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GEOMETRIC AND APPROXIMATION PROPERTIES OF GENERALIZED SINGULAR INTEGRALS IN THE UNIT DISK

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Abstract. The aim of this paper is to obtain several results in approximation by Jackson-type generalizations of complex Picard, Poisson-Cauchy and Gauss-Weierstrass singular integrals in terms of higher order moduli of smoothness. In addition, these generalized integrals preserve some sufficient conditions for starlikeness and univalence of analytic functions. Also approximation results for vector-valued functions defined on the unit disk are given.

1. Introduction

For $D = \{z \in \mathbb{C};\ |z| < 1\}$, let us denote $A(\overline{D}) = \{f : \overline{D} \to \mathbb{C};\ f$ is continuous on $\overline{D}$, analytic on D, $f(0) = 0,\ f'(0) = 1\}$. Therefore if $f \in A(\overline{D})$, then we can write $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ for all $z \in D$.

For $f \in A(\overline{D})$ and $\xi > 0$, let us consider the generalized complex singular integrals

$$P_{n,\xi}(f)(z) = -\frac{1}{2\xi}\int_{-\infty}^{+\infty}\left[e^{-|u|/\xi}\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}f(ze^{iku})\right]du,$$

$$Q_{n,\xi}(f)(z) = -\frac{1}{\frac{2}{\xi}\operatorname{arctg}\left(\frac{\pi}{\xi}\right)}\cdot\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}\int_{-\pi}^{\pi}\frac{f(ze^{iku})}{u^2+\xi^2}\,du,$$

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$$W_{n,\xi}(f)(z) = -\frac{1}{2C(\xi)} \cdot \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \int_{-\pi}^{\pi} f(ze^{iku}) e^{-u^2/\xi^2} \, du,$$

$$W^*_{n,\xi}(f)(z) = -\frac{1}{2C^*(\xi)} \cdot \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \int_{-\infty}^{+\infty} f(ze^{iku}) e^{-u^2/\xi^2} \, du,$$

$z \in \overline{D}$, $n \in \mathbb{N}$, $C(\xi) = \int_0^{\pi} e^{-u^2/\xi^2} \, du$, $C^*(\xi) = \int_0^{\infty} e^{-u^2/\xi^2} \, du$. Here $P_{n,\xi}(f)(z)$ is called of Picard type, $Q_{n,\xi}(f)(z)$ is called of Poisson-Cauchy type and $W_{n,\xi}(f)(z)$, $W^*_{n,\xi}(f)(z)$ are called of Gauss-Weierstrass type.

In the very recent paper [3], [4], [5], classes of convolution complex polynomials were introduced regarding rates, global smoothness preservation properties and some geometric properties like preservation of coefficients' bounds, positivity of real part, bounded turn, starlikeness, convexity, univalence, were proved.

In the very recent paper [2], we have obtained similar results for the complex singular integrals of Picard, Poisson-Cauchy, Gauss-Weierstrass.

The aim of the present paper is to obtain some similar properties for the generalized complex singular integrals defined above.

Approximation properties for vector-valued functions defined on the unit disk also are presented.

2. Approximation properties

In this section we study the approximation and global smoothness preservation properties.

THEOREM 2.1. (i) *For $z \in \overline{D}$ and $\xi \in (0, 1]$ we have*

$$|P_{n,\xi}(f)(z) - f(z)| \leq \left[\sum_{k=0}^{n+1} \binom{p+1}{k} k! \right] \omega_{n+1}(f; \xi)_{\partial D},$$

$$|W_{n,\xi}(f)(z) - f(z)| \leq C_n \omega_{n+1}(f; \xi)_{\partial D}, \; C_n = \frac{\int_0^{\infty} (1+u)^{n+1} e^{-u^2} \, du}{\int_0^{\pi} e^{-u^2} \, du},$$

$$|W^*_{n,\xi}(f)(z) - f(z)| \leq C^*_n \omega_{n+1}(f; \xi)_{\partial D}, \; C^*_n = \frac{\int_0^{\infty} (1+u)^{n+1} e^{-u^2} \, du}{\int_0^{\infty} e^{-u^2} \, du},$$

$$|Q_{n,\xi}(f)(z) - f(z)| \leq K(n, \xi) \omega_{n+1}(f; \xi)_{\partial D}, \; K(n, \xi) = \frac{\int_0^{\pi/\xi} \frac{(u+1)^{n+1}}{u^2+1} \, du}{\tan^{-1}\left(\frac{\pi}{\xi}\right)},$$

where

$$\omega_{n+1}(f;\xi)_{\partial D} = \sup\{|\Delta_u^{n+1} f(e^{ix})|; |x| \le \pi, |u| \le \xi\}.$$

(ii) *For all* $\delta > 0$, $\xi > 0$ *and* $n \in \mathbb{N}$, *we have*

$$\begin{aligned}\omega_1(P_{n,\xi}(f);\delta)_{\overline{D}} &\le (2^{n+1}-1)\omega_1(f;\delta)_{\overline{D}},\\ \omega_1(W_{n,\xi}(f);\delta)_{\overline{D}} &\le (2^{n+1}-1)\omega_1(f;\delta)_{\overline{D}},\\ \omega_1(W^*_{n,\xi}(f);\delta)_{\overline{D}} &\le (2^{n+1}-1)\omega_1(f;\delta)_{\overline{D}},\\ \omega_1(Q_{n,\xi}(f);\delta)_{\overline{D}} &\le (2^{n+1}-1)\omega_1(f;\delta)_{\overline{D}}.\end{aligned}$$

Proof. (i) Let $z \in \overline{D}$, $|z| = 1$, $\xi > 0$ be fixed. Because of the Maximum Modulus Principle, it suffices to estimate $|P_{n,\xi}(f)(z) - f(z)|$, for this $|z| = 1$, $z = e^{ix}$. We get

$$\begin{aligned} & f(z) - P_{n,\xi}(f)(z)\\ &= f(z)\cdot \frac{1}{2\xi}\int_{-\infty}^{+\infty} e^{-|u|/\xi}\,du\\ &\quad + \frac{1}{2\xi}\int_{-\infty}^{+\infty}\left[\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}\right] f(e^{i(x+ku)})e^{-|u|/\xi}\,du\\ &= \frac{1}{2\xi}\int_{-\infty}^{+\infty}(-1)^{n+1}\Delta_u^{n+1} f(e^{ix})e^{-|u|/\xi}\,du,\end{aligned}$$

where from

$$\begin{aligned}|f(z) - P_{n,\xi}(f)(z)| &\le \frac{1}{2\xi}\int_{-\infty}^{+\infty}\omega_{n+1}(f;|u|)_{\partial D}e^{-|u|/\xi}\,du\\ &= \frac{1}{\xi}\int_0^{+\infty}\omega_{n+1}\left(f;\frac{u}{\xi}\cdot\xi\right)_{\partial D} e^{-u/\xi}\,du\\ &\le \omega_{n+1}(f;\xi)_{\partial D}\frac{1}{\xi}\int_0^{+\infty}\left(1+\frac{u}{\xi}\right)^{n+1} e^{-u/\xi}\,du\\ &= \text{(reasoning exactly as in [6, p.254])}\\ &= \sum_{k=0}^{n+1}\binom{n+1}{k}k!\,\omega_{n+1}(f;\xi)_{\partial D}.\end{aligned}$$

As above, we obtain

$$f(z) - W_{n,\xi}(f)(z) = \frac{1}{2C(\xi)}\int_{-\pi}^{\pi}(-1)^{n+1}\Delta_n^{n+1} f(e^{ix})e^{-u^2/\xi^2}\,du,$$

which implies

$$
\begin{aligned}
|f(z) - W_{n,\xi}(f)(z)| &\leq \frac{1}{C(\xi)} \int_0^{\pi} \omega_{n+1}(f;u)_{\partial D} e^{-u^2/\xi^2}\, du \\
&\leq \frac{1}{C(\xi)} \omega_{n+1}(f;\xi)_{\partial D} \int_0^{\pi} \left[1 + \frac{u}{\xi}\right]^{n+1} e^{-u^2/\xi^2}\, du \\
&\quad \text{(reasoning exactly as in [6, p.260])} \\
&\leq \frac{\int_0^{+\infty} [1+u]^{n+1} e^{-u^2}\, du}{\int_0^{\pi} e^{-u^2}\, du} \cdot \omega_{n+1}(f;\xi)_{\partial D}.
\end{aligned}
$$

Similarly,

$$
f(z) - W^*_{n,\xi}(f)(z) = \frac{1}{2C^*(\xi)} \int_{-\infty}^{+\infty} (-1)^{n+1} \Delta_u^{n+1} f(e^{ix}) e^{-u^2/\xi^2}\, du,
$$

which implies as above

$$
\begin{aligned}
|f(z) - W^*_{n,\xi}(f)(z)| &\leq \frac{1}{C^*(\xi)} \omega_{n+1}(f;\xi)_{\partial D} \int_0^{\infty} \left[1 + \frac{u}{\xi}\right]^{n+1} e^{-u^2/\xi^2}\, du \\
&\leq \frac{\int_0^{+\infty} [1+u]^{n+1} e^{-u^2}\, du}{\int_0^{\infty} e^{-u^2}\, du} \omega_{n+1}(f;\xi)_{\partial D}.
\end{aligned}
$$

Finally, by the relation

$$
f(z) - Q_{n,\xi}(f)(z) = \frac{1}{\frac{2}{\xi} \operatorname{arctg} \frac{\pi}{\xi}} \int_{-\pi}^{\pi} \frac{(-1)^{n+1} \Delta_u^{n+1} f(e^{ix})}{u^2 + \xi^2}\, du,
$$

it follows (taking into account [1, p.518] too)

$$
\begin{aligned}
|f(z) - Q_{n,\xi}(f)(z)| &\leq \frac{\xi}{\operatorname{arctg} \frac{\pi}{\xi}} \int_0^{\pi} \frac{\omega_{n+1}(f;u)_{\partial D}}{u^2 + \xi^2}\, du \\
&\leq \frac{\xi}{\operatorname{arctg} \frac{\pi}{\xi}} \omega_{n+1}(f;\xi)_{\partial D} \int_0^{\pi} \left[\frac{u}{\xi} + 1\right]^{n+1} \cdot \frac{1}{u^2 + \xi^2}\, du \\
&= K(n,\xi) \omega_{n+1}(f;\xi)_{\partial D}
\end{aligned}
$$

which proves (i).

(ii) Let $|z_1 - z_2| \le \delta$, $z_1, z_2 \in \overline{D}$. We have

$$\begin{aligned}
&|P_{n,\xi}(f)(z_1) - P_{n,\xi}(f)(z_2)| \\
&\le \omega_1\big(f; |z_1 - z_2|\big)_{\overline{D}} \cdot \frac{1}{2\xi} \int_{-\infty}^{+\infty} \sum_{k=1}^{n+1} \binom{n+1}{k} e^{-|u|/\xi}\, du \\
&\le \sum_{k=1}^{n+1} \binom{n+1}{k} \omega_1(f;\delta)_{\overline{D}} \\
&= \big(2^{n+1} - 1\big)\omega_1(f;\delta)_{\overline{D}}.
\end{aligned}$$

As above, we obtain

$$\begin{aligned}
&|W_{n,\xi}(f)(z_1) - W_{n,\xi}(f)(z_2)| \\
&\le \frac{1}{2C(\xi)} \cdot \int_{-\pi}^{\pi} \sum_{k=1}^{n+1} \binom{n+1}{k} e^{-u^2/\xi}\, du \cdot \omega_1(f;\delta)_{\overline{D}} \\
&\le \sum_{k=1}^{n+1} \binom{n+1}{k} \omega_1(f;\delta)_{\overline{D}} \\
&= (2^{n+1} - 1)\omega_1(f;\delta)_{\overline{D}},
\end{aligned}$$

and analogously

$$|W^*_{n,\xi}(f)(z_1) - W^*_{n,\xi}(f)(z_2)| \le (2^{n+1} - 1)\omega_1(f;\delta)_{\overline{D}}.$$

Finally,

$$\begin{aligned}
&|Q_{n,\xi}(f)(z_1) - Q_{n,\xi}(f)(z_2)| \\
&\le \frac{1}{\frac{2}{\xi}\operatorname{arctg}\frac{\pi}{\xi}} \int_{-\pi}^{\pi} \frac{du}{u^2 + \xi^2} \cdot \sum_{k=1}^{n+1} \binom{n+1}{k} \omega_1(f;\delta)_{\overline{D}} \\
&= \big(2^{n+1} - 1\big)\omega_1(f;\delta)_{\overline{D}}.
\end{aligned}$$

Passing in all the above inequalities to sup with $|z_1 - z_2| \le \delta$, we obtain the required relations in (ii).

In what follows we extend the above results and some approximation results in [2]-[3] to vector-valued functions. For this purpose first we recall some known concepts and results.

Definition 2.2 (see e.g. [7, pp.92–93]). Let $(X, \|\cdot\|)$ be a complex Banach space and $f\colon \overline{D} \to X$. We say that f is holomorphic on D if for any $x^* \in B_1 = \{x^*\colon X \to \mathbb{C};\ x^* \text{ linear and continuous, } |||x^*||| \le 1\}$, the function $g\colon \overline{D} \to \mathbb{C}$ given by $g(z) = x^*[f(z)]$, is holomorphic on D. (Here $|||\cdot|||$ represents the usual norm in the dual space X^*).

We let us denote by $A(\overline{D}; X)$ the space of $f\colon \overline{D} \to X$ which are continuous on $\overline{D}$ and holomorphic on D.

Note that everywhere in this section $(X, \|\cdot\|)$ will be a complex Banach space.

THEOREM 2.3 (see e.g. [7, p.93]). *If $f\colon \overline{D} \to X$ is holomorphic on D, then $f(z)$ is continuous (as mapping between two metric spaces) and differentiable (in the sense that exists $f'(z) \in C$ given by $\lim_{h\to 0} \left\| \frac{f(z+h)-f(z)}{h} - f'(z) \right\| = 0$) uniformly with respect to z in any compact subset of D.*

THEOREM 2.4 (see e.g. [7, p.97]). *If $f\colon \overline{D} \to X$ is holomorphic on D, then we have the Taylor expansion*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \quad z \in D,$$

where the series converges uniformly on any compact subset of D.

Also, the following result in Functional Analysis is well-known.

THEOREM 2.5. *Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{R}$ of $\mathbb{C}$ and denote by X^* the conjugate of X. Then $\|x\| = \sup\{|x^*(x)|;\ x^* \in X^*, |||x^*||| \le 1\}$ for all $x \in X$.*

Now we are in a position to prove our results. We present

THEOREM 2.6. *Let $f \in A(\overline{D}; X)$, $(X, \|\cdot\|)$ a complex normed space.*
(i) *Define*

$$P_n(f)(z) = \frac{1}{2\pi n'[2(n')^2+1]} \int_{-\pi}^{\pi} f(ze^{iu}) K_n(u)\, du,$$

where $K_n(u) = [\sin(n'u/2)/\sin(u/2)]^4$, $n' = [n/2]+1$, and $\int_{-\pi}^{\pi}$ is the classical Riemann integral for vector-valued functions. Then $P_n(f)(z)$ is a polynomial in z of degree $\le n$, with coefficients in X, which satisfies the estimate

$$\|f(z) - P_n(f)(z)\| \le C\omega_2\left(f; \frac{1}{n}\right)_{\partial D}, \quad \forall z \in \overline{D},$$

where

$$\omega_p(f;\delta)_{\partial D} = \sup\{\|\Delta_u^p f(e^{ix})\|; |x| \le \pi, |u| \le \delta\}, \quad p = 2, 3, \ldots ;$$

(ii) *Define the polynomials in z (with coefficients in X)*

$$I_{n,p}(f)(z) = -\int_{-\pi}^{\pi} K_{n,r}(u) \sum_{k=1}^{p+1} (-1)^k \binom{p+1}{k} f(ze^{iku})\, du,$$

where $K_{n,r}(u) = \lambda_{n,r}[\sin(nu/2)/\sin(u/2)]^{2r}$, *$r$ is the smallest integer which satisfies* $r \geq (p+2)/2$ *and the constants* $\lambda_{n,r}$ *are chosen such that* $\int_{-\pi}^{\pi} K_{n,r}(u)\, du = 1$. *Then we have*

$$\|I_{n,p}(f)(z) - f(z)\| \leq C_p \omega_{p+1}\left(f; \frac{1}{n}\right)_{\partial D}, \quad \forall z \in \overline{D};$$

(iii) *Define*

$$V_n(f)(z) = 2L_{2n}(f)(z) - L_n(f)(z),$$

where

$$L_n(f)(z) = \frac{1}{2n\pi} \int_{-\pi}^{\pi} f(ze^{iu}) F_n(u)\, du,$$

$F_n(u) = [\sin(nu/2)/\sin(u/2)]^2$. *Then* $V_n(f)(z)$ *is a polynomial of degree* $\leq 2n-1$ *in z, with coefficients in X which satisfies the estimate*

$$\|f(z) - V_n(f)(z)\| \leq 4E_n(f)_\infty(\overline{D}), \quad \forall z \in \overline{D},$$

where $E_n(f)_\infty(\overline{D}) = \inf\{\|f - P\|_{\overline{D}};\ P$ *polynomial of degree* $\leq n$ *in z, with coefficients in* $X\}$, $\|f\|_{\overline{D}} = \sup\{\|f(z)\|;\ z \in \overline{D}\}$.

(iv) *Define* $P_{n,\xi}(f)(z)$ *and* $W_{n,\xi}(f)(z)$ *as in the Introduction but for* $f \in A(\overline{D}; X)$. *Then we have*

$$\|Q_{n,\xi}(f)(z) - f(z)\| \leq K(n,\xi)\omega_{n+1}(f;\xi)_{\partial D}, \quad \forall z \in \overline{D}, \xi \in (0,1],\ n \in \mathbb{N}$$

and

$$\|W_{n,\xi}(f)(z) - f(z)\| \leq C_n \omega_{n+1}(f;\xi)_{\partial D}, \quad \forall z \in \overline{D},\ \xi \in (0,1],\ n \in \mathbb{N},$$

where $K(n,\xi)$ *and* C_n *are as in Theorem 2.1,* (i).

(v) *If for* $f \in A(\overline{D}; X)$ *we consider the operators*

$$Q_\xi(f)(z) = \frac{\xi}{\pi} \int_{-\pi}^{\pi} \frac{f(ze^{iu})}{u^2 + \xi^2}\, du, \quad z \in \overline{D},\ \xi > 0,$$

$$W_\xi(f)(z) = \frac{1}{\sqrt{\pi\xi}} \int_{-\pi}^{\pi} f(ze^{iu}) e^{-u^2/\xi}\, du, \quad z \in \overline{D},\ \xi > 0,$$

then we have

$$\|Q_\xi(f)(z) - f(z)\| \leq C\frac{\omega_2(f;\xi)_{\partial D}}{\xi}, \quad \forall z \in \overline{D},\ \xi \in (0,1]$$

and

$$\|W_\xi(f)(z) - f(z)\| \leq C\frac{\omega_2(f;\xi)_{\partial D}}{\xi}, \quad \forall z \in \overline{D},\ \xi \in (0,1].$$

Proof. (i) By Theorem 2.4 and reasoning exactly as in [3, pp.419–420], we first easily get that $P_n(f)(z)$ is a generalized complex polynomial of degree $\leq n$ in z, with coefficients in X.

Let $x^* \in B_1$ and define $g(z) = x^*[f(z)]$, $g\colon \overline{D} \to \mathbb{C}$. By the estimate in [3, p.423] we have

$$|g(z) - P_n(g)(z)| \leq C\omega_2\left(g;\frac{1}{n}\right)_{\partial D}, \quad \forall z \in D.$$

But $\Delta_u^2 g(e^{ix}) = x^*[\Delta_u^2 f(e^{ix})]$ and

$$|\Delta_u^2 g(e^{ix})| \leq |||x^*||| \cdot \|\Delta_u^2 f(e^{ix})\| \leq \|\Delta_u^2 f(e^{ix})\|,$$

which immediately implies $\omega_2(g;\delta)_{\partial D} \leq \omega_2(f;\delta)_{\partial D}$, therefore

$$\begin{aligned}|x^*[f(z) - P_n(f)(z)]| &= |g(z) - P_n(g)(z)| \\ &\leq C\omega_2\left(f;\frac{1}{n}\right)_{\partial D}, \quad \forall z \in \overline{D},\ n \in \mathbb{N}.\end{aligned}$$

Passing to supremum with $x^* \in B_1$, by Theorem 2.5 we get

$$\|f(z) - P_n(f)(z)\| \leq C\omega_2\left(f;\frac{1}{n}\right)_{\partial D}, \quad \forall z \in \overline{D},\ n \in \mathbb{N}.$$

(ii) Let $g(z) = x^*[f(z)]$, $x^* \in B_1$. By [3, p.424] we have

$$|g(z) - I_{n,p}(g)(z)| \leq C_p\omega_{p+1}\left(g;\frac{1}{n}\right)_{\partial D},$$

which by similar reasonings gives the required conclusion in the statement. Here x^* commutes with the integral.

(iii) By the definition of $E_n(f)_\infty(\overline{D})$ as infimum, for any $m \in \mathbb{N}$, there exists P_m-polynomial of degree $\leq n$ in z, with coefficients in X, such that

$$E_n(f)_\infty(\overline{D}) \leq \|f - P_m\|_{\overline{D}} \leq E_n(f)_\infty(\overline{D}) + \frac{1}{m}, \quad m = 1, 2, \ldots .$$

Since it is easy to prove (as in e.g. [3, p.425]) that $V_n(P_m)(z) = P_m(z)$, $\forall z \in \overline{D}$, $n \in \mathbb{N}$ and

$$\begin{aligned}\|f(z) - V_n(f)(z)\| &= \|f(z) - P_m(z) + V_n(P_m - f)(z)\| \\ &\leq 4\|f(z) - P_m(z)\| \\ &\leq 4\left[E_n(f)_\infty(\overline{D}) + \frac{1}{m}\right], \quad \forall z \in \overline{D},\ m \in \mathbb{N},\end{aligned}$$

passing to the limit with $m \to \infty$, we get the desired relation. Here the key relation is $V_n(g)(z) = x^*[V_n(f)(z)]$, for $g(z) = x^*[f(z)]$, $x^* \in B_1$.

(iv), (v) Let $g(z) = x^*[f(z)]$, $x^* \in B_1$. Since by Theorem 2.1, (i) we have

$$|Q_{n,\xi}(g)(z) - g(z)| \leq K(n,\xi)\omega_{n+1}(g;\xi)_{\partial D}, \quad \forall z \in \overline{D},\ n \in \mathbb{N},\ \xi \in (0,1]$$

$$|W_{n,\xi}(g)(z) - g(z)| \leq C_n\omega_{n+1}(g;\xi)_{\partial D}, \quad \forall z \in \overline{D},\ n \in \mathbb{N},\ \xi \in (0,1]$$

by $\omega_{n+1}(g;\xi)_{\partial D} \leq \omega_{n+1}(f;\xi)_{\partial D}$, by

$$Q_{n,\xi}(g)(z) = x^*[Q_{n,\xi}(f)(z)], \quad W_{n,\xi}(g)(z) = x^*[W_{n,\xi}(f)(z)]$$

and by Theorem 2.5, as above we easily obtain the required results for $Q_{n,\xi}$ and $W_{n,\xi}$.

In the case of $Q_\xi(f)(z)$ and $W_\xi(f)(z)$ we take into account the estimates in [2]

$$|Q_\xi(g)(z) - g(z)| \leq \frac{C\omega_2(g;\xi)_{\partial D}}{\xi}, \quad \forall z \in D,\ \xi \in (0,1],$$

$$|W_\xi(g)(z) - g(z)| \leq \frac{C\omega_2(g;\xi)_{\partial D}}{\xi}, \quad \forall z \in D,\ \xi \in (0,1]$$

and reasoning as above, the theorem is proved. □

3. Geometric properties

In this section we present some geometric properties of the generalized complex singular integrals.

We present

THEOREM 3.1. (i) *If* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *is analytic in* D *and continuous in* $\overline{D}$, *then* $P_{n,\xi}(f)(z)$, $W_{n,\xi}(f)(z)$, $W^*_{n,\xi}(f)(z)$ *and* $Q_{n,\xi}(f)(z)$ *are analytic in* D *and continuous in* $\overline{D}$ *for all* $\xi > 0$ *and* $n \geq 2$.

Also, we can write

$$P_{n,\xi}(f)(z)=\sum_{p=0}^{\infty}a_p b_{p,n}(\xi)z^p,$$

with

$$b_{p,n}(\xi)=\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}\cdot\frac{1}{\xi^2k^2p^2+1},$$

$$W_{n,\xi}(f)(z)=\sum_{p=0}^{\infty}a_p c_{p,n}(\xi)z^p,$$

with

$$c_{p,n}(\xi)=\frac{1}{C(\xi)}\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}\cdot\int_0^{\pi}\cos(kpu)e^{-u^2/\xi^2}\,du,$$

$$W^*_{n,\xi}(f)(z)=\sum_{p=0}^{\infty}a_p c^*_{p,n}(\xi)z^p,$$

with

$$c^*_{p,n}(\xi)=\frac{1}{C^*(\xi)}\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}\cdot\int_0^{\infty}\cos(kpu)e^{-u^2/\xi^2}\,du$$

$$Q_{n,\xi}(f)(z)=\sum_{p=0}^{\infty}a_p d_{p,n}(\xi)z^p,$$

with

$$d_{p,n}(\xi)=\frac{\xi}{\operatorname{arctg}\frac{\pi}{\xi}}\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}\int_0^{\pi}\frac{\cos(kpu)}{u^2+\xi^2}\,du.$$

Here

$$b_{1,n}(\xi)=\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}\frac{1}{\xi^2k^2+1}>0,\quad \text{for } \xi\in(0,\xi_n],$$

$$c_{1,n}(\xi)=\frac{1}{C(\xi)}\cdot\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}\int_0^{\pi}\cos(ku)e^{-u^2/\xi^2}\,du$$
$$>0,\quad \xi\in(0,\xi_n],$$

$$c_{1,n}^*(\xi) = \frac{1}{C^*(\xi)} \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \int_0^\infty \cos(ku) e^{-u^2/\xi^2}\, du$$
$$= \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} e^{-k^2\xi^2/4} > 0, \quad \text{for } \xi \in (0, \xi_n]$$

and

$$d_{1,n}(\xi) = \frac{\xi}{\operatorname{arctg} \frac{\pi}{\xi}} \cdot \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \int_0^\pi \frac{\cos ku}{u^2 + \xi^2}\, du > 0, \ \ \forall \xi \in (0, \xi_n],$$

where $0 < \xi_n$ *is independent of* k *and* f(but may depend on n).

(ii) *Denote by* $S_M = \{f \in A(\overline{D}); |f'(z)| < M, \forall z \in D\}$, *with* $M > 1$. *Then, for all* $\xi > 0$, $n \in \mathbb{N}$ *we have*

$$\frac{1}{b_{1,n}(\xi)} P_{n,\xi}(S_M) \subset S_{M(2^{n+1}-1)/|b_{1,n}(\xi)|},$$
$$\frac{1}{c_{1,n}(\xi)} W_{n,\xi}(S_M) \subset S_{M(2^{n+1}-1)/|c_{1,n}(\xi)|},$$
$$\frac{1}{c_{1,n}^*(\xi)} W_{n,\xi}^*(S_M) \subset S_{M(2^{n+1}-1)/|c_{1,n}^*(\xi)|},$$
$$\frac{1}{d_{1,n}(\xi)} Q_{n,\xi}(S_M) \subset S_{M(2^{n+1}-1)/|d_{1,n}(\xi)|}.$$

Proof. (i) Let $f(z) = \sum_{p=0}^{\infty} a_p z^p$, $z \in D$. For fixed $z \in D$, we can write $f(ze^{iku}) = \sum_{p=0}^{\infty} a_p e^{ikpu} z^p$ and since $|a_p e^{ikpu}| = |a_p|$ for all $u \in \mathbb{R}$ and the series $\sum_{p=0}^{\infty} a_p z^p$ is convergent, it follows that the series $\sum_{p=0}^{\infty} a_p e^{ikpu} z^p$ is uniformly convergent with respect to $u \in \mathbb{R}$. Therefore the series can be integrated term by term (with respect to u), i.e.,

$$P_{n,\xi}(f)(z) = -\frac{1}{2\xi} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \sum_{p=0}^{\infty} a_p z^p \int_{-\infty}^{+\infty} e^{ikpu} e^{-|u|/\xi}\, du.$$

But

$$-\frac{1}{2\xi} \int_{-\infty}^{+\infty} e^{ikpu} e^{-|u|/\xi}\, du$$
$$= -\frac{1}{2\xi} \int_{-\infty}^{\infty} [\cos(kpu) + i \sin(kpu)] e^{-|u|/\xi}\, du$$

$$= -\frac{1}{\xi}\int_0^{\infty} \cos(kpu)e^{-u/\xi}\,du$$
$$= -\frac{1}{\xi}\cdot \frac{e^{-u/\xi}\left[-\frac{1}{\xi}\cos(kpu)+k\sin(kpu)\right]}{k^2p^2+\frac{1}{\xi^2}}\Bigg|_0^{+\infty}$$
$$= \frac{1}{\xi}\cdot\frac{-\frac{1}{\xi}}{\xi^2k^2p^2+1}\cdot\xi^2 = -\frac{1}{\xi^2k^2p^2+1}.$$

Therefore we can write

$$P_{n,\xi}(f)(z) = \sum_{p=0}^{\infty} a_p z^p \cdot \left[-\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}\cdot\frac{1}{\xi^2k^2p^2+1}\right]$$
$$= \sum_{p=0}^{\infty} a_p b_{p,n}(\xi) z^p,$$

with

$$b_{p,n}(\xi) = \sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}\cdot\frac{1}{\xi^2k^2p^2+1}$$

for all $z \in D$.

For the continuity property, let $z \in \overline{D}$ and $z_m \in \overline{D}$, $n \in \mathbb{N}$, with $\lim_{m\to\infty} z_m = z_0$. We have

$$|P_{n,\xi}(f)(z_m) - P_{n,\xi}(f)(z_0)| \le \frac{1}{2\xi}\sum_{k=1}^{n+1}\binom{n+1}{k}$$
$$\int_{-\infty}^{+\infty} |f(z_m e^{iku}) - f(z_0e^{iku})|e^{-|u|/\xi}\,du$$
$$\le (2^{n+1}-1)\omega_1(f;|z_m - z_0|)_{\overline{D}}.$$

Passing to limit with $m \to \infty$, we get that $P_{n,\xi}(f)(z)$ is continuous on $\overline{D}$.

The proofs for the other operators $W_{n,\xi}(f)\ (z)$, $W^*_{n,\xi}(f)(z)$, and $Q_{n,\xi}(f)(z)$ are similar. The formulas for $b_{1,n}(\xi)$, $c_{1,n}(\xi)$ and $d_{1,n}(\xi)$ are immediate from above.

Also, since $C^*(\xi) = \int_0^\infty e^{-u^2/\xi^2}\,du = \xi\int_0^\infty e^{-v^2}\,dv = \frac{\xi}{\sqrt{\pi}}$ (see e.g. [9, p.228]) and

$$\int_0^\infty \cos(ku)e^{-u^2/\xi^2}\,du = \xi\int_0^\infty \cos(k\xi v)e^{-v^2}\,dv = \frac{\xi\cdot e^{-k^2\xi^2/4}}{\sqrt{\pi}},$$

we get

$$C^*_{1,n}(\xi) = \frac{\sqrt{\pi}}{\xi} \cdot \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \cdot \frac{\xi}{\sqrt{\pi}} e^{-k^2\xi^2/4}$$
$$= \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} e^{-k^2\xi^2/4}.$$

Now, by

$$0 = (-1+1)^{n+1} = \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} = 1 + \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k},$$

it follows $\sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} = 1$.

Then, since

$$b_{1,n}(\xi) = \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \frac{1}{\xi^2 k^2 + 1}$$

and

$$c^*_{1,n}(\xi) = \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} e^{-k^2\xi^2/4}$$

are obviously continuous functions of $\xi \in \mathbb{R}$ and

$$b_{1,n}(0) = c^*_{1,n}(0) = \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} = 1,$$

there exists $\xi_n > 0$ such that $b_{1,n}(\xi) > 0$, $c^*_{1,n}(\xi) > 0$, $\forall \xi \in (0, \xi_n]$. Also, $c_{1,n}(\xi)$ and $d_{1,n}(\xi)$ are obviously continuous functions of $\xi \in \mathbb{R} \setminus \{0\}$.

Since

$$C(\xi) = \int_0^\pi e^{-u^2/\xi^2}\, du = \xi \int_0^{\pi/\xi} e^{-v^2}\, dv$$

and

$$\int_0^\pi \cos(ku) e^{-u^2/\xi^2}\, du = \xi \int_0^{\pi/\xi} \cos(k\xi v) e^{-v^2}\, dv,$$

we get

$$\lim_{\xi \downarrow 0} \left[\int_0^\pi \cos(ku) e^{-u^2/\xi^2}\, du / C(\xi) \right]$$
$$= \lim_{\xi \downarrow 0} \int_0^{\pi/\xi} \cos(k\xi v) e^{-v^2}\, dv / \lim_{\xi \downarrow 0} \int_0^{\pi/\xi} e^{-v^2}\, dv$$

$$= \lim_{\xi\downarrow 0} \int_0^{\pi/\xi} \cos(k\xi v) e^{-v^2}\, dv \Big/ \int_0^\infty e^{-v^2}\, dv$$
$$= 2\sqrt{\pi} \lim_{\xi\downarrow 0} \int_0^{\pi/\xi} \cos(k\xi v) e^{-v^2}\, dv.$$

By the substitution $\xi v = u$ we get

$$\int_0^{\pi/\xi} [1 - \cos(k\xi v)] e^{-v^2}\, dv = \frac{1}{\xi} \int_0^{\pi} [1 - \cos(ku)] e^{-(u/\xi)^2}\, du,$$

i.e.,

$$\left| \int_0^{\pi/\xi} \cos(k\xi v) e^{-v^2}\, dv - \int_0^\infty e^{-v^2}\, dv \right|$$
$$\leq \left| \int_0^{\pi/\xi} \cos(k\xi v) e^{-v^2}\, dv - \int_0^{\pi/\xi} e^{-v^2}\, dv \right|$$
$$+ \left| \int_0^{\pi/\xi} e^{-v^2}\, dv - \int_0^\infty e^{-v^2}\, dv \right|$$
$$\leq \int_0^{\pi} |1 - \cos(ku)| \xi^{-1} e^{-(u/\xi)^2}\, du + \left| \int_0^{\pi/\xi} e^{-v^2}\, dv - \int_0^\infty e^{-v^2}\, dv \right|.$$

Since

$$|1 - \cos(ku)| = 2\sin^2 \frac{ku}{2} \leq \frac{2k^2u^2}{4} = \frac{k^2u^2}{2},$$

we get

$$|1 - \cos(ku)| \xi^{-1} e^{-(u/\xi)^2} \leq \frac{k^2}{2} u^2 \xi^{-1} e^{-(u/\xi)^2},$$

where

$$\lim_{\xi\downarrow 0} u^2 \xi^{-1} e^{-(u/\xi)^2} = \lim_{\xi\downarrow 0} \frac{\frac{u^2}{\xi}}{e^{(u/\xi)^2}} = \lim_{\xi\downarrow 0} \frac{-\frac{u^2}{\xi^2}}{-\frac{2u^2}{\xi^3} e^{u^2/\xi^2}} = \lim_{\xi\downarrow 0} \frac{\xi}{e^{u^2/\xi^2}} = 0,$$

i.e., $|1 - \cos(ku)| \xi^{-1} e^{-(u/\xi)^2} \xrightarrow{\xi\to 0} 0$, uniformly with respect to $u \in [0, \pi]$ (We applied l'Hospital's rule). This immediately implies

$$\lim_{\xi\downarrow 0} \int_0^{\pi/\xi} \cos(ku) e^{-u^2/\xi^2}\, du / C(\xi) = 2\sqrt{\pi} \int_0^\infty e^{-v^2}\, dv = 1.$$

Therefore,

$$\lim_{\xi\downarrow 0} c_{1,n}(\xi) = \sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}\lim_{\xi\downarrow 0}\frac{\int_0^\pi \cos(ku)e^{-u^2/\xi^2}\,du}{C(\xi)}$$
$$= \sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k} = 1 > 0,$$

which implies that there exists $\xi_n > 0$ such that $c_{1,n}(\xi) > 0$, $\forall \xi \in (0, \xi_n]$.

In the case of $d_{1,n}(\xi)$, since

$$\frac{\xi}{\operatorname{arctg}\frac{\pi}{\xi}} = \frac{1}{\int_0^\pi \frac{du}{u^2+u^2}} = \frac{1}{\frac{1}{\xi}\int_0^{\pi/\xi}\frac{dv}{v^2+1}}$$

and

$$\int_0^\pi \frac{\cos ku}{u^2+\xi^2}\,du = \frac{1}{\xi}\int_0^{\pi/\xi}\frac{\cos(k\xi v)}{v^2+1}\,dv,$$

we get

$$\lim_{\xi\downarrow 0}\frac{\xi}{\operatorname{arctg}\frac{\pi}{\xi}}\cdot\int_0^\pi\frac{\cos ku}{u^2+\xi^2}\,du = \lim_{\xi\downarrow 0}\frac{\int_0^{\pi/\xi}\frac{\cos(k\xi v)}{v^2+1}\,dv}{\int_0^{\pi/\xi}\frac{dv}{v^2+1}}$$
$$= \frac{\lim_{\xi\downarrow 0}\int_0^{\pi/\xi}\frac{\cos(k\xi v)}{v^2+1}\,dv}{\int_0^\infty\frac{dv}{v^2+1}}$$
$$= \frac{2}{\pi}\lim_{\xi\downarrow 0}\int_0^{\pi/\xi}\frac{\cos(k\xi v)}{v^2+1}\,dv$$
$$= \frac{2}{\pi}\cdot\int_0^\infty\frac{dv}{v^2+1} = 1.$$

Here, as in the above case, we write

$$\left|\int_0^{\pi/\xi}\frac{\cos(k\xi v)}{v^2+1}\,dv - \int_0^\infty\frac{dv}{v^2+1}\right|$$
$$\le \left|\int_0^{\pi/\xi}\frac{\cos(k\xi v)}{v^2+1}\,dv - \int_0^{\pi/\xi}\frac{dv}{v^2+1}\right| + \left|\int_0^{\pi/\xi}\frac{dv}{v^2+1} - \int_0^\infty\frac{dv}{v^2+1}\right|$$
$$\le \int_0^{\pi/\xi}\frac{[1-\cos(k\xi v)]}{v^2+1}\,dv + \left|\int_0^{\pi/\xi}\frac{dv}{v^2+1} - \int_0^\infty\frac{dv}{v^2+1}\right|.$$

But

$$\int_0^{\pi/\xi} \frac{[1-\cos(k\xi v)]}{v^2+1}\,dv = \frac{1}{\xi}\int_0^{\pi} \frac{(1-\cos ku)}{1+\left(\frac{u}{\xi}\right)^2}\,du$$
$$= \frac{1}{\xi}\int_0^{\pi} \frac{2\sin^2\frac{ku}{2}}{1+\left(\frac{u}{\xi}\right)^2}\,du$$
$$\leq \frac{k^2}{2}\int_0^{\pi} \frac{u^2}{\xi}\cdot\frac{1}{1+\left(\frac{u}{\xi}\right)^2}\,du$$
$$= \frac{k^2}{2}\int_0^{\pi} \xi\frac{u^2}{u^2+\xi^2}\,du.$$

Denote $0 \leq g_\xi(u) = \xi\frac{u^2}{u^2+\xi^2} \leq \xi$. We obviously have $\lim_{\xi\downarrow 0} g_\xi(u) = 0$, uniformly with respect to $u \in [0,\pi]$, which implies

$$\frac{2}{\pi}\lim_{\xi\downarrow 0}\int_0^{\pi/\xi} \frac{\cos(k\xi v)}{v^2+1}\,dv = \frac{2}{\pi}\int_0^{\infty}\frac{dv}{v^2+1} = 1.$$

Therefore,

$$\lim_{\xi\downarrow 0} d_{1,n}(\xi) = \sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k} = 1 > 0,$$

which implies that there exists $\xi_n > 0$ such that $d_{1,n}(\xi) > 0$ for all $\xi \in (0,\xi_n]$.

Obviously, we can choose the same $\xi_n > 0$ for all the four operators $P_{n,\xi}(f)$, $W_{n,\xi}(f)$, $W^*_{n,\xi}(f)$ and $Q_{n,\xi}(f)$.

(ii) Let

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A(\overline{D}),\ |f'(z)| < M,\quad \forall z \in D.$$

Since $a_0 = 0$, by (i) we get

$$P_{n,\xi}(f)(0) = W_{n,\xi}(f)(0) = W^*_{n,\xi}(f)(0) = Q_{n,\xi}(f)(0) = 0.$$

Also, since $a_1 = 1$, by (i) we get

$$\frac{1}{b_{1,n}(\xi)}\cdot P'_{n,\xi}(f)(0) = \frac{1}{c_{1,n}(\xi)}\cdot W'_{n,\xi}(f)(0)$$
$$= \frac{1}{c^*_{1,n}(\xi)}[W^*_{n,\xi}(f)]'(0) = \frac{1}{d_{1,n}(\xi)}\cdot Q'_{n,\xi}(f)(0) = 1,$$

which implies that

$$\frac{1}{b_{1,n}(\xi)}P_{n,\xi}(f),\ \frac{1}{c_{1,n}(\xi)}W_{n,\xi}(f),\ \frac{1}{c^*_{1,n}(\xi)}\cdot W^*_{n,\xi}(f),\ \frac{1}{d_{1,n}(\xi)}Q_{n,\xi}(f)$$
$$\in A(\overline{D}).$$

Also, by

$$P'_{n,\xi}(f)(z) = -\frac{1}{2\xi}\int_{-\infty}^{+\infty} e^{-|u|/\xi}\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}f'(ze^{iku})e^{iku}\,du,$$

we obtain

$$\left|\frac{1}{b_{1,n}(\xi)}P'_{n,\xi}(f)(z)\right| < \frac{M}{|b_{1,n}(\xi)|}\cdot\sum_{k=1}^{n+1}|(-1)^k|\binom{n+1}{k} = \frac{M(2^{n+1}-1)}{|b_{1,n}(\xi)|},$$

i.e.,

$$\frac{1}{b_{1,n}(\xi)}\cdot P_{n,\xi}(f)(z) \in S_{M(2^{n+1}-1)/|b_{1,n}(\xi)|}.$$

The proofs for the other operators are similar, which proves the theorem.

REMARKS. 1) By e.g. [8, p.111, Exercise 5.4.1], $f \in S_M$, $M > 1$, implies that f is univalent in $\left\{z \in \mathbb{C};\ |z| < \frac{1}{M}\right\} \subset D$. Theorem 3.1 (ii) shows that $f \in S_M$ implies that $P_{n,\xi}(f)(z)$ is univalent in

$$\left\{z \in \mathbb{C};\ |z| < \frac{|b_{1,n}(\xi)|}{M(2^{n+1}-1)}\right\} \subset \left\{z \in \mathbb{C};\ |z| < \frac{1}{M}\right\} \subset D,$$

since by Theorem 3.1 (i), we have

$$\begin{aligned}|b_{p,n}(\xi)| &\le \sum_{k=1}^{n+1}\binom{n+1}{k}\cdot\frac{1}{\xi^2k^2p^2+1}\\ &< \sum_{k=1}^{n+1}\binom{n+1}{k} = 2^{n+1}-1,\ \forall p = 0, 1, \dots .\end{aligned}$$

Similar conclusions hold for the operators $W_{n,\xi}(f)(z)$, $W^*_{n,\xi}(f)(z)$ and $Q_{n,\xi}(f)(z)$, by replacing above $b_{1,n}(\xi)$ by $c_{1,n}(\xi)$, $c^*_{1,n}(\xi)$ and $d_{1,n}(\xi)$, respectively.

2) For any fixed $n \in \mathbb{N}$, let us denote

$$B_{1,n} = \inf\{|b_{1,n}(\xi)|; \xi \in (0, \xi_n]\},\ C_{1,n} = \inf\{|c_{1,n}(\xi)|; \xi \in (0, \xi_n]\},$$
$$C^*_{1,n} = \inf\{|c^*_{1,n}(\xi)|; \xi \in (0, \xi_n]\},\ D_{1,n} = \inf\{|d_{1,n}(\xi)|; \xi \in (0, \xi_n]\}.$$

If $B_{1,n}$, $C_{1,n}$, $C^*_{1,n}$, $D_{1,n} > 0$, then by Theorem 3.1 (ii), the following properties hold:

$f \in S_M$ implies that $P_{n,\xi}(f)$ is univalent in $\left\{z \in \mathbb{C};\, |z| < \frac{B_{1,n}}{M(2^{n+1}-1)}\right\}$ for all $x \in (0, \xi_n]$,

$f \in S_M$ implies that $W_{n,\xi}(f)$ is univalent in $\left\{z \in \mathbb{C};\, |z| < \frac{C_{1,n}}{M(2^{n+1}-1)}\right\}$, for all $\xi \in (0, \xi_n]$,

$f \in S_M$ implies that $W^*_{n,\xi}(f)$ is univalent in $\left\{z \in \mathbb{C};\, |z| < \frac{C^*_{1,n}}{M(2^{n+1}-1)}\right\}$, for all $\xi \in (0, \xi_n]$,

$f \in S_M$ implies that $Q_{n,\xi}(f)$ is univalent in $\left\{z \in \mathbb{C};\, |z| < \frac{D_{1,n}}{M(2^{n+1}-1)}\right\}$, for all $\xi \in (0, \xi_n]$.

Therefore, it remains to calculate (for each fixed $n \in \mathbb{N}$), $B_{1,n}$, $C_{1,n}$, $C^*_{1,n}$, $D_{1,n}$, to check if $B_{1,n} > 0$, $C_{1,n} > 0$, $C^*_{1,n} > 0$, problems which are left to the reader as open questions.

3) It would be interesting to investigate for other geometric properties of the operators in this paper.

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