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ON WEYL SPECTRA OF ALGEBRAICALLY TOTALLY-PARANORMAL OPERATORS

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Dedicated to Professor Yong Tae Kim on his 65th birthday

ABSTRACT. In this paper we show that Weyl's theorem holds for $f(T)$ when an Hilbert space operator T is "algebraically totally-paranormal" and f is any analytic function on an open neighborhood of the spectrum of T.

1. Introduction

Throughout this paper let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators acting on an infinite dimensional Hilbert space $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$ write $N(T)$ and $R(T)$ for the null space and range of T; $\sigma(T)$ for the spectrum of T; $\pi_0(T)$ for the set of eigenvalues of T; $\pi_{00}(T)$ for the isolated points of $\sigma(T)$ which are eigenvalues of finite multiplicity. Recall ([5], [7]) that an operator $T \in \mathcal{L}(\mathcal{H})$ is called *Fredholm* if it has closed range with finite dimensional null space and its range of finite co-dimension. The *index* of a Fredholm operator $T \in \mathcal{L}(\mathcal{H})$ is given by

$$\text{ind}\,(T) = \dim N(T) - \dim R(T)^{\perp} \ (= \dim N(T) - \dim N(T^*)).$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *Weyl* if it is Fredholm of index zero. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *Browder* if it is Fredholm "of finite ascent and descent": equivalently, if T is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. The essential spectrum $\sigma_e(T)$, the

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Weyl spectrum $\omega(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in \mathcal{L}(\mathcal{H})$ are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\},$$
$$\omega(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\},$$
$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\},$$
$$\sigma_e(T) \subseteq \omega(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \operatorname{acc} \sigma(T),$$

where we write $\operatorname{acc} \mathbf{K}$ for the accumulation points of $\mathbf{K} \subseteq \mathbb{C}$. Following Coburn ([1]) we say that *Weyl's theorem holds for* $T \in \mathcal{L}(\mathcal{H})$ if there is equality

$$\sigma(T) \setminus \omega(T) = \pi_{00}(T).$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T. Recall ([8]) that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *totally-paranormal* if

$$||(T-\lambda)x||^2 \leq ||(T-\lambda)^2 x||\,||x|| \text{ for all } x \in \mathcal{H} \text{ and } \lambda \in \mathbb{C}.$$

We shall say that the operator $T \in \mathcal{L}(\mathcal{H})$ is *algebraically totally-paranormal* if there exists a nonconstant complex polynomial p such that $p(T)$ is totally-paranormal. Evidently,

$$\{\text{hyponormal operators}\} \subseteq \{\text{totally-paranormal operators}\}$$

and

$$\{\text{algebraically hyponormal operators}\}$$
$$\subseteq \{\text{algebraically totally-paranormal operators}\}.$$

From well-known facts (cf. [8]) of totally-paranormal operators we easily see that

(a) If $T \in \mathcal{L}(\mathcal{H})$ is algebraically totally-paranormal, then so is $T - \lambda I$ for each $\lambda \in \mathbb{C}$.

(b) If $T \in \mathcal{L}(\mathcal{H})$ is algebraically totally-paranormal and $\mathcal{M} \subseteq \mathcal{H}$ is invariant under T, then $T|\mathcal{M}$ is algebraically totally-paranormal.

(c) Unitary equivalence preserves algebraical totally-paranormality.

In [4] Han and Lee showed that Weyl's theorem holds for $f(T)$ when T is an algebraically hyponormal operator and f is an analytic function on an open neighborhood of $\sigma(T)$.

In this paper we extend this result to algebraically totally-paranormal operators: our proof however differs from the correspondence in [4], in that we employ techniques from local spectral theory.

The following is our main result.

THEOREM. *If $T \in \mathcal{L}(\mathcal{H})$ is algebraically totally-paranormal, then for every $f \in H(\sigma(T))$, Weyl's theorem holds for $f(T)$, where $H(\sigma(T))$ denotes the set of analytic functions on an open neighborhood of $\sigma(T)$.*

2. Proofs

The following two lemmas give important and essential facts for algebraically totally-paranormal operators but its proofs are routine and similar to that of Han and Lee ([4]). Thus we shall just state them without proofs.

The following result is an extension of [4, Lemma 1] to algebraically totally-paranormal operators.

LEMMA 1. *Suppose $T \in \mathcal{L}(\mathcal{H})$.*

(i) *If T is algebraically totally-paranormal and quasinilpotent, then T is nilpotent.*

(ii) *If T is algebraically totally-paranormal, then T is isoloid.*

(iii) *If T is algebraically totally-paranormal, then T has finite ascent.*

The following result is an extension of [4, Theorem 3] to algebraically totally-paranormal operators.

LEMMA 2. *If $T \in \mathcal{L}(\mathcal{H})$ is algebraically totally-paranormal, then*

$$\omega(f(T)) = f(\omega(T)) \quad \text{for every } f \in H(\sigma(T)).$$

To state next lemma we need some notions from local spectral theory. We say that $T \in \mathcal{L}(\mathcal{H})$ has the *single valued extension property* (*SVEP*) if there is implication, for arbitrary open sets $U \subseteq \mathbb{C}$ and holomorphic functions $f : U \to \mathcal{H}$,

$$(T - zI)f(z) = 0 \text{ on } U \implies f(z) = 0 \text{ on } U.$$

If this holds for a neighborhood U of $\lambda \in \mathbb{C}$ we say that T has the SVEP at λ.

We introduce two important subsets of $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$ and F is a closed set in $\mathbb{C}$, we define

$$\mathcal{H}_T(F) = \{x \in \mathcal{H} : \text{there exists an analytic } \mathcal{H}\text{-valued function } f : \mathbb{C} \setminus F \longrightarrow \mathcal{H} \text{ such that } (T - \lambda)f(\lambda) = x\}.$$

Then $\mathcal{H}_T(F)$ is said to be the *spectral manifold* of T. If T has the SVEP, then the above definition is identical with $\mathcal{H}_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subseteq F\}$, where $\sigma_T(x)$ is the local spectrum of T at x. (see [2], [3], [8], [9] for details)

Let $H_\circ(T) = \{x \in \mathcal{H} : ||T^n x||^{\frac{1}{n}} \rightarrow 0\}$. If $H_\circ(T) = \mathcal{H}$, then T is a quasinilpotent operator on $\mathcal{H}$ ([2, p.28. Lemma]).

Now we are ready for the following result.

LEMMA 3. *Weyl's theorem holds for every algebraically totally-paranormal operator.*

Proof. Suppose $p(T)$ is totally-paranormal for some nonconstant polynomial p. We first prove that $\pi_{00}(T) \subseteq \sigma(T) \setminus \omega(T)$. Without loss of generality, it suffices to show that

$$0 \in \pi_{00}(T) \Longrightarrow T \text{ is Weyl but not invertible.}$$

Suppose $0 \in \pi_{00}(T)$. Since $0 \in \text{iso}\sigma(T)$, we can consider the Riesz spectral projection P_0 with respect to 0 ([7, Theorem 49.1; Proposition 49.1]) such that

$$R(P_0) = H_\circ(T),\ (T)|_{N(P_0)} \text{ is invertible, and } H = R(P_0) \oplus N(P_0).$$

It is well known ([8, Proposition 1.8]) that if T has finite ascent, then it has the SVEP at 0. It is well known ([8, Corollary 2.4]) that if T has the SVEP at 0, then

$$\mathcal{H}_T(\{0\}) = H_\circ(T).$$

Thus we have

$$R(P_0) = H_\circ(T) = \mathcal{H}_T(\{0\}).$$

By hypothesis $R(T)$ is closed and $0 \in \pi_0(T)$, and so T is semi-Fredholm. Then since $\mathcal{H}_T(\{0\})$ is closed, we have by [9, Theorem 2]

$$R(P_0) = \mathcal{H}_T(\{0\}) \text{ is finite dimensional.}$$

Thus the restrictions of T to reducing subsets $R(P_0)$ and $N(P_0)$ are finite dimensional and invertible operators, respectively. So we can see that T is Weyl but not invertible. Hence we have that $\pi_{00}(T) \subseteq \sigma(T) \setminus \omega(T)$.

For the reverse inclusion, suppose $0 \in \sigma(T) \setminus \omega(T)$. Thus T is Weyl. Since T has a finite ascent, T has also a finite descent by [10, Theorem 1(4)]. So T is Weyl of finite ascent and descent, and then it is Browder. Therefore $0 \in \pi_{00}(T)$. This completes the proof. $\square$

Now we conclude with the proof of Theorem.

Proof of Theorem. Remembering [12, Lemma] that if T is isoloid, then

$$f\big(\sigma(T)\setminus\pi_{00}(T)\big)=\sigma(f(T))\setminus\pi_{00}(f(T))\quad\text{for every } f\in H(\sigma(T));$$

it follows from Lemma 1 (ii), Lemma 2 and Lemma 3 that

$$\sigma(f(T))\setminus\pi_{00}(f(T))=f\big(\sigma(T)\setminus\pi_{00}(T)\big)=f(\omega(T))=\omega(f(T)),$$

which implies that Weyl's theorem holds for $f(T)$. □

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