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Numerical analysis of quantization-based optimization

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Abstract

We propose a number-theory-based quantized mathematical optimization scheme for various NP-hard and similar problems. Conventional global optimization schemes, such as simulated and quantum annealing, assume stochastic properties that require multiple attempts. Although our quantization-based optimization proposal also depends on stochastic features (i.e., the white-noise hypothesis), it provides a more reliable optimization performance. Our numerical analysis equates quantization-based optimization to quantum annealing, and its quantization property effectively provides global optimization by decreasing the measure of the level sets associated with the objective function. Consequently, the proposed combinatorial optimization method allows the removal of the acceptance probability used in conventional heuristic algorithms to provide a more effective optimization. Numerical experiments show that the proposed algorithm determines the global optimum in less operational time than conventional schemes.

KEYWORDS

combinatorial optimization, global optimization, number theory, quantization, quantum annealing

1 | INTRODUCTION

Developing an effective combinatorial optimization-algorithm for solving NP-hard and similar problems is a crucial prerequisite for using machine learning to overcome engineering challenges [1–3]. The classic Markov chain Monte Carlo (MCMC) algorithm developed in the 1950s is an early algorithmic approach [4–8]. Simulated annealing (SA) is an advanced MCMC method developed in the 1980s for thermodynamic challenges [9, 10]. SA assumes stochastic properties when seeking global convergence [11–14]. Such stochastic analyses have inspired the development of new heuristic combinatorial optimization algorithms (e.g., genetic and evolutionary [15, 16] and particle swarm types [17, 18]; analysis of the particle swarm algorithm can be found in [19, 20]). Another approach for solving an NP-hard problem is an optimization technique based on a numerical differential evolutionary algorithm. Recent research on such optimization techniques involves the transformation of a comprehensive multimodal objective function with multivariable parameters into a single-objective optimization problem using fuzzy processing. Following this transformation, we can obtain a feasible solution using differential evolutionary optimization for a simplified problem [21, 22].

From the viewpoint of stochastic optimization, stochastic features require related search algorithms to be written as Langevine stochastic equations with independent drift and diffusion terms. The transition probabilities of these search algorithms converge to Gibbs or

Boltzmann distributions under appropriate conditions. Hence, based on Laplace's theorem [13], their global optimization convergence performance is weak owing to the unnecessary divergence permitted in the early stages of the search [12, 14, 23]. Recently, researchers exploring global optimization in modern artificial intelligence rediscovered Laplace's theorem [17, 20]. Therefore, the acceptance probability given by weak convergence can be excessively high. Such divergence is still a powerful way to avoid local minima; however, it drastically slows convergence and reduces optimization performance.

Many researchers have proposed schemes to improve convergence speed, but their global convergence performance has never been proven [24–28]. Therefore, this study applies number and measure theories to analyze the quantized optimization algorithm. Traditional optimization analyses have rarely exploited number theory, leading quantization-based optimization to remain out of reach [29].

Analyzing the effects of quantization based on number theory provides an intuitive numerical interpretation from which exciting results can be obtained based on the quantization of the objective function. In other words, the size of the objective function's level set can be redefined based on measurement theory optimization: the generalization and formalization of geometrical measures.

From these foundations, we propose an effective optimization algorithm to improve performance and speed. Additionally, we confirm the validity of the quantized-based optimization algorithm using the simulation results of a continuous multimodal function and the traveling salesman problem (TSP) involving more than 100 cities.

2 | PRELIMINARIES

2.1 | Quantization definitions

Before describing the proposed algorithm, we establish the following definitions and assumptions.

Definition 1. For $f \in \mathbf{R}$, we define the quantization of f as

$$f^Q \triangleq \frac{1}{Q_p}\left\lfloor Q_p \cdot \left(f + 0.5 \cdot Q_p^{-1}\right)\right\rfloor = f + \varepsilon Q_p^{-1}, \tag{1}$$

where $\lfloor f \rfloor \in \mathbf{Z}$ denotes the floor function such that $\lfloor f \rfloor \le f$ for all $f \in \mathbf{R}$, $Q_p \in \mathbf{Q}^+$ denotes the quantization parameter, and $\varepsilon \in \mathbf{R}$ denotes the quantization error.

Definition 2. We define quantization parameter $Q_p \in \mathbf{Q}^+$ as a monotonically increasing function, $Q_p : \mathbf{R}^{++} \mapsto \mathbf{Z}^+$, such that

$$Q_p(t) = \eta \cdot b^{\bar{h}(t)}, \tag{2}$$

where $\eta \in \mathbf{Q}^{++}$ denotes the fixed constant part of the quantization parameter, b represents the base, and $\bar{h} : \mathbf{R}^{++} \mapsto \mathbf{Z}^+$ denotes the power function such that $\bar{h}(t) \uparrow \infty$ as $t \to \infty$.

To discuss the main algorithm, we consider the optimization problem for an objective function, f, such that

$$\text{minimize} \, f : \mathbf{R}^n \mapsto \mathbf{R}^+. \tag{3}$$

Algorithm 1 Blind random search (BRS) with the proposed quantization scheme

given $f(x) \in \mathbf{R}^+$
Initialize $t \leftarrow 0$ and $\bar{h}(0) \leftarrow 0$
Select initial candidate x_0 randomly, and $x_{opt} \leftarrow x_0$
Compute the initial value of $f(x_t)$
Set $b = 2$ and $\eta = b^{-\lfloor \log_b(f(x_0)+1) \rfloor}$
Compute initial quantization parameter $Q_p(t) \leftarrow \eta$
$f_{opt}^Q \leftarrow \frac{1}{Q_p}\lfloor Q_p \cdot (f + 0.5 \cdot Q_p^{-1})\rfloor$
repeat
    $t \leftarrow t + 1$
    Select candidate x_t randomly
    Compute $f(x_t)$
    $f^Q \leftarrow \frac{1}{Q_p}\lfloor Q_p \cdot (f + 0.5 \cdot Q_p^{-1})\rfloor$
    if $f^Q \le f_{opt}^Q$ **then**
        $x_{opt} \leftarrow x_t$
        $\bar{h}(t) \leftarrow \bar{h}(t) + 1$
        $Q_p \leftarrow \eta \cdot b^{\bar{h}(t)}$
        $f_{opt}^Q \leftarrow \frac{1}{Q_p}\lfloor Q_p \cdot (f + 0.5 \cdot Q_p^{-1})\rfloor$
    end if
until Stopping condition is satisfied

In various combinatorial optimization problems, the actual input is considered as $x^r \in [0,1]^m$. Thus, there exists a proper transformation from a binary input to a proper real vector space such that $\mathcal{T} : [0,1]^m \to \mathcal{X} \subseteq \mathbf{R}^n$, where $\mathcal{X}$ represents the virtual domain of the objective function f. Under the transformation assumption, we assume that $f \in C^\infty$ fulfills the Lipschitz continuity as follows:

Assumption 1. For $x_t \in B^o(x^*, \rho)$, there exists a positive value, L, with respect to the scalar field, $f(x) : \mathbf{R}^n \to \mathbf{R}$, such that

$$\|f(x_t) - f(x^*)\| \le L\|x_t - x^*\|, \ \forall t > t_0, \tag{4}$$

where $B^o(x^*, \rho)$ denotes an open ball, $B^o(x^*, \rho) = \{x \| \|x - x^*\| < \rho\}$, for all $\rho \in \mathbf{R}^{++}$, and $x^* \in \mathbf{R}^n$ denotes the global optimum.

2.2 | Primitive algorithm

For the most elementary implementation, we apply the proposed quantization technique to the BRS algorithm. First, as shown in Algorithm 1, we randomly select an input point, x_t, and quantize the value of the objective function, $f(x_t)$, such that $f^Q(x_t)$ with the quantization parameter, $Q_p(t-1)$.

Comparing the quantization values, $f^Q(\overline{x}_{t-1})$ and $f^Q(x_t)$, if $f^Q(\overline{x}_{t-1})$ is greater than or equal to $f^Q(x_t)$, we establish x_t as the optimal value and replace $\overline{x}_t$ with x_t. Following this procedure, we update the quantization parameter, $Q_p(t-1)$, by increasing the power function, $\overline{h}(t)$, defined in (2). We denote this as the requantization process. Because we update the quantization parameter, the quantization value of $f^Q(x_t)$ is requantized using $Q_p(t)$. Consequently, we select another input point as part of the BRS, as shown in Figure 2.

We also propose a simple initialization of the quantization parameter to implement BRS with the proposed scheme. We want the transition probability of the initial state, $\mathbb{P}(x_1|x_0)$, to be maximized such that $\mathbb{P}(x_1|x_0) = 1$. Therefore, the quantization of the other objective function values, $f^Q(x_1)$, $\forall x_1 \ne x_0$, should be lower than the quantization of the initial objective function. For this purpose, we establish the initial quantization parameter, η, as represented by the following theorem.

Theorem 1. *Suppose the initial value of a given objective function,* $f(x_0) \in \mathbf{R}$, *is* $\sup_{x \in \mathbf{R}} f(x)$. *As shown by the proposed algorithm, the probability of transitioning to the next step,* $\mathbb{P}(x_1|x_0)$, *is one when the initial parameter,* $\eta \in \mathbf{Q}^+$, *satisfies the following equation:*

$$\eta = b^{-\lfloor \log_b(f(x_0)+1) \rfloor}, \tag{5}$$

where b represents the base of Definition 2 for $Q_p(t)$.

Proof. Using the proposed algorithm and its assumptions, we establish the following inequality for all $x_1 \ne x_0$:

$$f(x_0) + Q_p^{-1}(0) \ge f(x_1) + Q_p^{-1}(1). \tag{6}$$

From the definition of quantization parameter Q_p, we note that $Q_p(0) = \eta b^0 = \eta$ and $Q_p(1) = \eta b^{-1}$. Thus,

$$f(x_0) + \eta^{-1} \ge f(x_1) + \eta^{-1} b \Rightarrow f(x_0) - f(x_1) \ge \eta^{-1}(b-1). \tag{7}$$

We assume that η is the power of b; that is, $\eta = b^k$, where $k \in \mathbf{Z}^+$. By substituting η with the power, b, we can rewrite (7) such that

$$\begin{aligned} & f(x_0) - f(x_1) \ge b^{-k}(b-1) \Rightarrow \frac{f(x_0) - f(x)}{b-1} \ge b^{-k} \\ & \Rightarrow -\log_b \frac{f(x_0) - f(x_1)}{b-1} \le k \\ & \Rightarrow k \ge \log_b(b-1) - \log_b(f(x_0) - f(x)). \end{aligned} \tag{8}$$

As $\log_b(b-1) \ge 0$ and $\log_b(f(x_0) - f(x_1)) \ge \log_b f(x_0)$ for all $x_1 \ne x_0$, we obtain the following inequality:

$$\begin{aligned} k & \ge \log_b(b-1) - \log_b(f(x_0) - f(x)) \\ & > -1 - \log_b f(x_0) \ge -\lfloor 1 + \log_b f(x_0) \rfloor. \end{aligned} \tag{9}$$

Therefore, because $f(x_0) \in \mathbf{R}$ is $\sup_{x \in \mathbf{R}} f(x)$ and the initial transition probability is one, we can establish the initial value of the quantization parameter, $\eta = Q_p(0)$, as

$$\eta = b^{-\lfloor \log_b(f(x_0)+1) \rfloor}. \tag{10}$$

□

3 | ANALYSIS OF THE PROPOSED ALGORITHM

3.1 | Corresponding to quantum annealing (QA)

The proposed optimization scheme uses a monotonically increasing quantization resolution to provide an embedded QA effect. After updating the optimal point, we increase the quantization parameter to provide an effect equivalent to QA tunneling. For conformation, we set the conceptual Hamiltonian, which is used as the objective function in QA according to [30, 31] and [32], as follows:

$$H(s) = A(s)H_0 + B(s)H_1, \quad H, H_0, \text{ and } H_1 \in \mathbf{R}^+, \tag{11}$$

where $s \in \mathbf{R}[0,1]$ is a scheduling parameter that depends on time $t \in \mathbf{R}[0, t_f]$, H_0 is a predetermined Hamiltonian in which the lowest-energy state is easily determined, and H_1 is a user-input Hamiltonian that is the real objective function to be optimized. In (11), $A(s)$ is a scheduler function of H_0 that depends on s. $A(s)$ is a monotonically decreasing function from $A(0)$ to $A(1)$, and $B(s)$ is the other scheduler function, which monotonically increases from $B(0)$ to $B(1)$. In QA, an optimizer (e.g., SA) finds a feasible solution to the Hamiltonian, (11), instead of a direct optimization [33-35] and [36].

Returning to the quantization scheme, we write the objective function, $f \in \mathbf{R}[0,2)$, defined in (3) as the power series of base b in the quantization parameter, such that

$$f(x) = \sum_{k=0}^{\infty} a_k(x) b^{-k}, \tag{12}$$

where $a_k(x) \in \mathbf{Z}^+$ is the coefficient of the power of b^{-k}, which is less than b. That is, $a_k(x) < b$, $\forall x \in \mathbf{R}$. Intuitively, we can find a local minimum of the quantized objective function with a low-resolution quantization parameter, $Q_p(t)$ and high quantization step, $Q_p(t)^{-1}$. Conversely, it is difficult to find minima with a high quantization resolution because the quantized objective function is asymptotically equal to the original. To prove this intuitive proposition, we establish the supremum of the objective function with the lowest quantization resolution by using the following lemma and theorem (Figure 1):

Lemma 2. *For a given quantization parameter,* $Q_p(n) = \eta \cdot b^n|_{\eta=1}$*, the supremum of the objective function, written as* (12) *at* $n = 0$*, is*

$$\sup_{x \in \mathbf{R}} f^{Q_p(0)}(x) = a_0 + \left\lfloor \frac{b}{b-1} + \frac{1}{2} \right\rfloor. \tag{13}$$

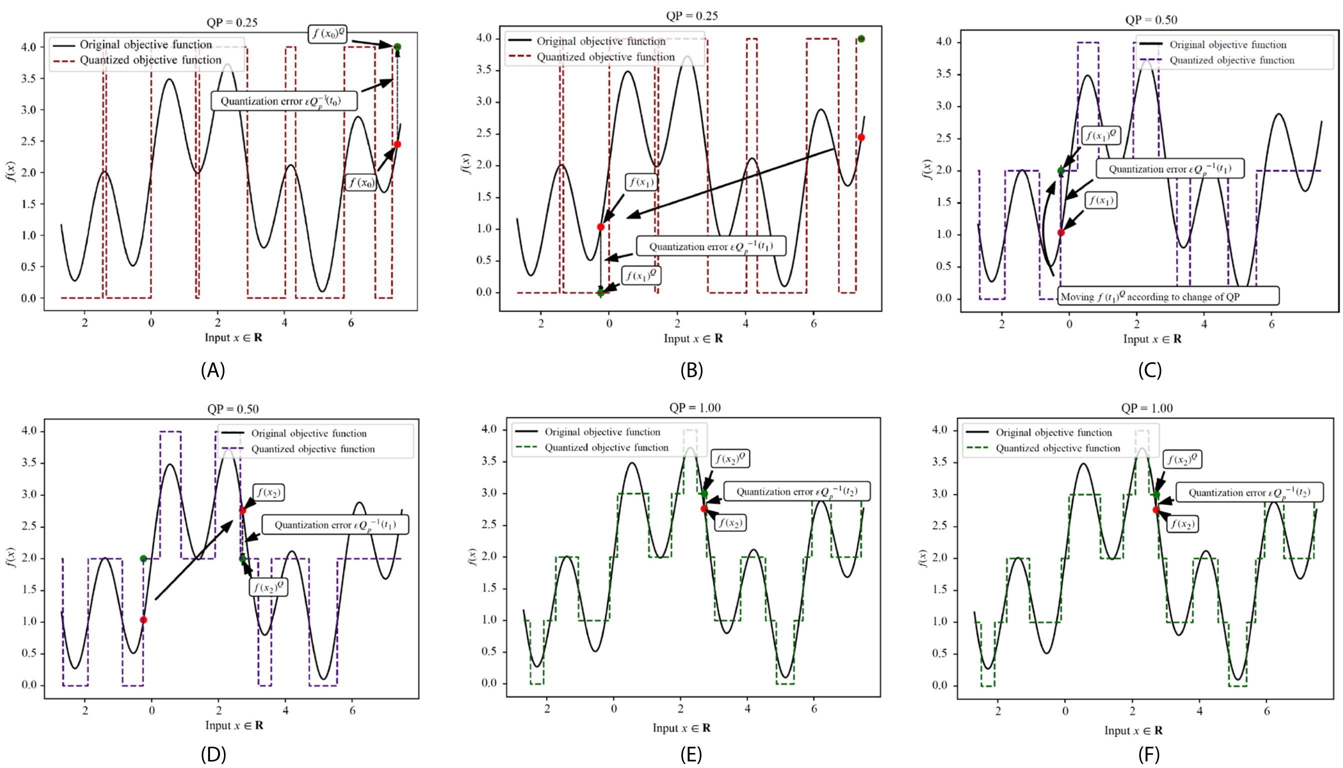


FIGURE 1 Conceptual operation diagram of the proposed algorithm: (A) set initial point and quantization with $Q_p = 0.25$, (B) blind random search similar to quantum annealing, (C) requantization with $Q_p = 0.5$, (D) annealing effect by an equal quantization level, (E) requantization with $Q_p = 1.0$, and (F) find global minima with quantization error. The red point denotes the value of the objective function, $f(x_t)$; the green point denotes the value of the quantized objective function, $f^Q(x_t)$; the solid line represents the curve of the objective function, $f(x_t)$ $\forall x_t \in \mathbf{R}$; and the dashed line represents the quantized value of the objective function, $f^Q(x_t) \forall x_t \in \mathbf{R}$, coinciding with $Q_p(t)$.

Proof. Let $A(s)=s, B(s)=1-s$. Thus, we obtain the following Hamiltonian for QA:

$$H(s)=sH_0+(1-s)H_1,\ s\in[0,1],\ H,H_0,H_1\in\mathbf{R}^+. \quad (14)$$

Suppose that $H=\sum_{k=0}^{\infty}a_k b^{-k}$ and $0<H\le 2$, where $b>a$, $\forall a,b\in\mathbf{N}$. Using the power series for H, we can rewrite Hamiltonian (14) such that

$$H=a_0+\sum_{k=1}^{\infty}a_k b^{-k}=\sum_{k=0}^{n-1}a_k b^{-k}+\sum_{k=n}^{\infty}a_k b^{-k}. \quad (15)$$

According to Definition 1 (i.e., quantization), we quantize (15) using $Q_p(0)=b^{-n}|_{n=0}=1$ as follows:

$$\begin{aligned} H^{Q_p(0)}(s) &= \left\lfloor a_0+\sum_{k=1}^{\infty}a_k b^{-k}+\frac{1}{2}Q_p^{-1}|_{Q_p=1}\right\rfloor \\ &< a_0+\left\lfloor \sum_{k=0}^{\infty}b\cdot b^{-k}+\frac{1}{2}\right\rfloor = a_0+\left\lfloor \sum_{k=1}^{\infty}b^{-k}+\frac{1}{2}\right\rfloor \\ &= a_0+\left\lfloor \frac{b}{b-1}+\frac{1}{2}\right\rfloor. \end{aligned} \quad (16)$$

Thus, (16) fulfills the lemma. □

Theorem 3. *For $n>0$, $n\in\mathbf{N}$, let a finite-order power series exist such that $\tilde{f}^{Q_p(n)}=\sum_{k=0}^{n}a_k b^{-k}$. We denote the supremum of the quantized objective functions as $\sup_{x\in\mathbf{R}}f^{Q_p(n)}(x)\triangleq\overline{f}^{s}Q_p(n)(x)$. Then, we obtain the following equation with respect to quantization order n:*

$$\sup_{x\in\mathbf{R}}f^{Q_p(n)}(x)=(a_0+\gamma)\cdot b^{-n}+\tilde{f}^{Q_p(n)}, \quad (17)$$

where $\gamma\in[1,2]$ is the maximum integer given by the round-off error of b such that $\gamma\triangleq\lfloor b/(b-1)+1/2\rfloor$.

Theorem 3 indicates that as quantization order n increases, the supremum to $f^{Q_p(n)}(x)$ decreases to zero. This property leads $\tilde{f}^{Q_p(n)}(x)$ to converge to the objective function, $f(x)$, asymptotically. Note that the increase in quantization resolution is equivalent to the tunneling effect in QA.

Proof. Let quantization parameter $Q_P(n)=\eta\cdot b^n|_{\eta=1}=b^n$. As shown in Definition 1, we set the quantization of the Hamiltonian as follows:

$$H^{Q_p(n)}=\frac{1}{Q_p}\left\lfloor Q_P\cdot\left(H+\frac{1}{2}Q_p^{-1}\right)\right\rfloor. \quad (18)$$

By expanding (18) to $Q_p(n)=b^n$, we obtain

$$\begin{aligned} H^{Q_p(n)} &= b^{-n}\left\lfloor b^n\cdot\left(\sum_{k=0}^{\infty}a_k b^{-k}+\frac{1}{2}b^{-n}\right)\right\rfloor \\ &= b^{-n}\left\lfloor \sum_{k=0}^{n}a_k b^{n-k}+\sum_{k=n+1}^{\infty}a_k b^{n-k}+\frac{1}{2}\right\rfloor \\ &= b^{-n}\ \sum_{k=0}^{n}a_k b^{n-k}+\left\lfloor \sum_{k=n+1}^{\infty}a_k b^{n-k}+\frac{1}{2}\right\rfloor\Bigg). \end{aligned} \quad (19)$$

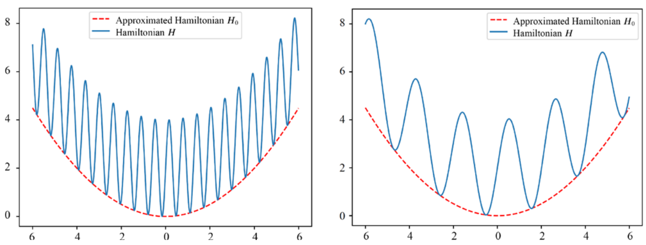
Because $a_k\le b^{n-k}$ for all $k\in\mathbf{Z}[n+1,\infty)$, we can rewrite (19) as

$$\begin{aligned} H^{Q_p(n)} &< b^{-n}\left(\sum_{k=0}^{n}a_k b^{n-k}+\left\lfloor \sum_{k=n+1}^{\infty}b^{n+1-k}+\frac{1}{2}\right\rfloor\right) \\ &= \sum_{k=0}^{n}a_k b^{-k}+b^{-n}\left\lfloor \sum_{k=0}^{\infty}b^{-k}+\frac{1}{2}\right\rfloor \\ &= \sum_{k=0}^{n}a_k b^{-k}+b^{-n}\left\lfloor \frac{b}{b-1}+\frac{1}{2}\right\rfloor. \end{aligned} \quad (20)$$

From the assumptions in Theorem 3, we note the definitions of γ and $\overline{H}^{Q_p(n)}=\sup H^{Q_p(n)}$. We also let $\tilde{H}^{Q_p(n)}=\sum_{k=0}^{n}a_k b^{-k}$ without loss of generality. With these definitions and (19), we induce the supremum of $H^{Q_p(n)}$ for each $n\in\mathbb{N}$ such that

$$\begin{aligned} \overline{H}^{Q_p(0)} &= a_0+\gamma \\ \overline{H}^{Q_p(1)} &= a_0+a_1 b^{-1}+\gamma\cdot b^{-1}=a_0+(a_1+\gamma)\cdot b^{-1} \\ \overline{H}^{Q_p(2)} &= a_0+a_1 b^{-1}+a_2 b^{-2}+\gamma\cdot b^{-2} \\ &= \sum_{k=0}^{1}a_k b^{-k}+(a_2+\gamma)b^{-2}=\tilde{H}^{Q_p(1)}+(a_2+\gamma)b^{-2} \\ &\dots \\ \overline{H}^{Q_p(n)} &= \sum_{k=0}^{n-1}a_k b^{-k}+(a_n+\gamma)b^{-n}=\tilde{H}^{Q_p(n-1)}+(a_n+\gamma)b^{-n}. \end{aligned} \quad (21)$$

For an arbitrary $n>n_0, n, n_0\in\mathbf{N}$, we assume that $a_n=a_0$ without loss of generality. Therefore, we rewrite (21) as



(A) (B)

FIGURE 2 Parabolic washboard potential functions given by [33, 35] with the bandwidth parameter (A) $\alpha = 10.0$ and (B) $\alpha = 0.3$. The bold line represents the H_1 Hamiltonian, and the red line represents the H_0 Hamiltonian.

$$\begin{aligned}\overline{H}^{Q_p(n)} &= \tilde{H}^{Q_p(n-1)} + (a_n+\gamma)b^{-n} = \tilde{H}^{Q_p(n-1)} + (a_0+\gamma)b^{-n} \\ &= \tilde{H}^{Q_p(n-1)} + \overline{H}^{Q_p(0)} b^{-n} = \overline{H}^{Q_p(0)} b^{-n} + \tilde{H}^{Q_p(n-1)}.\end{aligned} \tag{22}$$

Given $\sup_{x \in \mathbf{R}} f^{Q_p(n)}(x)$ and $\tilde{f}^{Q_p(n)}$, we note that Equation (22) is equivalent to Equation (17), thereby proving the theorem. □

Moreover, when we set a Hamiltonian such that $H(s) = A(s)H_0 + B(s)H_1$ for QA with the parameter s increasing from zero to one, $A(s)$ monotonically decreases to zero (Figure 2). Thus, we obtain the following equivalence:

$$\begin{aligned} H(s) &= \overline{H}^{Q_p(n)},\ H_0 = \overline{H}^{Q_p(0)}.\ A_0 = b^{-n} = Q_p(n) \\ B(s)H_1 &= \tilde{H}^{Q_p(n-1)} = \sum_{k=0}^{n-1} a_k b^{-k}. \end{aligned} \tag{23}$$

Because the parameter s increases from zero to one as the time index increases from zero to ∞, the proposed quantization scheme satisfies the property of the Hamiltonian in QA. This equivalence indicates that the proposed algorithm provides a global optimum [37]. In the next section, we analyze this property from the viewpoint of convergence.

3.2 | Brief analysis of the convergence property for the proposed algorithm

We can establish a stochastic differential equation (SDE) for the proposed algorithm, as follows [37]:

$$d\boldsymbol{X}_s = -\nabla f(\boldsymbol{X}_s)ds + \sqrt{C_q} Q_p^{-1}(s) d\boldsymbol{W}_s,\ s \in \mathbf{R}(t, t+1). \tag{24}$$

where $\boldsymbol{X} \in R^n$ denotes a random variable corresponding to the input parameter for the L-Lipschitz continuous objective function $f: \mathbf{R}^n \to \mathbf{R}^+$, $\boldsymbol{W}_s$ denotes an i.i.d. standard Wiener process with zero mean and variance of one, and C_q is a constant variable corresponding to the quantization parameter $Q_p(s)$. In addition, we establish another standard Wiener process $\{\tilde{\boldsymbol{X}}_s\}$ as follows:

$$d\tilde{\boldsymbol{X}}_s = \sqrt{C_q} Q_p^{-1}(s) d\boldsymbol{W}_s,\ s \in \mathbf{R}(t, t+1). \tag{25}$$

To calculate the transition probability for an arbitrary value, we introduce the following Girsanov theorem [38, 39].

$$\begin{aligned}\frac{dP_x}{dS_x} &= \exp\left(-\int_t^{t+1} \frac{C_q^{-1}\nabla f(X_s)}{Q_p^{-2}(s)} d\tilde{X}_s \right. \\ &\quad \left. -\frac{1}{2}\int_t^{t+1} \frac{C_q^{-1}\|\nabla f(X_s)\|^2}{Q_p^{-2}(s)} ds \right),\end{aligned} \tag{26}$$

where P_x denotes the probability density for $\boldsymbol{X}_s$ and S_x denotes the probability density for $\tilde{\boldsymbol{X}}_s$. According to Assumption 1 and $f \in C$, we can suppose that there exists a positive value $\tilde{L}$ such that $\|\nabla f(x) - \nabla f(x^*)\| \le \tilde{L}\|x - x^*\|$ for $x \in \mathbf{R}^n$ and the global optimum $x^* \in \mathbf{R}^n$ satisfying $\nabla f(x^*) = 0$. Hence, by making these assumptions, we can calculate the bound of $\nabla f(x)$ for $\mathbf{R}[t, t+1]$ as follows.

$$\|\nabla f(x_s)\| = \|\nabla f(x_s) - \nabla f(x^*)\| < \tilde{L}\|x_s - x^*\| \le \tilde{L}\rho. \tag{27}$$

Using the upper bound of $\|\nabla f(x_s)\|$, we can obtain the upper bound of the first term in (26), such that

$$\begin{aligned}&\left\| \int_t^{t+1} \frac{C_q^{-1}}{Q_p^{-2}(s)} \nabla f(X_s) d\overline{X}_s \right\| \le \int_t^{t+1} \frac{C_q^{-1}}{Q_p^{-2}(s)} \|\nabla f(X_s)\| d\overline{\boldsymbol{X}}_s \\ &\le \frac{\sqrt{C_q^{-1}}}{Q_p^{-1}(s)} \tilde{L}\rho \left\| W_t - \frac{1}{2} \right\| \le \frac{\sqrt{C_q^{-1}}}{Q_p^{-1}(s)} \tilde{L}\rho(\rho+1) \le \frac{C_1}{Q_p^{-1}(s)},\end{aligned} \tag{28}$$

and second term

$$\frac{1}{2}\int_t^{t+1} \frac{C_q^{-1}}{Q_p^{-2}(s)} \|\nabla f(X_s)\|^2 ds \le \frac{C_q^{-1} L\rho}{2Q_p^{-2}(s)} \le \frac{C_2}{2Q_p^{-2}(s)}, \tag{29}$$

where $C_1 > 0$ denotes a constant value such that $C_1 > \sqrt{C_q^{-1}\tilde{L}\rho(\rho+1)}$ and the constant value $C_2 > 0$ satisfies $C_2 > C_q^{-1}L\rho$. Substituting (28) and (29) into (26), we can obtain the lower bound of the Radon–Nykodym derivation:

$$\frac{dP_x}{dS_x} \geq \exp\left(-\frac{1}{Q_p^{-1}(s)}\left(C_1 + \frac{C_2}{2Q_p^{-1}(s)}\right)\right) \geq \exp\left(-\frac{C_3}{Q_p^{-1}(s)}\right), \tag{30}$$

where $C_3 > 2\sigma(0)C_2 + C_1$. Therefore, for any $\rho > 0$ and $x_t, x^* \in \mathbf{R}^n$, we note that (30) yields the infimum of $P_x(|\boldsymbol{X}_t - x^*| < \varepsilon)$ as follows:

$$P_x(|\boldsymbol{X}_t - x^*| < \rho) \geq \exp\left(-\frac{C_3}{Q_p^{-1}(t)}\right) S_x(|\boldsymbol{X}_t - x^*| < \rho), \tag{31}$$

where S_x denotes a Gaussian distribution obtained using the standard Wiener process, $\boldsymbol{W}_t$. To prove convergence in the distribution using Laplace's method [11, 13], we establish the infimum of the transition probability from t to $t+1$ such that

$$\inf_{x,y \in \mathbf{R}^n} p(t, x_t, t+1, x^*) = \inf_{x,y \in \mathbf{R}^n} \lim_{\rho \to 0} \frac{1}{\rho} P_x(|X_t - x^*| < \rho) \geq \inf_{x,y \in \mathbf{R}^n} \lim_{\rho \to 0} \frac{1}{\rho} \cdot \exp\left(-\frac{C_3}{Q_p^{-1}(t)}\right) S_x(|X_t - x^*| < \rho). \tag{32}$$

As S_x is the probability distribution derived by the standard Wiener process $\boldsymbol{W}_t$, we can obtain the infimum of the transition probability $p(t, x_t, t+1, x^*)$ as follows [29, 37]:

$$S_x(|X_t - x^*| < \rho) \geq \rho \cdot Q_p(0)\sqrt{2/\pi C_q} = \rho \cdot C_4. \tag{33}$$

By substituting (35) into (34), we obtain

$$\inf_{x,y \in \mathbf{R}^n} p(t, x_t, t+1, x^*) \geq \inf_{x,y \in \mathbf{R}^n} \lim_{\rho \to 0} \frac{1}{\rho} \cdot \exp\left(-\frac{C_3}{Q_p^{-1}(t)}\right) \rho \cdot C_4 = C_4 \cdot \exp\left(-C_3 Q_p(t)\right). \tag{34}$$

The convergence in the distribution means that if we start the algorithm at any point, the transition probability converges in the sense of a Cauchy sequence such that

$$\lim_{t \to \infty} \sup_{v,w \in \mathbf{R}^n} |p(s, v, t, x^*) - p(s, w, t, x^*)| = 0. \tag{35}$$

For convenience, we define a transition probability from t to $t+1$ such that $\delta_t \triangleq \inf_{x,y \in \mathbf{R}^n} p(t, x, t+1, y)$. From the lemma of the difference between the transition probabilities based on the Kolmogorov equality in [29, 37], we note that

$$\lim_{t \to \infty} \sup_{v,w \in \mathbf{R}^n} |p(s, v, t, x^*) - p(s, w, t, x^*)| \leq \prod_{k=0}^{\infty} (1 - \delta_{t+k}). \tag{36}$$

Because $(1-a) \leq exp(-a)$, $\forall a \in \mathbf{R}$, we can rewrite the final term of (36) such that

$$\prod_{k=0}^{\infty} (1 - \delta_{t+k}) \leq \prod_{k=1}^{\infty} \exp(-\delta_{t+k}) = \exp\left(-\sum_{k=0}^{\infty} \delta_{t+k}\right). \tag{37}$$

Equation (37) implies that, if the inside term in the exponent increases to infinity, that is, $\sum_{k=0}^{\infty} \delta_{t+k} \uparrow \infty$, the transition probability converges with respect to the distribution. Assume that there exists an upper bound for $Q_p(t)$ such that $Q_p(t) = \eta \cdot b^{\overline{h}(t)} \leq C_3^{-1} \log(t+a)$ provided by the algorithm, where a denotes an arbitrary value $a \in \mathbf{R}^+$. Under this assumption, by calculating the final term in (37) using (34), we obtain

$$\sum_{k=0}^{\infty} \delta_{t+k} \geq \sum_{k=0}^{\infty} C_4 \cdot \exp(-C_3 C_3^{-1} \log(t+a)) = \sum_{k=0}^{\infty} C_4/(t+a). \tag{38}$$

Because the final term in (38) increases to infinity as t increases, we can calculate the limit value of (36) such that

$$\lim_{t \to \infty} \sup_{v,w \in \mathbf{R}^n} |p(s, v, t, x^*) - p(s, w, t, x^*)| = 0. \tag{39}$$

This indicates that the proposed algorithm converges in the distribution as t increases to infinity.

4 | SIMULATION RESULTS

4.1 | Standard continuous test functions

To confirm the similarities between QA and the proposed quantization scheme, we tested the algorithms on the optimization performance of the parabolic washboard potential function given by

$$f(x) = 0.125x^2 + 2\sin(\alpha x) + 2, \ \forall x \in \mathbf{R}, \tag{40}$$

where $\alpha \in \mathbf{R}$ is a tunneling band parameter. A small α denotes a wider band for the tunneling effect, whereas a large value indicates a narrow band. For the QA algorithm, we set $0.125x^2$ as the primitive Hamiltonian, H_0, for $s = 0$ and $f(x)$ as H_1 for $s = 1$. The performance was tested using a traditional scheduling function (e.g., $s = t/t_f$), where t_f is the final time (Figure 3).

The function (40) in [33, 35] was used as the benchmark function to verify the superiority of the QA algorithm in terms of the optimization performance. In this simulation, we set t_f 1000. We set the stop condition as the optimization error, $f(x) - f(x^*)$, becoming less than or equal to the quantization error, $Q_p(n) = 1/4096 = 2^{-12}$, and applied the stop condition equally to all three testing algorithms.

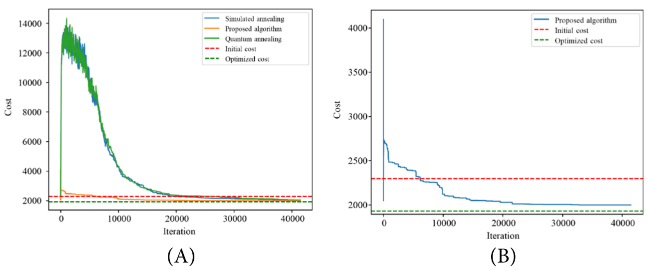


FIGURE 3 (A) Error trends of each algorithm in the TSP experiment with 100 cities. (B) The error trend shows that the proposed algorithm results in lower overshoot, fast searching speed, and better optimization performance with stable convergence properties.

Contrary to expectations, all the tested algorithms showed adequate optimization performance for the objective function with a narrow band such that $\alpha = 10$. However, for the wideband function, the improvement ratio of the QA algorithm significantly decreased compared with that of the narrowband case. This result indicates that the QA algorithm often fails to find the global minimum with a wide band, whereas the SA and proposed algorithms do so successfully.

Table 1 lists the simulation results for the parabolic washboard function with narrow ($\alpha = 10.0$) and wide ($\alpha = 3.0$) bandwidths. Furthermore, we conducted additional experiments on well-known continuous benchmark functions, such as Xin-She Yang N4, Salomon, Drop-Wave, and Shaffel N2. Because all additional test functions are continuous, simulated annealing, QA, and the proposed quantization-based optimization are able to determine the global minima within a finite number of iterations, except for the experiments for Xin-She-Yang N4 function with QA.

As shown in Table 2, the proposed quantization-based optimization performs better than the simulated and QA methods with fewer search iterations.

4.2 | TSP

We solved the simulated TSP problem for 100 cities in a two-dimensional square space with a $[0, 200]$ range. We used the 2-OPT local search algorithm, which is the standard city selection method for cost evaluation [2].

For QA, we set the primitive Hamiltonian, H_0, as the initial route led by the nearest-neighbor algorithm [1]. In all attempts, we used fixed city locations to guarantee generality. We set the number of iterations to 10 000 for each attempt and simulated 100 attempts to obtain average results for comparison. The results listed in Table 3 indicate that the average optimization performance of the proposed algorithm is superior to those of the classical and QA algorithms.

We also evaluated the optimization performance of the proposed algorithm for the TSP under severe and

TABLE 1 Simulation results of the parabolic washboard potential function.

	Narrow band α = 10.0			Wide band α = 3.0		
Criterion	**SA**	**Quantum annealing**	**Proposed**	**SA**	**Quantum annealing**	**Proposed**
Average minimum cost	0.0031	0.0031	0.0036	0.034	0.216	0.034
Improvement ratio to the initial setting	99.93	99.93	99.92	97.75	85.73	97.75

TABLE 2 Simulation results of standard nonlinear optimization functions.

Function	Equation	Criterion	SA	QA	Proposed
Xin-She Yang N4	$f(x)=2.0+\left(\sum_{i=1}^{d}\sin^2(x_i)-\exp\left(-\sum_{i=1}^{d}x_i^2\right)\exp\left(-\sum_{i=1}^{d}\sin^2\sqrt{\lvert x_i\rvert}\right)\right)$	Iteration	6420	17*	3144
		Improvement ratio	54.57%	35.22%	54.57%
Salomon	$f(x)=1-\cos\left(2\pi\sqrt{\sum_{i=1}^{d}x_i^2}\right)+0.1\sqrt{\sum_{i=1}^{d}x_i^2}$	Iteration	1312	7092	1727
		Improvement ratio	99.99%	99.99%	100.0%
Drop-Wave	$f(x)=1-\frac{1=\cos\left(12+\sqrt{x^2+y^2}\right)}{0.5(x^2+y^2)+2}$	Iteration	907	3311	254
		Improvement ratio	100.0%	100.0%	100.0%
Shaffel N2	$0.5+\frac{\sin^2(x^2-y^2)-0.5}{(1+0.001(x^2+y^2))^2}$	Iteration	7609	9657	2073
		Improvement ratio	100.0%	100.0%	100.0%

TABLE 3 Simulation results of the traveling salesman problem with 100 cities.

Criterion	SA	Quantum annealing	Proposed
Average minimum cost	1729.50	1721.07	1648.26
Improvement ratio to the initial setting	19.90%	20.29%	23.67%

TABLE 4 Simulation results of the traveling salesman problem with more than 100 cities.

Number of cities	Nearest neighbor (initial)	Simulated annealing	Quantum annealing	Proposed algorithm	Improvement ratio
100	2159.27	1729.50	1721.07	1648.26	23.67
125	2297.86	2027.52	2028.20	1923.65	16.28
150	2497.65	2255.15	2252.82	2032.21	18.63
175	2380.52	2380.52	2380.29	2147.17	9.80
200	2769.73	2769.34	2769.42	2366.72	14.55

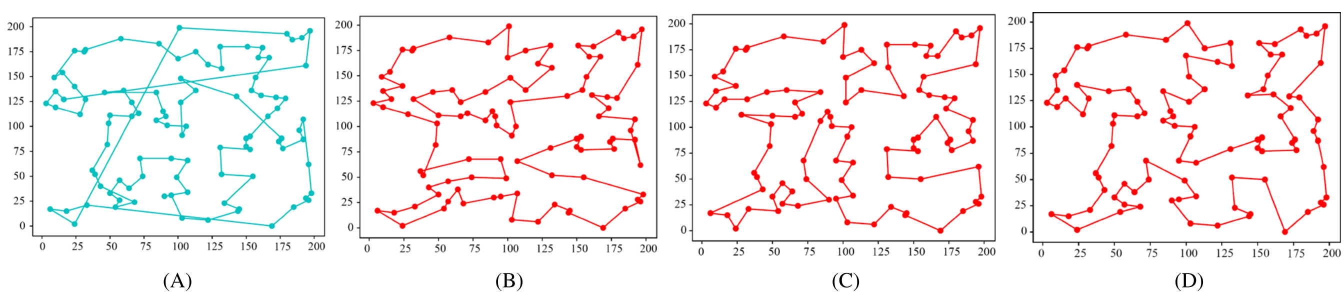


FIGURE 4 Comparison of traveling salesman problem routes generated by each optimization algorithm: (A) Initial path given by the nearest neighborhood algorithm ($cost = 2159$), final path given by the simulated annealing algorithm (B) $minimumcost = 1731$ and (C) $minimumcost = 1706$, (D) final path given by the quantization-based optimization algorithm ($minimumcost = 1636$).

challenging conditions. In most cases, the 100-city TSP is used as the benchmark. However, the TSP difficulty increases dramatically beyond 100 cities; it increases by approximately 100 times for each additional city. This drastic increase often causes combinatorial optimization failure.

In our simulation, conventional SA and QA could not find feasible solutions because they became stuck at unreasonable points at which the values of the objective functions were significantly higher. However, the list in Table 4 shows that the proposed algorithm can find feasible TSP solutions even when the number of cities exceeds 100, which conventional algorithms cannot achieve. Moreover, our proposed algorithm can find feasible solutions for 200 cities, whereas conventional algorithms cannot operate in such cases and become stuck at the initial point. Therefore, the proposed algorithm outperforms conventional algorithms such as SA and QA in terms of optimization performance.

5 | CONCLUSION

We presented a quantization-based optimization scheme with an increased quantization resolution to globally optimize an objective function. The tunneling and hill-climbing occurred at equal quantization levels, enabling the proposed algorithm to search for minima more effectively. We also proved that the quantization error variance was equal to "temperature" in classical SA, which relies on stochastic analyses. The proposed algorithm automatically provides a predetermined objective function for optimization with a low quantization resolution, whereas the user must set this function manually in QA. Notably, the proposed algorithm can be applied to discrete combinatorial and classical continuous optimization problems. In future work, we plan to develop a more effective gradient-based optimization algorithm based on this quantization method, because the proposed algorithm currently applies to any objective function, regardless of its continuity. We expect future work to establish an optimization algorithm with a discrete learning equation, such as the ADAM (ADAptive Momentum) optimizer. We also plan to develop an advanced scheduling function for quantization resolution and combine the proposed scheme with other optimization algorithms to improve overall search performance (Figure 4).

AUTHOR CONTRIBUTIONS

Jinwuk Seok developed the theoretical formalism, performed the analytic calculations, and performed the numerical simulations. Both authors, Jinwuk Seok and Chang Sik Cho, contributed to the final version of the manuscript. Chang Sik Cho supervised the project.

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CONFLICT OF INTEREST STATEMENT

The authors declare that there are no conflicts of interest.

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