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ON THE HYERS-ULAM STABILITY OF THE BANACH SPACE-VALUED DIFFERENTIAL EQUATION $y' = \lambda y$

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Dedicated to Professor Jyunji Inoue on his retirement from Hokkaido University

ABSTRACT. Let I be an open interval and X a complex Banach space. Let $\varepsilon \geq 0$ and λ a non-zero complex number with $\operatorname{Re}\lambda \neq 0$. If φ is a strongly differentiable map from I to X with $\|\varphi'(t) - \lambda\varphi(t)\| \leq \varepsilon$ for all $t \in I$, then we show that the distance between φ and the set of all solutions to the differential equation $y' = \lambda y$ is at most $\varepsilon/|\operatorname{Re}\lambda|$.

1. Introduction

Let φ be a differentiable function from $\mathbb{R}$ to $\mathbb{R}$, the real number field. Alsina and Ger [1] proved the following result: if a differentiable function $\varphi\colon \mathbb{R} \to \mathbb{R}$ satisfies $|\varphi'(t) - \varphi(t)| \leq \varepsilon$ for all $t \in \mathbb{R}$, then there exists a constant c such that $|\varphi(t) - ce^t| \leq 3\varepsilon$ for all $t \in \mathbb{R}$. That is, the distance between φ and the set of all solutions to the differential equation $y' = y$ is at most 3ε.

Let $X \neq \{0\}$ be a complex Banach space, $\varepsilon \geq 0$ and λ a non-zero complex number. Let I be an open interval of $\mathbb{R}$. We say that Hyers-Ulam stability holds for the differential equation $y' = \lambda y$ on I, if there exists a constant $k \geq 0$ with the following property: for every strongly differentiable map $\varphi\colon I \to X$ with $\|\varphi'(t) - \lambda\varphi(t)\| \leq \varepsilon$ there corresponds an $x_\varphi \in X$ so that $\|\varphi(t) - e^{\lambda t}x_\varphi\| \leq k\varepsilon$. Note that the general solution to the (X-valued) differential equation $y' = \lambda y$ is of the form $e^{\lambda t}x$ for some $x \in X$. We say that the constant $k \geq 0$ with the property stated above is a Hyers-Ulam constant for the differential equation $y' = \lambda y$, or simply Hyers-Ulam constant.

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In [2, 3] we considered some specific Banach spaces; the Banach space of all real-valued bounded continuous functions on a topological space; uniformly closed linear subspace of the Banach space of all complex-valued bounded continuous functions on a topological space. Then we proved the Hyers-Ulam stability for $y' = \lambda y$.

In this note we consider a strongly differentiable map from I to a general Banach space X. We study the Hyers-Ulam stability of the X-valued differential equation $y' = \lambda y$ on I. And the result is summarized as follows: if $\operatorname{Re}\lambda$, the real part of λ, is not zero then the Hyers-Ulam stability holds; if $\operatorname{Re}\lambda = 0$ then two cases occur: the Hyers-Ulam stability holds if the diameter $\delta(I)$ of I is finite and does not hold if $\delta(I)$ is infinite.

2. Main results

Let us define for $I \subset \mathbb{R}$ and $\lambda \in \mathbb{C}$

$m(I,\lambda) = \inf\{e^{-\operatorname{Re}\lambda t} : t \in I\}$ and $M(I,\lambda) = \sup\{e^{-\operatorname{Re}\lambda t} : t \in I\}$.

Clearly $0 \le m(I,\lambda) < \infty$ and $0 < M(I,\lambda) \le \infty$. Then we have the following result.

THEOREM 2.1. *Let $\varepsilon > 0$ and $\varphi\colon I \to X$ a strongly differentiable function such that $\|\varphi'(t) - \lambda\varphi(t)\| \le \varepsilon$ for all $t \in I$. Then the following assertions are true:*

(i) *If $\operatorname{Re}\lambda \ne 0$, then there exists an element $x_\varphi \in X$ such that*

$$\|\varphi(t) - e^{\lambda t}x_\varphi\| \le |\operatorname{Re}\lambda|^{-1}\left(1 - \frac{m(I,\lambda)}{M(I,\lambda)}\right)\varepsilon$$

for all $t \in I$. In particular, if $m(I,\lambda) = 0$ then x_φ with the property $\sup_{t\in I}\|\varphi(t) - e^{\lambda t}x_\varphi\| < \infty$ is unique.

(ii) *If $\operatorname{Re}\lambda = 0$ and the diameter $\delta(I)$ of I is finite, then there exists an $x_\varphi \in X$ such that*

$$\|\varphi(t) - e^{\lambda t}x_\varphi\| \le \varepsilon\delta(I)$$

for all $t \in I$.

(iii) *If $\operatorname{Re}\lambda = 0$ and $\delta(I) = \infty$, then the Hyers-Ulam stability of the differential equation $y' = \lambda y$ does not hold.*

Proof. Let X^* be the dual space of X. For each $f \in X^*$ we define the map $\varphi_f\colon I \to \mathbb{C}$ by

$$\varphi_f(t) = f(\varphi(t)), \quad (t \in I).$$

Fix $f \in X^*$ arbitrarily. Then we have $(\varphi_f)'(t) = f(\varphi'(t))$ for every $t \in I$. Also

$$\begin{aligned} |(\varphi_f)'(t) - \lambda\varphi_f(t)| &= |f(\varphi'(t)) - f(\lambda\varphi(t))| \\ &\leq \|f\| \, \|\varphi'(t) - \lambda\varphi(t)\| \leq \varepsilon\|f\| \end{aligned}$$

holds for every $t \in I$. Put $h(t) = e^{-\lambda t}\varphi_f(t)$ for each $t \in I$. Then we see that $h'(t) = \{(\varphi_f)'(t) - \lambda\varphi_f(t)\}e^{-\lambda t}$. Hence $|h'(t)| \leq \varepsilon\|f\||e^{-\lambda t}|$ for all $t \in I$. Let $s, t \in I$ with $s < t$. Then h' is integrable on $[s,t]$. Although h' need not be continuous, it is well-known that $h(s) - h(t) = \int_s^t h'(\tau)d\tau$ (cf. [4, Theorem 7.21]). Therefore we have

$$\begin{aligned} |h(s) - h(t)| &= \left| \int_s^t h'(\tau)d\tau \right| \\ &\leq \varepsilon\|f\| \int_s^t |e^{-\lambda\tau}|d\tau \qquad (1) \\ &= \varepsilon\|f\| \int_s^t e^{-\operatorname{Re}\lambda\tau}d\tau. \end{aligned}$$

(i) Suppose $\operatorname{Re}\lambda \neq 0$. By the inequality (1) we obtain

$$\left| f\left(e^{-\lambda t}\varphi(t) - e^{-\lambda s}\varphi(s)\right) \right| \leq \frac{\varepsilon\|f\|}{|\operatorname{Re}\lambda|} |e^{-\operatorname{Re}\lambda t} - e^{-\operatorname{Re}\lambda s}|$$

for all $s, t \in I$. Since f is arbitrary, it follows that

$$\|e^{-\lambda t}\varphi(t) - e^{-\lambda s}\varphi(s)\| \leq \frac{\varepsilon}{|\operatorname{Re}\lambda|} |e^{-\operatorname{Re}\lambda t} - e^{-\operatorname{Re}\lambda s}|$$

for all $s, t \in I$. This implies that if $e^{-\operatorname{Re}\lambda s} \searrow m(I, \lambda)$, then $e^{-\lambda s}\varphi(s)$ converges to an element, say $x_\varphi \in X$. Then we have

$$\begin{aligned} \|\varphi(t) - e^{\lambda t}x_\varphi\| &\leq e^{\operatorname{Re}\lambda t}\|e^{-\lambda t}\varphi(t) - e^{-\lambda s}\varphi(s)\| \\ &\quad + e^{\operatorname{Re}\lambda t}\|e^{-\lambda s}\varphi(s) - x_\varphi\| \\ &\leq \frac{\varepsilon}{|\operatorname{Re}\lambda|} \left| 1 - \frac{e^{-\operatorname{Re}\lambda s}}{e^{-\operatorname{Re}\lambda t}} \right| + e^{\operatorname{Re}\lambda t}\|e^{-\lambda s}\varphi(s) - x_\varphi\|. \end{aligned}$$

Letting $e^{-\operatorname{Re}\lambda s} \searrow m(I, \lambda)$ we obtain

$$\|\varphi(t) - e^{\lambda t}x_\varphi\| \leq \frac{\varepsilon}{|\operatorname{Re}\lambda|} \left(1 - \frac{m(I,\lambda)}{M(I,\lambda)}\right)$$

for all $t \in I$.

Now suppose that $x \in X$ is such that $\sup_{t\in I} \|\varphi(t) - e^{\lambda t}x\| = c < \infty$. If $m(I,\lambda) = 0$ then we have

$$\begin{aligned} \|x - x_\varphi\| &\leq |e^{-\lambda t}| \left\{ \|e^{\lambda t}x - \varphi(t)\| + \|\varphi(t) - e^{\lambda t}x_\varphi\| \right\} \\ &\leq \left(c + \frac{\varepsilon}{|\mathrm{Re}\,\lambda|} \right) e^{-\mathrm{Re}\,\lambda t} \\ &\to 0 \quad (\text{as } e^{-\mathrm{Re}\,\lambda t} \to 0). \end{aligned}$$

Hence $x = x_\varphi$.

(ii) Suppose $\mathrm{Re}\,\lambda = 0$ and $\delta(I) < \infty$. Then by the inequality (1) we have

$$\begin{aligned} |f(e^{-\lambda t}\varphi(t) - e^{-\lambda s}\varphi(s))| &\leq \varepsilon\|f\| \left| \int_s^t e^{-\mathrm{Re}\,\lambda\tau} d\tau \right| \\ &= \varepsilon|t-s|\|f\| \\ &\leq \varepsilon\delta(I)\|f\| \end{aligned}$$

for all $s, t \in I$. Since f is arbitrary and since $\mathrm{Re}\,\lambda = 0$, it follows that

$$\|\varphi(t) - e^{\lambda t}e^{-\lambda s}\varphi(s)\| \leq \varepsilon\delta(I)$$

for all $s, t \in I$. Then $e^{-\lambda s}\varphi(s)$ is an element with the property stated above for every $s \in I$.

(iii) Suppose that $\mathrm{Re}\,\lambda = 0$ and $\delta(I) = \infty$. We can find an element $x_0 \in X$ with $\|x_0\| = 1$. Put $\varphi_0(t) = \varepsilon\, te^{\lambda t}x_0$ for each $t \in I$. Then $\|\varphi_0{}'(t) - \lambda\varphi_0(t)\| = \varepsilon\,|e^{\lambda t}|\|x_0\| = \varepsilon$ for all $t \in I$. Assume that there exist a constant $k \geq 0$ and an element $y_0 \in X$ such that $\|\varphi_0(t) - e^{\lambda t}y_0\| \leq k\varepsilon$ for all $t \in I$. Then $\|\varepsilon\, tx_0 - y_0\| = \|\varphi_0(t) - e^{\lambda t}y_0\| \leq k\varepsilon$. Hence $|t| \leq k + \|y_0\|/\varepsilon$ for all $t \in I$. This contradicts $\delta(I) = \infty$. □

Remark 2.1. Suppose that $\mathrm{Re}\,\lambda \neq 0$ and $m(I,\lambda) = 0$. Then the constant $|\mathrm{Re}\,\lambda|^{-1}\{1 - m(I,\lambda)/M(I,\lambda)\} = |\mathrm{Re}\,\lambda|^{-1}$ in (i) of Theorem 2.1 is best possible. To see this, let $x_0 \in X$ with $\|x_0\| = 1$. We define $\varphi(t) = \varepsilon\,(\mathrm{Re}\,\lambda)^{-1}e^{i\,\mathrm{Im}\,\lambda t}x_0$ for each $t \in I$. Then

$$\|\varphi'(t) - \lambda\varphi(t)\| = \varepsilon \left| \frac{i\mathrm{Im}\,\lambda - \lambda}{\mathrm{Re}\,\lambda} \right| = \varepsilon, \quad (t \in I).$$

By Theorem 2.1, we can find a unique $x_\varphi \in X$ such that $\|\varphi(t) - e^{\lambda t}x_\varphi\| \leq |\mathrm{Re}\,\lambda|^{-1}\varepsilon$ for all $t \in I$. Then we have

$$\|x_\varphi\| \leq |e^{-\lambda t}| \left(\frac{1}{|\mathrm{Re}\,\lambda|}\varepsilon + \|\varphi(t)\| \right) = e^{-\mathrm{Re}\,\lambda t}\frac{2\varepsilon}{|\mathrm{Re}\,\lambda|}$$

for all $t \in I$. This implies $x_\varphi = 0$. Since x_φ is unique, it follows that $|\mathrm{Re}\,\lambda|^{-1}$ is no greater than any Hyers-Ulam stability constant.

REMARK 2.2. Suppose that $\operatorname{Re}\lambda \neq 0$ and $m(I,\lambda) > 0$. Then the uniqueness of $x_\varphi \in X$ with the property $\sup_{t\in I} \|\varphi(t) - e^{\lambda t}x_\varphi\| < \infty$ need not be true. Indeed, let $x_0 \in X$. Put $\varphi(t) = e^{\lambda t}x_0$ for each $t \in I$. For every $x \in X$ with

$$\|x_0 - x\| \le \frac{m(I,\lambda)\varepsilon}{|\operatorname{Re}\lambda|}\left(1 - \frac{m(I,\lambda)}{M(I,\lambda)}\right),$$

we have the following inequality.

$$\|\varphi(t) - e^{\lambda t}x\| = e^{\operatorname{Re}\lambda t}\|x_0 - x\| \le \frac{\varepsilon}{|\operatorname{Re}\lambda|}\left(1 - \frac{m(I,\lambda)}{M(I,\lambda)}\right).$$

REMARK 2.3. By using the Bochner integral, we can give another simple proof of Theorem 2.1 with an explicit formula for x_φ. Put $\psi(t) = \varphi'(t) - \lambda\varphi(t)$ for each $t \in I$. Then $\psi(t)$ is locally Bochner integrable. In fact, firstly it is separably valued since so is $\varphi'(t)$ as a derivative of a continuous function, secondly it is weakly measurable, and lastly it is assumed to be bounded (see [5, pp. 130–133]). Fix $a \in I$. Then we obtain the following equality for every $t \in I$, where the integral should be interpreted as a Bochner integral:

$$e^{-\lambda t}\varphi(t) - e^{-\lambda a}\varphi(a) = \int_a^t e^{-\lambda s}\psi(s)ds.$$

This equality can be justified by reducing it to the following scalar equality by considering the composition with an arbitrary $f \in X^*$:

$$e^{-\lambda t}f(\varphi(t)) - e^{-\lambda a}f(\varphi(a)) = \int_a^t e^{-\lambda s}f(\psi(s))ds.$$

(This scalar equality is nothing but the one already verified in the proof of Theorem 2.1.) If $\operatorname{Re}\lambda > 0$ then

$$\int_a^{\sup I} e^{-\lambda s}\psi(s)ds = \lim_{t\nearrow \sup I}\int_a^t e^{-\lambda s}\psi(s)ds$$

exists since $\|e^{-\lambda s}\psi(s)\| \le \varepsilon e^{-\mathrm{Re}\,\lambda s}$. By a simple calculation we have

$$\begin{aligned}
&\left\|\varphi(t) - e^{\lambda t}\Big\{e^{-\lambda a}\varphi(a) + \int_a^{\sup I} e^{-\lambda s}\psi(s)ds\Big\}\right\| \\
&= \left\|e^{\lambda t}\int_t^{\sup I} e^{-\lambda s}\psi(s)ds\right\| \\
&\le e^{\mathrm{Re}\,\lambda t}\int_t^{\sup I} e^{-\mathrm{Re}\,\lambda s}\|\psi(s)\|ds \\
&\le e^{\mathrm{Re}\,\lambda t}\frac{\varepsilon}{\mathrm{Re}\,\lambda}(e^{-\mathrm{Re}\,\lambda t} - e^{-\mathrm{Re}\,\lambda \sup I}) \\
&\le \frac{\varepsilon}{\mathrm{Re}\,\lambda}\Big(1 - \frac{m(I,\lambda)}{M(I,\lambda)}\Big)
\end{aligned}$$

for every $t \in I$, thereby we obtain Theorem 2.1 with an explicit formula for x_φ. If $\mathrm{Re}\,\lambda < 0$, then we can prove the Hyers-Ulam stability in a way similar to the above. If $\mathrm{Re}\,\lambda = 0$ then we have

$$\begin{aligned}
&\left\|\varphi(t) - e^{\lambda t}\Big\{e^{-\lambda a}\varphi(a) + \int_a^{u} e^{-\lambda s}\psi(s)ds\Big\}\right\| \\
&= \left\|e^{\lambda t}\int_t^{u} e^{-\lambda s}\psi(s)ds\right\| \\
&\le \varepsilon|t-u| \\
&\le \varepsilon\delta(I)
\end{aligned}$$

for every $t, u \in I$.

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