\alpha(x) = \alpha(y)$. Since $\alpha(y) \leq y$, it follows from (a9) that $\alpha(x - y) = \alpha(x) - y \leq \alpha(x) - \alpha(y) = 0$ so that $\alpha(x - y) = 0$. Hence $x - y \in \ker(\alpha)$, and so $x - y = 0$. Similarly, $y - x = 0$. This proves that $x = y$. Therefore α is one-to-one. □

Theorem 4.6. *Let α be a right fixed map of X. Then α is one-to-one if and only if α is the identity map.*

Proof. Sufficiency is obvious. Suppose that α is one-to-one. For every $x \in X$, we have

$$\alpha(x - \alpha(x)) = \alpha(x) - \alpha(x) = 0 = \alpha(0)$$

and so $x - \alpha(x) = 0$, i.e., $x \leq \alpha(x)$. Since $\alpha(x) \leq x$ for all $x \in X$, it follows that $\alpha(x) = x$ so that α is the identity map. □

Theorem 4.7. *Let α be a right fixed map of X. If α is idempotent, i.e., $\alpha(\alpha(x)) = \alpha(x)$ for all $x \in X$, then*

(i) $(\forall x \in X)\, (\alpha(x) = x \Leftrightarrow x \in \mathrm{Im}(\alpha))$.

(ii) $\ker(\alpha) \cap \mathrm{Im}(\alpha) = \{0\}$.

Proof. (i) Necessity is obvious. If $x \in \mathrm{Im}(\alpha)$, then $\alpha(y) = x$ for some $y \in X$. Thus $\alpha(x) = \alpha(\alpha(y)) = \alpha(y) = x$.

(ii) If $x \in \ker(\alpha) \cap \mathrm{Im}(\alpha)$, then $\alpha(x) = 0$ and $\alpha(y) = x$ for some $y \in X$. It follows that

$$0 = \alpha(x) = \alpha(\alpha(y)) = \alpha(y) = x$$

so that $\ker(\alpha) \cap \mathrm{Im}(\alpha) = \{0\}$. □

The following example shows that a WS-algebra X does not satisfy the assertion (a8) in general.

Example 4.8. Let $X = \{0, a, b, c, d\}$ be a WS-algebra, which is not a subtraction algebra, described in Example 4.2(4). We know that $b \leq c$, but there does not exist $w \in X$ such that $b = c - w$.

Theorem 4.9. *Let α be a right fixed map of a subtraction algebra X. Then*

(i) $(\forall x \in X)\ (\exists y \in \ker(\alpha), \exists z \in \mathrm{Im}(\alpha))\ (z = x - y)$.
(ii) *α is idempotent.*

Proof. Since $\alpha(x) \leq x$ for all $x \in X$, it follows from (a8) that $\exists w \in X$ such that $\alpha(x) = x - w$ so from (a6) that

$$x - (x - \alpha(x)) = x - (x - (x - w)) = x - w = \alpha(x).$$

Noticing that $x - \alpha(x) \in \ker(\alpha)$ and $\alpha(x) \in \mathrm{Im}(\alpha)$, we have the result (i). Moreover, using (a1) implies that

$$\alpha(\alpha(x)) = \alpha(x - w) = \alpha(x) - w = (x - w) - w = x - w = \alpha(x)$$

for all $x \in X$, which proves (ii). □

Corollary 4.10. *If α is a right fixed map of a subtraction algebra X, then*

(i) $(\forall x \in X)\, (\alpha(x) = x \Leftrightarrow x \in \mathrm{Im}(\alpha))$.
(ii) $\ker(\alpha) \cap \mathrm{Im}(\alpha) = \{0\}$.

Denote by $RF(X)$ the set of all right fixed maps of X. Let $\ominus$ be a binary operation on $RF(X)$ defined by $(\alpha \ominus \beta)(x) = \alpha(x) - \beta(x)$ for all $\alpha, \beta \in RF(X)$ and $x \in X$. It is easy to verify that if X is a WS-algebra, then $(RF(X), \ominus)$ is a WS-algebra. Let $IRF(X)$ denote the set of all idempotent right fixed maps of X.

Theorem 4.11. *For every $\alpha, \beta \in IRF(X)$, if $\alpha \ominus \beta = 0$ in $RF(X)$, then $\mathrm{Im}(\alpha) \subset \mathrm{Im}(\beta)$.*

Proof. Let $\alpha, \beta \in IRF(X)$ satisfy $\alpha \ominus \beta = 0$. If $y \in \mathrm{Im}(\alpha)$, then $\alpha(y) = y$ by Theorem 4.7, and hence

$$0 = (\alpha \ominus \beta)(y) = \alpha(y) - \beta(y) = y - \beta(y),$$

i.e., $y \leq \beta(y)$. Combining this with Proposition 4.3(iii), we get $y = \beta(y) \in \mathrm{Im}(\beta)$. Hence $\mathrm{Im}(\alpha) \subset \mathrm{Im}(\beta)$. □

Theorem 4.12. *Let $\alpha, \beta \in IRF(X)$. Then*

(i) $\alpha \ominus \beta \in RF(X)$.
(ii) *If $\alpha(\beta(x)) = \beta(\alpha(x))$ for all $x \in X$, then $\alpha \ominus \beta \in IRF(X)$.*
(iii) *If $\mathrm{Im}(\alpha) \subset \mathrm{Im}(\beta)$ and $\alpha(\beta(x)) = \beta(\alpha(x))$ for all $x \in X$, then $\alpha \ominus \beta = 0$ in $RF(X)$.*
(iv) $\mathrm{Im}(\alpha) \cap \ker(\beta) \subset \mathrm{Im}(\alpha \ominus \beta)$.

Proof. (i) For every $x, y \in X$, we have

$$\begin{aligned}(\alpha \ominus \beta)(x-y) &= \alpha(x-y) - \beta(x-y)\\ &= (\alpha(x)-y) - (\beta(x)-y)\\ &= (\alpha(x)-\beta(x)) - y\\ &= (\alpha \ominus \beta)(x) - y,\end{aligned}$$

and so $\alpha \ominus \beta \in RF(X)$.

(ii) Assume that $\alpha(\beta(x)) = \beta(\alpha(x))$ for all $x \in X$. Let $x \in X$. Then

$$\begin{aligned}(\alpha \ominus \beta)((\alpha \ominus \beta)(x)) &= (\alpha \ominus \beta)(\alpha(x) - \beta(x))\\ &= \alpha(\alpha(x)-\beta(x)) - \beta(\alpha(x)-\beta(x))\\ &= (\alpha(\alpha(x)) - \beta(x)) - (\beta(\alpha(x)) - \beta(x))\\ &= (\alpha(x)-\beta(x)) - (\alpha(\beta(x)) - \beta(x))\\ &= (\alpha(x)-\beta(x)) - \alpha(\beta(x)-\beta(x))\\ &= (\alpha(x)-\beta(x)) - \alpha(0)\\ &= \alpha(x) - \beta(x)\\ &= (\alpha \ominus \beta)(x),\end{aligned}$$

that is, $\alpha \ominus \beta$ is idempotent. Hence $\alpha \ominus \beta \in IRF(X)$.

(iii) Suppose that $\mathrm{Im}(\alpha) \subset \mathrm{Im}(\beta)$ and $\alpha(\beta(x)) = \beta(\alpha(x))$ for all $x \in X$. Since $\alpha(x) \in \mathrm{Im}(\alpha) \subset \mathrm{Im}(\beta)$ for all $x \in X$, it follows from Theorem 4.7 that

$$\begin{aligned}(\alpha \ominus \beta)(x) &= \alpha(x) - \beta(x) = \beta(\alpha(x)) - \beta(x)\\ &= \alpha(\beta(x)) - \beta(x) = \alpha(\beta(x) - \beta(x))\\ &= \alpha(0) = 0\end{aligned}$$

for all $x \in X$. Therefore $\alpha \ominus \beta = 0$.

(iv) If $y \in \mathrm{Im}(\alpha) \cap \ker(\beta)$, then $\beta(y) = 0$ and $\alpha(x) = y$ for some $x \in X$. It follows from (a2) that

$$y = \alpha(x) = \alpha(\alpha(x)) - 0 = \alpha(y) - \beta(y) = (\alpha \ominus \beta)(y) \in \mathrm{Im}(\alpha \ominus \beta).$$

Therefore $\mathrm{Im}(\alpha) \cap \ker(\beta) \subset \mathrm{Im}(\alpha \ominus \beta)$. □

We pose a problem: If $\alpha \in RF(X)$, then is $\ker(\alpha)$ an order system (or, an ideal) of X?

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