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GEOMETRIC PROBABILITY IN THE MINKOWSKI PLANE

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ABSTRACT. In this paper, we get the measure of strips and lines in the Minkowski plane M^2 that meet a fixed compact convex body in M^2. From this we also investigate the probability that strips and lines meet a fixed compact convex body in M^2.

1. Introduction

In [2], G. D. Chakerian computed the measure of lines that meet a fixed curve, compact convex body in the Minkowski plane. The study on integral geometry was initiated by W. Blaschke. Many problems treated in integral geometry had their roots in geometric probability ([4]). Now many problems in integral geometry for the Minkowski spaces are open. In this paper, we have some integral formulas for the strip in the Minkowski plane. We also get the measure of strips and lines in the Minkowski plane M^2 that meet a fixed compact convex body in M^2. From this we get the probability that strips and lines meet a fixed compact convex body in M^2.

2. Basic concepts for the Minkowski plane

The Euclidean plane E^2 can be viewed as a normed space with an arbitrary norm $\|\cdot\|$. Then the norm $\|\cdot\|$ makes E^2 a new metric space $(M^2, \|\cdot\|)$ called the Minkowski plane. Let $K = \{\mathrm{x} \in M^2 : \|\mathrm{x}\| \leq 1\}$ be a unit ball in M^2. And we denote the boundary of K by bdK. We shall assume throughout that bdK is sufficiently differentiable and has positive finite curvature everywhere. Then bdK is a centrally symmetric

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closed convex curve. Assume that bdK has enclosing Euclidean area π. Now we parametrize bdK as follows:

We write the equation of bdK as $t = t(\phi)$, $0 \le \phi \le 2\pi$. Then $\|\overrightarrow{t}\| = 1$, where $\overrightarrow{t}$ is the vector from the origin O to the point corresponding to $t(\phi)$ on bdK. And the sectorial area from the positive x-axis to $\overrightarrow{t}$ is $\frac{\phi}{2}$([1]).

From now on we do not distinguish $\overrightarrow{t}$ from t. Now we define $n(\phi)$ by $n(\phi) = \frac{dt(\phi)}{d\phi}$, $0 \le \phi \le 2\pi$. We denote by bdK^* the curve with equation $n = n(\phi)$, $0 \le \phi \le 2\pi$. Then bdK^* is the polar dual of bdK with respect to the Euclidean unit circle, rotated through $\frac{\pi}{2}$. The function $\lambda = \lambda(\phi)$, $0 \le \phi \le 2\pi$, defined by $\frac{dn(\phi)}{d\phi} = -\frac{t(\phi)}{\lambda(\phi)}$ is called the Minkowskian curvature of bdK^* at a point where the tangent has direction $t(\phi)$([2]). For $\mathrm{x} = (x_1, x_2)$, $\mathrm{y} = (y_1, y_2)$ in M^2 we denote $[\mathrm{x}, \mathrm{y}] = x_1 y_2 - x_2 y_1$. Put

$$\mathcal{A}(p, t(\phi)) = (0, \infty) \times [0, 2\pi) \cup 0 \times [0, \pi)$$

and let $\mathcal{L}(M^2)$ be the set of all straight lines on M^2. Then there is a one-to-one correspondence T from $\mathcal{L}(M^2)$ onto $\mathcal{A}(p, t(\phi))$ as follows:

If $G \in \mathcal{L}(M^2)$ is a line parallel to the direction $t(\phi)$ and the Euclidean area of the parallelogram with the sides $t(\phi)$ and $\mathrm{x} \in G$ is p, then $T(G) = (p, t(\phi))$.

Here the equation of G is:

$$[t(\phi), \mathrm{x}] = p.$$

Then the density of G(see [2]) is

$$dG = \lambda^{-1}(\phi) dp d\phi.$$

3. Main results

By a strip $B \subset M^2$ we mean the closed region between two parallel straight lines. The position of B can be determined by the coordinates p and $t(\phi)$ of its midparallel. So the density for sets of strips is

$$dB = \lambda^{-1}(\phi) dp d\phi.$$

For a compact convex body U with the origin in its interior, the Minkowski length $L(bdU)$ of bdU is defined by

$$L(bdU) = \int_{-\pi}^{\pi} \left(h_U(\theta) + h_U''(\theta)\right) \rho(\theta + \frac{\pi}{2}\epsilon) d\theta,$$

where $\epsilon = \pm 1$, $h_U(\cdot)$ is a support function of U, and $\rho(\cdot)$ is a support function of K^*([3]). From now on we denote the Minkowski length by $L(\cdot)$. We have the following theorem.

THEOREM 1. *For a compact convex body $U \subset M^2$ and a strip B of breadth Br, we have*

$$\int_{B\cap U\neq\emptyset} dB = L(bdU) + \frac{1}{2}BrP(K^*),$$

where $P(\cdot)$ is the Euclidean perimeter.

Proof. B meets U if and only if the midparallel of B meets the parallel body $U_{\frac{1}{2}Br}$ of U at distance $\frac{1}{2}Br$. Let G be a midparallel of B. For the equality $\int_{G\cap U_{\frac{1}{2}Br}\neq\emptyset} dG = L(bdU_{\frac{1}{2}Br})$, let $\mathrm{x}(s)$ be the equation of $bdU_{\frac{1}{2}Br}$ with Minkowski arc length s. Now we can assume that x is on the intersection of $bdU_{\frac{1}{2}Br}$ with G of the equation $[t(\phi), \mathrm{x}] = p$. Let $\overrightarrow{u} = \frac{d\mathrm{x}}{ds}$ be the Minkowski unit tangent vector at $\mathrm{x}(s)$. Then

$$\begin{aligned}
(1) \qquad \int_{G\cap U_{\frac{1}{2}Br}\neq\emptyset} dG &= \frac{1}{2}\int_{bdU_{\frac{1}{2}Br}}\int_0^{\pi} |[t(\phi), \overrightarrow{u}]|\lambda^{-1}(\phi)d\phi ds \\
&= \int_{bdU_{\frac{1}{2}Br}} ds \\
&= L(bdU_{\frac{1}{2}Br}).
\end{aligned}$$

In the equation (1) we can assume that $\overrightarrow{u}$ moves from the direction $t(0)$ to the direction $t(\pi)$ without loss of generality. Thus we get

$$\begin{aligned}
\int_{B\cap U\neq\emptyset} dB &= \int_{G\cap U_{\frac{1}{2}Br}\neq\emptyset} dG \\
&= L(bdU_{\frac{1}{2}Br}) \\
&= \int_{-\pi}^{\pi}\left(h_{U_{\frac{1}{2}Br}}(\theta) + h''_{U_{\frac{1}{2}Br}}(\theta)\right)\rho(\theta + \frac{\pi}{2}\epsilon)d\theta \\
&= \int_{-\pi}^{\pi}\left(\frac{1}{2}Br + h_U(\theta) + h''_U(\theta)\right)\rho(\theta + \frac{\pi}{2}\epsilon)d\theta \\
&= L(bdU) + \frac{1}{2}BrP(K^*).
\end{aligned}$$

This completes the proof. □

From the Theorem 1 we have immediately the following corollary.

COROLLARY 1. *For a fixed point* x $\in M^2$ *and a strip* B *of breadth* Br, *we have*

$$\int_{\mathrm{x}\in B} dB = \frac{1}{2} BrP(K^*).$$

Now we compute the integral of some geometric quantity on the set of strips that meet a fixed compact convex body in M^2. First, we have the following theorem.

THEOREM 2. *Let* U *be a compact convex body in* M^2 *and let* B *be a strip of breadth* Br. *Then*

$$\int_{B\cap U\neq\emptyset} A(B\cap U)dB = \frac{1}{2} BrP(K^*)A(U),$$

where $A(\cdot)$ *is the area.*

Proof. Let G_1 and G_2 be any lines in M^2. Now we consider the integral

$$\int_{G_1\cap G_2\cap B\cap U\neq\emptyset} dG_1 dG_2 dB.$$

Fixing the lines G_1 and G_2 and then integrating over the strips B, gives

$$\begin{aligned}
&\int_{G_1\cap G_2\cap B\cap U\neq\emptyset} dG_1 dG_2 dB \\
=\ & \int_{G_1\cap U\neq\emptyset}\int_{G_2\cap(G_1\cap U)\neq\emptyset}\int_{B\cap(G_1\cap G_2\cap U)\neq\emptyset} dB dG_2 dG_1 \\
=\ & \frac{1}{2} BrP(K^*)\int_{G_1\cap U\neq\emptyset}\int_{G_2\cap(G_1\cap U)\neq\emptyset} dG_2 dG_1 \\
=\ & BrP(K^*)\int_{G_1\cap U\neq\emptyset} L(G_1\cap U)dG_1 \\
=\ & BrP(K^*)A(K^*)A(U).
\end{aligned}$$

The second equality follows from Corollary 1 immediately. On the other hand, first fixing the strip B and then integrating over the lines G_1 and

G_2 gives

$$\begin{aligned}
&\int_{G_1 \cap G_2 \cap B \cap U \neq \emptyset} dG_1 dG_2 dB \\
= &\int_{B \cap U \neq \emptyset} \int_{G_1 \cap (B \cap U) \neq \emptyset} \int_{G_2 \cap (G_1 \cap B \cap U) \neq \emptyset} dG_2 dG_1 dB \\
= &\ 2 \int_{B \cap U \neq \emptyset} \int_{G_1 \cap (B \cap U) \neq \emptyset} L(G_1 \cap B \cap U) dG_1 dB \\
= &\ 2A(K^*) \int_{B \cap U \neq \emptyset} A(B \cap U) dB.
\end{aligned}$$

From this we have the result. □

Now we investigate the measure of the set of lines and strips that meet a fixed compact convex body U.

THEOREM 3. *If B_1 and B_2 are independent strips in M^2 of breadth Br_1, Br_2, respectively that meet a fixed compact convex body U and the line $G \in \mathcal{L}(M^2)$ meets U, then the probability that $B_1 \cap B_2 \cap G \cap U \neq \emptyset$ is*

$$Prob = \frac{P(K^*)A(K^*)A(U)[Br_1 + Br_2] + \frac{1}{4}P^2(K^*)L(bdU)Br_1 Br_2}{L(bdU)[L(bdU) + \frac{1}{2}Br_1 P(K^*)][L(bdU) + \frac{1}{2}Br_2 P(K^*)]}.$$

Proof. We compute the measure of B_1, B_2, and G such that $B_1 \cap B_2 \cap G \cap U \neq \emptyset$.

$$\begin{aligned}
&\int_{B_1 \cap B_2 \cap G \cap U \neq \emptyset} dG dB_1 dB_2 \\
= &\int_{B_1 \cap U \neq \emptyset} \int_{G \cap (B_1 \cap U) \neq \emptyset} \int_{B_2 \cap (B_1 \cap G \cap U) \neq \emptyset} dB_2 dG dB_1 \\
= &\int_{B_1 \cap U \neq \emptyset} \int_{G \cap (B_1 \cap U) \neq \emptyset} [2L(G \cap B_1 \cap U) + \frac{1}{2} Br_2 P(K^*)] dG dB_1 \\
= &\int_{B_1 \cap U \neq \emptyset} [2A(K^*)A(B_1 \cap U) + \frac{1}{2} Br_2 P(K^*) L(bd(B_1 \cap U))] dB_1.
\end{aligned}$$

From Theorem 2 we have

$$\int_{B_1 \cap U \neq \emptyset} 2A(K^*)A(B_1 \cap U) dB_1 = P(K^*)A(K^*)A(U)Br_1.$$

Now for a line $G \in \mathcal{L}(M^2)$ and a strip B, consider the measure of G and B such that $G \cap B \cap U \neq \emptyset$.

$$\begin{aligned}\int_{G\cap B\cap U\neq\emptyset} dBdG &= \int_{B\cap U\neq\emptyset}\int_{G\cap(B\cap U)\neq\emptyset} dGdB \\ &= \int_{B\cap U\neq\emptyset} L(bd(B\cap U))dB.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{G\cap B\cap U\neq\emptyset} dBdG &= \int_{G\cap U\neq\emptyset}\int_{B\cap(G\cap U)\neq\emptyset} dBdG \\ &= \int_{G\cap U\neq\emptyset}[2L(G\cap U)+\frac{1}{2}BrP(K^*)]dG \\ &= 2A(K^*)A(U)+\frac{1}{2}BrP(K^*)L(bdU).\end{aligned}$$

Thus we have

$$\int_{B\cap U\neq\emptyset} L(bd(B\cap U))dB = 2A(K^*)A(U)+\frac{1}{2}BrP(K^*)L(bdU).$$

So we have

$$\begin{aligned}&\int_{B_1\cap B_2\cap G\cap U\neq\emptyset} dGdB_1dB_2 \\ &= P(K^*)A(K^*)A(U)[Br_1+Br_2]+\frac{1}{4}P^2(K^*)L(bdU)Br_1Br_2.\end{aligned}$$

From this we get the result easily. □

If B_1 and B_2 are strips of breadth Br, then we have the following corollary.

COROLLARY 2. *Let U be a compact convex body in M^2 and let B_1 and B_2 be strips of breadth Br. Then for a line $G \in \mathcal{L}(M^2)$ we have*

$$\begin{aligned}&\int_{B_1\cap B_2\cap G\cap U\neq\emptyset} dGdB_1dB_2 \\ &= 2P(K^*)A(K^*)A(U)Br+\frac{1}{4}P^2(K^*)L(bdU)(Br)^2.\end{aligned}$$

In Theorem 3 we have

$$\int_{B\cap U\neq\emptyset} L(bd(B\cap U))dB = 2A(K^*)A(U)+\frac{1}{2}BrP(K^*)L(bdU).$$

Now we consider the measure $\int_{B\cap U\neq\emptyset} L^2(bd(B\cap U))dB$.

THEOREM 4. *Let U be a compact convex body in M^2 and let B be a strip of breadth Br. Then we have*

$$\begin{aligned} & \int_{B\cap U\neq\emptyset} L^2(bd(B\cap U))dB \\ = & \left[2A(K^*)A(U)+\frac{1}{2}BrP(K^*)L(bdU)\right]L(bd(B\cap U)). \end{aligned}$$

Proof. Let G_1, G_2 be any lines in $\mathcal{L}(M^2)$. Then we have

$$\begin{aligned} & \int_{G_1\cap B\cap U\neq\emptyset, G_2\cap B\cap U\neq\emptyset} dG_1dG_2dB \\ = & \int_{B\cap U\neq\emptyset}\int_{G_1\cap(B\cap U)\neq\emptyset, G_2\cap(B\cap U)\neq\emptyset} dG_1dG_2dB \\ = & \int_{B\cap U\neq\emptyset}\int_{G_1\cap(B\cap U)\neq\emptyset} dG_1\int_{G_2\cap B\cap U\neq\emptyset} dG_2dB \\ = & \int_{B\cap U\neq\emptyset} L^2(bd(B\cap U))dB. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \int_{G_1\cap B\cap U\neq\emptyset, G_2\cap B\cap U\neq\emptyset} dG_1dG_2dB \\ = & \int_{G_1\cap B\cap U\neq\emptyset} dG_1\int_{G_2\cap B\cap U\neq\emptyset} dBdG_2. \end{aligned}$$

Here we get

$$\begin{aligned} \int_{G_2\cap B\cap U\neq\emptyset} dBdG_2 &= \int_{G_2\cap U\neq\emptyset}\int_{B\cap(G_2\cap U)\neq\emptyset} dBdG_2 \\ &= \int_{G_2\cap U\neq\emptyset}\left[2L(G_2\cap U)+\frac{1}{2}BrP(K^*)\right]dG_2 \\ &= 2A(K^*)A(U)+\frac{1}{2}BrP(K^*)L(bdU). \end{aligned}$$

Thus we have

$$\begin{aligned} & \int_{G_1\cap B\cap U\neq\emptyset, G_2\cap B\cap U\neq\emptyset} dG_1dG_2dB \\ = & L(bd(B\cap U))\left[2A(K^*)A(U)+\frac{1}{2}BrP(K^*)L(bdU)\right]. \end{aligned}$$

This completes the proof. □

THEOREM 5. *Let U be a compact convex body in M^2 and let B be a strip of breadth Br. Then for $n \geq 1$ we have*

$$\frac{\int_{B\cap U\neq\emptyset} L^{n+1}(bd(B\cap U))dB}{\int_{B\cap U\neq\emptyset} L^{n}(bd(B\cap U))dB} = L(bd(B\cap U)).$$

Proof. Let $G_1, G_2, \cdots, G_n$ be any lines in $\mathcal{L}(M^2)$. Then for $1 \leq i \leq n$, we have

$$\begin{aligned}
&\int_{G_i\cap B\cap U\neq\emptyset} dG_1 dG_2\cdots dG_n dB\\
&= \int_{B\cap U\neq\emptyset}\int_{G_i\cap(B\cap U)\neq\emptyset} dG_1 dG_2\cdots dG_n dB\\
&= \int_{B\cap U\neq\emptyset}\int_{G_1\cap(B\cap U)\neq\emptyset} dG_1\cdots\int_{G_n\cap B\cap U\neq\emptyset} dG_n dB\\
&= \int_{B\cap U\neq\emptyset} L^n(bd(B\cap U))dB.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{G_i\cap B\cap U\neq\emptyset} dG_1 dG_2\cdots dG_n dB\\
&= \int_{G_1\cap B\cap U\neq\emptyset} dG_1\cdots\int_{G_{n-1}\cap B\cap U\neq\emptyset} dG_{n-1}\int_{G_n\cap B\cap U\neq\emptyset} dB dG_n\\
&= L^{n-1}(bd(B\cap U))\left[2A(K^*)A(U)+\frac{1}{2}BrP(K^*)L(bdU)\right].
\end{aligned}$$

This completes the proof. □

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