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GEOMETRY OF SOME SUBMANIFOLDS IN A 3-DIMENSIONAL WALKER MANIFOLD

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Abstract. In this work, we study the geometric properties of curves and surfaces in a Walker 3-dimensional manifold. We characterize and give a geometric description of normal and binormal surfaces passing through a giving curve. We deduce the geometries of such surfaces from those of the curve.

1. Introduction

The accurate knowledge of submanifolds of a given manifold is of great help while studying the features (geometric and topological properties) of the ambient manifold. The geometry of the Grassmanian of any manifold gives a total description of objects lying in the later. Therefore it is interesting to get a natural and easy read of geometric properties of different level submanifolds of a given manifold. Some of the tools helping study these submanifolds are the geometric evolution equations which provides new ways to address a variety of non-linear problems in Riemannian geometry, and gives numerous applications either in mathematics or in physics. They are divided into classes of intrinsic and extrinsic curvature flows.

The curve flows are studied by many authors. Mohamed [12], studied and gave the general description of the motions of spacelike curves with spacelike normal vector in a 3-dimensional de-Sitter space $S^{2,1}$ and provided some explicit examples of motions of these curves in $S^{2,1}$. Schief and Rogers [15] studied the binormal motion of curves with constant curvatures. Abdel-All et al. [1], constructed and studied new geometrical models of flows of curves and surfaces.

On a Riemannian manifold, the shortest path between any two given points is called a geodesic while the surface (in general submanifold) with smallest area (volume) of any given is called a minimal surface (submanifold). It is well

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known that the mean curvature flow is the greatest flow to assess a minimal submanifold.

In this work the ambient space we will consider is a Lorentzian three-manifold admitting a parallel null vector field called a strict Walker manifold. It is known that Walker metrics have served as a powerful tool of constructing interesting indefinite metrics which exhibit various aspects of geometric properties not given by any positive definite metrics. For more detail on Walker manifold see [3, 6].

The study of differential geometry of surfaces captured many researchers' attention. In [17], Tamura showed that complete surfaces of constant mean curvature in E^3 on which there exist two helical geodesics through each point are planes, spheres or circular cylinders. In [13], the authors construct two special families of ruled surfaces in a three dimensional strict Walker manifold. The local degeneracy (resp. non-degeneracy) to one of this family has a strong consequence on the geometry of the ambient Walker manifold. In [14], the same authors study minimal translations surfaces in a strict Walker 3-manifold. Based on the existence of two isometries, they classify minimal translation surfaces on this class of manifold.

Motivated by the above works, in this paper we study the geometric properties of some type of surfaces in a strict Walker 3-dimensional manifold. We characterize and give a geometric description of normal and binormal surfaces passing through a giving curve.

The paper is organized as follows: In Section 2, we give some preliminaries about the geometry of Walker 3-manifold. In Section 3 we define the geometry of curves in a Walker 3-manifold. In Section 4 we study the geometry of surfaces in a 3-dimensional Walker manifold and in Section 5, we study the normal and the binormal surfaces in a strict Walker 3-manifold.

2. Background material

2.1. Walker structures in 3-dimension

A Walker n-manifold is a pseudo-Riemannian manifold, which admits a field of null parallel r-planes with $r \leqslant \frac{n}{2}$. The canonical forms of the metrics were investigated by Walker [3] who derived adapted coordinates to a parallel plane field. Therefore, a Walker structure is modeled on $\mathbb{R}^n$ with such coordinates. The Walker structure g_f^ϵ (of parameters f and $\epsilon = \pm 1$) of a 3-dimensional manifold M with adapted coordinates (x, y, z) is expressed as

$$g_f^\epsilon = dx \circ dz + \epsilon dy^2 + f(x, y, z)dz^2 \tag{1}$$

and in matrix form as

$$(2)\quad (g_f^\epsilon)=\begin{pmatrix}0&0&1\\0&\epsilon&0\\1&0&f\end{pmatrix}\quad\text{with inverse}\quad (g_f^\epsilon)^{-1}=\begin{pmatrix}-f&0&1\\0&\epsilon&0\\1&0&0\end{pmatrix}$$

for some function $f(x,y,z)$. If the function f depends only on y and z coordinates $\big(f=f(y,z)\big)$ the structures g_f^ϵ is called Strict Walker.

When $\epsilon=1$ $\big($ resp. $\epsilon=-1\big)$ the Walker manifold has signature $(2,1)$ $\big($ resp. $(1,2)$ $\big)$ therefore is Lorentzian in both cases.

2.2. Curvatures of a 3-dimensional Walker manifold

Let M be a manifold equipped with a 3-dimensional Walker structure g_f^ϵ (as in (1)). The Levi-Civita connection of g_f^ϵ is given by:

$$\begin{aligned}\nabla_{\partial_x}\partial z &= \frac{1}{2}f_x\partial_x,\quad \nabla_{\partial_y}\partial z=\frac{1}{2}f_y\partial_x,\\ (3)\qquad \nabla_{\partial_z}\partial z &= \frac{1}{2}(ff_x+f_z)\partial_x-\frac{\epsilon}{2}f_y\partial_y-\frac{1}{2}f_x\partial_z,\end{aligned}$$

where ∂_x, ∂_y and ∂_z are the coordinate vector fields $\frac{\partial}{\partial_x}$, $\frac{\partial}{\partial_y}$ and $\frac{\partial}{\partial_z}$, respectively. Hence, if (M,g_f^ϵ) is a strict Walker manifolds i.e., $f(x,y,z)=f(y,z)$, then the associated Levi-Civita connection satisfies

$$(4)\qquad \nabla_{\partial_y}\partial z=\frac{1}{2}f_y\partial_x,\ \ \nabla_{\partial_z}\partial z=\frac{1}{2}f_z\partial_x-\frac{\epsilon}{2}f_y\partial_y.$$

The non-zero components of the curvature tensor of (M,g_f^ϵ) are given by

$$\begin{aligned}R(\partial_x,\partial_z)\partial_x &= \frac{1}{2}f_{xx}\partial_x,\ \ R(\partial_x,\partial_z)\partial_y=\frac{1}{2}f_{xy}\partial_x,\ \ R(\partial_y,\partial_z)\partial_y=\frac{1}{2}f_{yy}\partial_x,\\ R(\partial_x,\partial_z)\partial_z &= \frac{1}{2}ff_{xx}\partial_x-\frac{\epsilon}{2}ff_{xy}\partial_y-\frac{1}{2}ff_{xx}\partial_z,\ \ R(\partial_y,\partial_z)\partial_x=\frac{1}{2}f_{xy}\partial_x,\\ R(\partial_y,\partial_z)\partial_z &= \frac{1}{2}ff_{xy}\partial_x-\frac{\epsilon}{2}ff_{yy}\partial_y-\frac{1}{2}ff_{xy}\partial_z.\end{aligned}$$

Note that the existence of a null parallel vector field (i.e $f=f(y,z)$) simplifies the non-zero components of the Christoffel symbols and the curvature tensor of the metric g_f^ϵ as follows:

$$\Gamma_{23}^1=\Gamma_{32}^1=\frac{1}{2}f_y\ ,\qquad \Gamma_{33}^1=\frac{1}{2}f_z\ ,\qquad \Gamma_{33}^2=-\frac{\epsilon}{2}f_y$$

and respectively

$$R(\partial_y,\partial_z)\partial_y=\frac{1}{2}ff_{yy}\partial_x,\quad R(\partial_y,\partial_z)\partial_z=-\frac{\epsilon}{2}ff_{yy}\partial_y.$$

Let U and V be two vector fields over (M,g_f^ϵ). The vector (cross) product of U and V in TM with respect to the metric g_f^ϵ is the vector field denoted by

$U \times V$ and defined by

$$g_f^\epsilon(U \times V, W) = \det(U, V, W) \qquad \forall\, W \in TM, \tag{5}$$

where $\det(U, V, W)$ is the determinant function associated to the canonical basis of $\mathbb{R}^3$. So by setting $U = (U^x, U^y, U^z)$ and $V = (V^x, V^y, V^z)$ with respect to the adapted basis $(\partial_x, \partial_y, \partial_z)$ and using (5), we obtain:

$$U \times V = (U^xV^y - U^yV^x - f(U^yV^z - U^zV^y), -\epsilon(U^xV^z - U^zV^x), U^yV^z - U^zV^y)\,.$$

3. Geometry of Curves in 3-dimensional Walker manifold

In this section, we present some of the geometric quantities specifically attached to curves in a 3-dimensional Walker manifold (M, g_f^ϵ) with adapted coordinates (x, y, z) so that the metric has the form (1).

Let I be an open interval in $\mathbb{R}$ and $\gamma : u \in I \longmapsto \gamma(u) \in (M, g_f^\epsilon)$ a unit speed curve in (M, g_f^ϵ). with nowhere vanishing curvature function κ. Recall that the function κ measures the potential of a curve to bend from a straight line in the manifold.

For each $u \in I$, if we write $\gamma(u) = \big(x(u), y(u), z(u)\big)$ we have

$$\gamma'(u) = \frac{d}{du}\gamma(u) = \frac{dx}{du}\frac{\partial}{\partial x} + \frac{dy}{du}\frac{\partial}{\partial y} + \frac{dz}{du}\frac{\partial}{\partial z} = x'(u)\partial_x + y'(u)\partial_y + z'(u)\partial_z.$$

Denoting by T the (unit) tangent vector (field) to γ, the vector (field) $N = \dfrac{1}{\kappa}\dfrac{d}{du}T$ (resp. $B = T \times N$) is the normal (resp. binormal) vector (field) to γ. The family (T, N, B) is the Frenet frame along the curve γ.

In the sequel, the numbers $\varepsilon_1 = g_f^\epsilon(T, T)$; $\varepsilon_2 = g_f^\epsilon(N, N)$ and $\varepsilon_3 = g_f^\epsilon(B, B)$ will denote the causal character of T, N and B respectively and the following hold:

$$\begin{aligned} g_f^\epsilon(T, N) &= g_f^\epsilon(T, B) = g_f^\epsilon(N, B) = 0, \\ T &= \varepsilon_1\, N \times B, \quad N = \varepsilon_2\, B \times T, \quad B = \varepsilon_3\, T \times N. \end{aligned}$$

Definition 3.1.
Let $\gamma : I \longrightarrow (M, g_f^\epsilon)$ be a curve such that $\gamma'(u) \times \gamma(u) \neq 0$ for all $u \in I$. The torsion of γ at $\gamma(u)$ (at time u) is given by:

$$\tau = \frac{\det\big(\gamma'(u), \gamma''(u), \gamma'''(u)\big)}{\|\gamma'(u) \times \gamma''(u)\|}.$$

The evolution of the Frenet frame along the curve γ is given by the Frenet formulas:

$$\left\{\begin{array}{lcl} \nabla_T T &=& \varepsilon_2 \kappa N \\ \nabla_T N &=& -\varepsilon_1 \kappa T + \varepsilon_3 \tau B \\ \nabla_T B &=& -\varepsilon_2 \tau N \end{array}\right. , \tag{6}$$

where κ and τ are respectively the curvature and the torsion of the curve γ. This can be written as:

$$\frac{d}{du}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_2\kappa & 0 \\ -\varepsilon_1\kappa & 0 & \varepsilon_3\tau \\ 0 & -\varepsilon_2\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{7}$$

.

4. Geometry of surfaces in a 3-dimensional Walker manifold

In this section we study the differential geometry of surfaces in a Walker manifold. Let $\mathcal{U}$ be an open subset of the plane $\mathbb{R}^2$ satisfying this interval condition: horizontal or vertical lines intersect $\mathcal{U}$ in intervals (if at all).

A *two-parameter map* is a smooth map $\varphi : \mathcal{U} \longrightarrow M$. Thus φ is composed of two interwoven families of *parameter curves*:

1. the u-parameter curves $v = v_0$ of φ is $u \longmapsto \varphi(u, v_0)$,
2. the v-parameter curves $u = u_0$ of φ is $v \longmapsto \varphi(u_0, v)$.

The partial velocities $\varphi_u = d\varphi(\partial_u)$ and $\varphi_v = d\varphi(\partial_v)$ are vector fields on φ. Evidently $\varphi_u(u_0, v_0)$ is the velocity vector at u_0 of the u-parameter curve $v = v_0$, and symmetrically for $\varphi_v(u_0, v_0)$. If φ lies in the domain of a coordinate system $(x^1, \cdots, x^n)$, then its coordinate functions $x^i \circ \varphi$ $(1 \leqslant i \leqslant n)$ are real-valued functions on $\mathcal{U}$ and

$$\begin{cases} \varphi_u &= \displaystyle\sum_i \frac{\partial x^i}{\partial u}\partial_{x^i} \\ \varphi_v &= \displaystyle\sum_i \frac{\partial x^i}{\partial v}\partial_{x^i}. \end{cases}$$

So far, M could be a smooth manifold; assume it is pseudo-Riemannian. If Z is a smooth vector field on φ, its partial covariant derivatives are:

$Z_u = \nabla_{\frac{\partial}{\partial u}} Z$, the covariant derivative of Z along the u-parameter curves,
$Z_v = \nabla_{\frac{\partial}{\partial v}} Z$, the covariant derivative of Z along the v-parameter curves.

Explicitly, $Z_u(u_0, v_0)$ is the covariant derivative at u_0 of the vector field $u \longmapsto Z(u, v_0)$ on the curve $u \mapsto \varphi(u, v_0)$. In terms of coordinates, $Z = \sum_i Z^i \partial_{x^i}$, where each $Z^i = Z(x^i)$ is a real valued function. Then

$$Z_u = \sum_k \left\{ \frac{\partial Z^k}{\partial u} + \sum_{i,j} \Gamma_{ij}^k Z^i \frac{\partial x^j}{\partial u} \right\} \partial_{x^k}. \tag{8}$$

In the special case $Z = \varphi_u$, the derivative $Z_u = \varphi_{uu}$ gives the accelerations of the u-parameter curves, while φ_{vv} gives the v-parameter accelerations. With coordinate notation as above, we have:

$$\varphi_{uv} = \sum_k \left\{ \frac{\partial^2 x^k}{\partial v \partial u} + \sum_{i,j} \Gamma^k_{ij} \frac{\partial x^i}{\partial u} \frac{\partial x^j}{\partial v} \right\} \partial_{x^k}. \tag{9}$$

Now we will assume that φ is an isometric immersion. The first fundamental form of the immersion φ is given by:

$$\begin{cases} E = g_f^\epsilon\left(\varphi_u, \varphi_u\right) \\ F = g_f^\epsilon\left(\varphi_u, \varphi_v\right) \\ G = g_f^\epsilon\left(\varphi_v, \varphi_v\right). \end{cases} \tag{10}$$

The coefficients of the second fundamental form of φ are

$$\begin{cases} L &= \varepsilon_4 g_f^\epsilon\left(\varphi_{uu}, \mathcal{N}\right) \\ M &= \varepsilon_4 g_f^\epsilon\left(\varphi_{uv}, \mathcal{N}\right) \\ P &= \varepsilon_4 g_f^\epsilon\left(\varphi_{vv}, \mathcal{N}\right) \end{cases}, \tag{11}$$

where $\varepsilon_4 = g_f^\epsilon(\mathcal{N}, \mathcal{N})$ denotes the sign of the unit normal $\mathcal{N}$ along φ.

To end this section, we recall the two most important curvatures functions for submanifolds: the mean curvature and the Gauss curvature. The mean curvature is given by

$$H = \frac{\varepsilon_4}{2}\Big(\frac{LG - 2MF + PE}{EG - F^2}\Big). \tag{12}$$

As a submanifolds of (M, g_f^ϵ), the geometric support Σ of the immersion $\varphi : U \to (M, g_f^\epsilon)$, satisfies the Gauss equation. That is, if ∂_u and ∂_v span the tangent space to Σ at the point (u, v), then the sectional curvature $K(\partial_u, \partial_v)$ of Σ and the sectional curvature $\overline{K}(\partial_u, \partial_v)$ of (M, g_f^ϵ) are related by

$$K(\partial_u, \partial_v) = \overline{K}(\partial_u, \partial_v) + \varepsilon_1 \frac{LN - M^2}{EG - F^2}. \tag{13}$$

5. Geometry of Normal and binormal surfaces through a curve

Loosely speaking, the ruled surface formed by the line moving with the normal/binormal direction and base curve is a normal/binormal surface. Introduced by Hellmuth Kneser [11] to prove the prime decomposition theorem for 3-dimensional manifolds, the concept of normal surface was extend and refined by Wolfgang Haken [8] to create normal surface theory which is the backbone of many algorithms in 3-dimensional manifolds theory.

Besides, binormal motion has received many authors attention for used made of it (see e.g. [7, 2]). In 1972, Hasimoto derived the celebrated nonlinear Schrödinger equation in an approximation to the self-induced motion of a thin, isolated, vortex filament travelling without stretching in an incompressible fluid.

Our daily lives compound surfaces as its part. Some of them can be defined by integrable equations. Known as the smoke ring equation or localized induction equation of a regular space curve, Hasimoto surfaces are some example. Because the normal/binormal line moves on the curve, this surface is called the normal/binormal motion of the curve of constant curvature or torsion, respectively, and is shown to lead to integrable extensions of the Dym and classical sine–Gordon equations.

In this section we study the geometry of normal and binormal surfaces through a given curve in (M, g_f^ϵ). Let γ be a curve on a Walker 3-dimensional manifold (M, g_f^ϵ); denote by (T, N, B) the Frenet-Serret frame along γ and by u its arc-length parameter.

Definition 5.1. *A surface $\mathcal{S}$ passing through a curve γ is a normal (resp. binormal) surface of the curve γ if $\mathcal{S}$ can be parameterized by*

$$\chi(u,v) = \gamma(u) + vN(u) \qquad (\textit{resp.} \quad \chi(u,v) = \gamma(u) + vB(u)), \tag{14}$$

where N (resp. B) is the normal (resp. binormal) vector to γ at time u.

5.1. Geometry of a normal surface

We compute some of the geometric quantities of the normal surface through a curve and study some of its properties.

5.1.1. *The fundamental forms of a normal surface.* To measure the length of curve or to compute the distance between two points on a surface, one uses the first fundamental form, while quantifying how far the surface bends from the plane, the curvatures are the tools.

Theorem 5.2. *The first and the second fundamental forms of a normal surface χ through a curve γ in a 3-dimensional Walker manifold are respectively given by*

$$\mathcal{I} = \begin{pmatrix} (1-\varepsilon_2\kappa v)^2 + \varepsilon_3(\tau v)^2 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}$$

and

$$\mathcal{II} = \frac{1}{\sqrt{\varepsilon_1(\tau v)^2 + \varepsilon_3(1-\varepsilon_1\kappa v)^2}} \begin{pmatrix} -\varepsilon_3 v\left(\varepsilon_1\tau v\dfrac{d\kappa}{du} + (1-\varepsilon_2\kappa v)\dfrac{d\tau}{du}\right) & -\dfrac{\varepsilon_3\tau}{2} \\ -\dfrac{\varepsilon_3\tau}{2} & \varepsilon_1\varepsilon_3\tau v\mathcal{A} + \mathcal{C}(1-\varepsilon_1\kappa v) \end{pmatrix},$$

where $\mathcal{A} = cT_1 + \epsilon bT_2 + (a + fc)T_3$ *and* $\mathcal{C} = cB_1 + \epsilon bN_2 + (a + fc)N_3$ *with the coefficients*

$$\begin{aligned} a &= N_1(N_{1x} + \frac{1}{2}N_3 f_x) + N_2(N_{1y} + \frac{1}{2}N_3 f_y) + N_3\big(N_{1z} + \frac{1}{2}N_3(ff_x + f_z)\big) \\ b &= N_1 N_{2x} + N_2 N_{2y} + N_3(N_{2z} - \frac{\epsilon}{2}N_3 f_y) \\ c &= N_1 N_{3x} + N_2 N_{3y} + N_3(N_{3z} - \frac{1}{2}N_3 f_x). \end{aligned}$$

In the particular case of strict Walker metric, the coefficients a*,* b *and* c *reduce to*

$$\begin{aligned} a &= N_1 N_{1x} + N_2 N_{1y} + N_3 N_{1z} + \frac{1}{2}N_2 N_3 f_y + \frac{1}{2}N_3^2 f_z \\ b &= N_1 N_{2x} + N_2 N_{2y} + N_3(N_{2z} - \frac{\epsilon}{2}N_3 f_y), \\ c &= N_1 N_{3x} + N_2 N_{3y} + N_3 N_{3z}. \end{aligned}$$

Proof. Let $(u, v) \in \mathcal{U}$ and consider $\chi(u, v) = \gamma(u) + vN(u)$ where N is the normal vector to γ at u. From the Frenet formulas in (6) we obtain

$$(15)\quad \begin{cases} \chi_u = \frac{\partial}{\partial u}\chi(u, v) = T(u) + v(-\varepsilon_2\kappa T - \varepsilon_3\tau B) = (1 - \varepsilon_2\kappa v)T(u) - \varepsilon_3\tau v B(u) \\ \chi_v = \frac{\partial}{\partial v}\chi(u, v) = N(u). \end{cases}$$

Then we have the coefficients of the first fundamental form

$$\begin{aligned} E &= g_f^\epsilon\Big((1 - \varepsilon_2\kappa v)T(u) - \varepsilon_3\tau v B(u), (1 - \varepsilon_2\kappa v)T(u) - \varepsilon_3\tau v B(u)\Big) \\ &= \varepsilon_1(1 - \varepsilon_2\kappa v)^2 + \varepsilon_3(\tau v)^2, \qquad (16) \end{aligned}$$

$$(17)\quad F = g_f^\epsilon(\chi_u, \chi_v) = 0,$$

$$(18)\quad G = g_f^\epsilon(N(u), N(u)) = \varepsilon_2.$$

Therefore, the first fundamental form is given by:

$$(19)\quad \mathcal{I} = Edu^2 + Fdu \cdot dv + Gdv^2 = \big[(1 - \varepsilon_2\kappa v)^2 + \varepsilon_3(\tau v)^2\big]du^2 + \varepsilon_2 dv^2.$$

In order to compute the second fundamental form, we need the second order derivative and the normal to the surface χ. To this end, let us express T, N, and B in the basis $(\partial_x, \partial_y, \partial_z)$

$$\begin{cases} T = T_1\partial_x + T_2\partial_y + T_3\partial_z \\ N = N_1\partial_x + N_2\partial_y + N_3\partial_z \\ B = B_1\partial_x + B_2\partial_y + B_3\partial_z \end{cases},$$

and write

$$\nabla_{\chi_v}\chi_v = \nabla_N N = a\partial_x + b\partial_y + c\partial_z.$$

One easily gets

$$
\begin{aligned}
a &= N_1(N_{1x}+\frac{1}{2}N_3f_x)+N_2(N_{1y}+\frac{1}{2}N_3f_y)+N_3\big(N_{1z}+\frac{1}{2}N_3(ff_x+f_z)\big),\\
b &= N_1N_{2x}+N_2N_{2y}+N_3(N_{2z}+\frac{\epsilon}{2}N_3f_y),\\
c &= N_1N_{3x}+N_2N_{3y}+N_3(N_{3z}-\frac{\epsilon}{2}N_3f_x).
\end{aligned}
$$

If the metric is strict Walker, we get

$$
\begin{aligned}
a &= N_1N_{1x}+N_2N_{1y}+N_3N_{1z}+\frac{1}{2}N_2N_3f_y+\frac{1}{2}N_3^2f_z,\\
b &= N_1N_{2x}+N_2N_{2y}+N_3N_{2z}+\frac{\epsilon}{2}N_3^2f_y,\\
c &= N_1N_{3x}+N_2N_{3y}+N_3N_{3z}.
\end{aligned}
$$

Therefore in the Frenet frame (T,N,B), the second order derivatives of χ are expressed as follow:

$$
\left\{
\begin{aligned}
\chi_{uu} &= \frac{\partial^2}{\partial u^2}\chi(u,v)=-\varepsilon_2 v\frac{d\kappa}{du}T(u)+\Big((1-\varepsilon_2\kappa v)\varepsilon_2\kappa-\varepsilon_2\varepsilon_3\tau^2v\Big)N(u)-\varepsilon_3 v\frac{d\tau}{du}B(u)\\
\chi_{uv} &= \frac{\partial}{\partial v}\chi_u=\frac{\partial}{\partial u}\chi_v=\frac{\partial^2}{\partial u\partial v}\chi(u,v)=-\varepsilon_1\kappa T(u)-\varepsilon_3\tau B(u)\\
\chi_{vv} &= \frac{\partial^2}{\partial v^2}\chi(u,v)=\mathcal{A}T(u)+\mathcal{B}N(u)+\mathcal{C}B(u)
\end{aligned}
\right. ,
$$

where

$$
\left\{
\begin{aligned}
\mathcal{A} &= \varepsilon_1\big(cT_1+\epsilon bT_2+(a+fc)T_3\big)\\
\mathcal{B} &= \varepsilon_2\big(cN_1+\epsilon bN_2+(a+fc)N_3\big)\\
\mathcal{C} &= \varepsilon_3\big(cB_1+\epsilon bB_2+(a+fc)B_3\big)
\end{aligned}
\right.
$$

and the unit normal vector to χ:

$$
\mathcal{N}=\frac{\chi_u\times\chi_v}{\|\chi_u\times\chi_v\|}=\frac{1}{\sqrt{\varepsilon_1(\tau v)^2+\varepsilon_3(1-\varepsilon_1\kappa v)^2}}\Big(\varepsilon_1\varepsilon_3\tau vT(u)+\varepsilon_3(1-\varepsilon_1\kappa v)B(u)\Big).
$$

Thus we compute the coefficients of the second fundamental form to obtain:

$$
\begin{aligned}
L &= g_f^\epsilon(\mathcal{N},\mathcal{X}_{uu})=-\frac{\varepsilon_3 v}{\sqrt{\varepsilon_1(\tau v)^2+\varepsilon_3(1-\varepsilon_1\kappa v)^2}}\left(\varepsilon_1\tau v\frac{d\kappa}{du}+(1-\varepsilon_1\kappa v)\frac{d\tau}{du}\right),\\
M &= g_f^\epsilon(\mathcal{N},\mathcal{X}_{uv})=-\frac{\varepsilon_3\tau}{\sqrt{\varepsilon_1(\tau v)^2+\varepsilon_3(1-\varepsilon_1\kappa v)^2}},\\
P &= g_f^\epsilon(\mathcal{N},\mathcal{X}_{vv})=\frac{1}{\sqrt{\varepsilon_1(\tau v)^2+\varepsilon_3(1-\varepsilon_1\kappa v)^2}}\big(\varepsilon_3\tau v\mathcal{A}+(1-\varepsilon_1\kappa v)\mathcal{C}\big).
\end{aligned}
$$

□

5.1.2. *Curvatures of a normal surface.*

We give, hereafter, the curvatures of the normal surface through a curve in a 3-dimensional Walker manifold.

Theorem 5.3. *Let γ be a curve in a 3-dimensional Walker manifold and χ be a normal surface passing through γ. The principles curvature, the Gaussian curvature and the mean curvature of χ are given by:*

$$k_i = \frac{1}{2\sqrt{\varepsilon_1(\tau v)^2 + \varepsilon_3(1-\varepsilon_1\kappa v)^2}}\left(\beta - \alpha + \sqrt{(\alpha+\beta)^2 + \tau^2}\right) \qquad \forall\, i = 1,2,$$

$$H = \frac{-1}{2\sqrt{\varepsilon_1(\tau v)^2 + \varepsilon_3(1-\varepsilon_1\kappa v)^2}}\left(\varepsilon_3 v\left(\varepsilon_1\tau v\frac{d\kappa}{du} + (1-\varepsilon_1\kappa v)\frac{d\tau}{du}\right) - \varepsilon_1\varepsilon_3\tau v\mathcal{A} - \mathcal{C}(1-\varepsilon_1\kappa v)\right),$$

$$K = -\frac{4\alpha\beta + \tau^2}{\varepsilon_1(\tau v)^2 + \varepsilon_3(1-\varepsilon_1\kappa v)^2},$$

where $\alpha = \varepsilon_3 v\left(\varepsilon_1\tau v\frac{d\kappa}{du} + (1-\varepsilon_1\kappa v)\frac{d\tau}{du}\right)$ *and* $\beta = \varepsilon_1\varepsilon_3\tau v\mathcal{A} + \mathcal{C}(1-\varepsilon_1\kappa v)$.

Proof. Let us write the second fundamental form

$$\mathcal{II} = \frac{1}{\delta}\begin{pmatrix} -\alpha & -\frac{\varepsilon_3\tau}{2} \\ -\frac{\varepsilon_3\tau}{2} & \beta \end{pmatrix}$$

with $\delta = \sqrt{\varepsilon_1(\tau v)^2 + \varepsilon_3(1-\varepsilon_1\kappa v)^2}$ and α, β as expressed in Theorem 5.3. One obtains

$$P_{\mathcal{II}}(\lambda) = \det(\mathcal{II} - \lambda I_2) = 0 \Longleftrightarrow \lambda^2 + \frac{\beta-\alpha}{\delta}\lambda - \frac{4\alpha\beta+\tau^2}{4\delta^2} = 0,$$

from which we deduce the principles curvature that give the mean and Gaussian curvature. □

Remark 5.4. *A normal surface passing through a given curve in a 3-dimensional Walker manifold is umbilical.*

Corollary 5.5. *Let γ be a planar curve (everywhere vanishing torsion $\tau = 0$) in a 3-dimensional Walker manifold. Then the normal surface passing through γ is umbilical, minimal and flat. In other words, the principles curvature, the Gaussian curvature and the mean curvature of χ are all zero*

Proof. It is easy to find that when $\tau = 0$, all the principal curvatures vanish. □

5.2. Geometry of binormal surface

In this subsection, we compute the geometric invariants of the binormal surface through a curve and study some of its properties.

5.2.1. *The fundamental forms of a binormal surface.* Likely for the normal surface, we compute the first fundamental form which is used to measure distance between the points on the surface and the second fundamental form to appreciate how much the surface bends from the plane.

Theorem 5.6. *The first and the second fundamental forms of a binormal surface χ through a curve γ in a 3-dimensional Walker manifold are respectively represented by the following matrices:*

$$\mathcal{I} = \begin{pmatrix} \varepsilon_1 + (\tau v)^2\varepsilon_3 & 0 \\ 0 & \varepsilon_3 \end{pmatrix} \tag{20}$$

and

$$\mathcal{II} = \frac{-1}{\sqrt{|\varepsilon_1(\tau v)^2 + \varepsilon_2|}} \begin{pmatrix} \varepsilon_1\kappa(\tau v)^2 + \varepsilon_2\Big(\kappa - v\dfrac{d\tau}{du}\Big) & -\dfrac{\varepsilon_2\tau}{2} \\ -\dfrac{\varepsilon_2\tau}{2} & \varepsilon_2\tau v\mathcal{A} + \varepsilon_2\varepsilon_3\mathcal{B} \end{pmatrix}. \tag{21}$$

Proof. Using the Frenet formulas in (6) we obtain

$$\begin{cases} \chi_u &= \dfrac{\partial}{\partial u}\chi(u,v) = T(u) + v\frac{d}{du}B(u) = T(u) - \varepsilon_2 v\tau N(u) \\ \chi_v &= \dfrac{\partial}{\partial v}\chi(u,v) = B(u). \end{cases} \tag{22}$$

Then the coefficients of the first fundamental form are

$$E = \varepsilon_1 + \varepsilon_2(\tau v)^2, \qquad F = 0 \quad \text{and} \quad G = \varepsilon_3.$$

Thus the first fundamental form is : $g(u,v) = (\varepsilon_1 + \varepsilon_2(\tau v)^2)du^2 + \varepsilon_3 dv^2$.

In order to compute the second fundamental form, one needs the unit normal vector to χ and the second order derivative. The unit normal vector to χ is:

$$\mathcal{N} = \frac{\chi_u \times \chi_v}{\|\chi_u \times \chi_v\|} = \frac{1}{\sqrt{|\varepsilon_1(\tau v)^2 + \varepsilon_2|}}\Big(- \varepsilon_1\varepsilon_2\tau v T(u) - \varepsilon_2 N(u)\Big). \tag{23}$$

For the second order derivatives, let us expressed the Frenet frame's vectors in the basis $(\partial_x, \partial_y, \partial_z)$

$$\begin{cases} T &= T_1\partial_x + T_2\partial_y + T_3\partial_z \\ N &= N_1\partial_x + N_2\partial_y + N_3\partial_z \\ B &= B_1\partial_x + B_2\partial_y + B_3\partial_z \end{cases},$$

and write

$$\nabla_{\chi_v}\chi_v = \nabla_B B = a\partial_x + b\partial_y + c\partial_z.$$

The coefficients a, b and c are given by

$$\begin{aligned} a &= B_1(B_{1x}+\frac{1}{2}B_3f_x)+B_2(B_{1y}+\frac{1}{2}B_3f_y)+B_3\Big(B_{1z}+\frac{1}{2}B_3(ff_x+f_z)\Big)\\ b &= B_1B_{2x}+B_2B_{2y}+B_3(B_{2z}-\frac{\epsilon}{2}B_3f_y)\\ c &= B_1B_{3x}+B_2B_{3y}+B_3(B_{3z}-\frac{1}{2}B_3f_x)\end{aligned}$$

and in the strict Walker structure by

$$\begin{aligned} a &= B_1B_{1x}+B_2B_{1y}+B_3B_{1z}+\frac{1}{2}B_2B_3f_y+\frac{1}{2}B_3^2f_z\\ b &= B_1B_{2x}+B_2B_{2y}+B_3(B_{2z}-\frac{\epsilon}{2}B_3f_y)\\ c &= B_1B_{3x}+B_2B_{3y}+B_3B_{3z}.\end{aligned}$$

Therefore, in the Frenet frame, the second order derivatives of χ are:

$$(24)\quad \begin{cases} \chi_{uu} &= \varepsilon_1\varepsilon_2 v\kappa\tau T(u)+\varepsilon_2\left(\kappa-v\frac{d\tau}{du}\right)N(u)-\varepsilon_2\varepsilon_3 v\tau^2B(u)\\ \chi_{uv} &= -\varepsilon_2\tau N(u)\\ \chi_{vv} &= \mathcal{A}T(u)+\mathcal{B}N(u)+\mathcal{C}B(u),\end{cases}$$

where

$$\begin{cases} \mathcal{A} &= \varepsilon_1\big(cT_1+\epsilon bT_2+(a+cf)T_3\big)\\ \mathcal{B} &= \varepsilon_2\big(cN_1+\epsilon bN_2+(a+cf)N_3\big)\\ \mathcal{C} &= \varepsilon_3\big(cB_1+\epsilon bB_2+(a+cf)B_3\big).\end{cases}$$

Thus the coefficients of the second fundamental form are given by:

$$\begin{aligned} L &= g_f^\epsilon(\mathcal{N},\mathcal{X}_{uu})=\frac{-1}{\sqrt{\varepsilon_1(\tau v)^2+\varepsilon_2}}\Big(\varepsilon_1\kappa(\tau v)^2+\varepsilon_2(\kappa-v\frac{d\tau}{du})\Big),\\ M &= g_f^\epsilon(\mathcal{N},\mathcal{X}_{uv})=\frac{\varepsilon_2\tau}{\sqrt{\varepsilon_1(\tau v)^2+\varepsilon_2}},\\ P &= g_f^\epsilon(\mathcal{N},\mathcal{X}_{vv})=\frac{-1}{\sqrt{\varepsilon_1(\tau v)^2+\varepsilon_2}}\big(\varepsilon_2\tau v\mathcal{A}+\varepsilon_2\varepsilon_3\mathcal{B}\big).\end{aligned}$$

□

5.2.2. *Curvatures.* We compute the curvature of the binormal surface through a curve in a 3-dimensional Walker manifold.

Theorem 5.7. *Let γ be a curve in a 3-dimensional Walker manifold and χ be a binormal surface through γ. The principles curvature, the mean curvature* *and the Gaussian curvature of* χ *are respectively given by:*

$$\begin{aligned} k_i &= \frac{1}{2\sqrt{\varepsilon_1(\tau v)^2+\varepsilon_2}}\Big(-\alpha-\beta\pm\sqrt{(\alpha+\beta)^2+\tau^2}\Big) \qquad \forall\, i=1,2,\\ H &= \frac{-1}{2\sqrt{\varepsilon_1(\tau v)^2+\varepsilon_2}}\left(\varepsilon_1\kappa(\tau v)^2+\varepsilon_2\kappa-\varepsilon_2 v\frac{d\tau}{du}+\varepsilon_2\tau v\mathcal{A}+\varepsilon_1\varepsilon_2\mathcal{B}\right),\\ K &= \frac{-\tau^2}{4(\varepsilon_1(\tau v)^2+\varepsilon_2)}, \end{aligned}$$

where

$$\begin{aligned} \alpha &= \varepsilon_1\kappa(\tau v)^2+\varepsilon_2(\kappa-v\frac{d\tau}{du}) \quad \text{and} \quad \beta=\varepsilon_2\tau v\mathcal{A}+\varepsilon_2\varepsilon_3\mathcal{B}\\ \mathcal{A} &= \varepsilon_1\big(cT_1+\epsilon bT_2+(a+cf)T_3\big) \quad \text{and} \quad \mathcal{B}=\varepsilon_2\big(cN_1+\epsilon bN_2+(a+cf)N_3\big). \end{aligned}$$

Proof. Putting the second fundamental form

$$\mathcal{II}=\frac{1}{\delta}\begin{pmatrix} -\alpha & \frac{\varepsilon_2\tau}{2}\\ \frac{\varepsilon_2\tau}{2} & -\beta \end{pmatrix}$$

with $\delta=\sqrt{\varepsilon_1(\tau v)^2+\varepsilon_2}$ and α, β as in Theorem 5.7, one obtains

$$P_{\mathcal{II}}(\lambda)=\det(\mathcal{II}-\lambda I_2)=0 \iff \lambda^2+\frac{\alpha+\beta}{\delta}\lambda+\frac{4\alpha\beta-\tau^2}{4\delta^2}=0.$$

from which we deduce the principles curvature that give the mean and Gaussian curvature. □

Proposition 5.8. *Let* χ *be a binormal surface through a curve* γ *(with constant torsion) in a 3-dimensional Walker manifold. The principles curvature, the Gaussian and the mean curvature of* χ *are given by:*

$$\begin{aligned} k_i &= \frac{1}{2\sqrt{\varepsilon_1(\tau v)^2+\varepsilon_2}}\Big(-\alpha-\beta\pm\sqrt{(\alpha+\beta)^2+\tau^2}\Big) \qquad \forall\, i=1,2,\\ H &= \frac{-1}{2\sqrt{\varepsilon_1(\tau v)^2+\varepsilon_2}}\big(\varepsilon_1\kappa(\tau v)^2+\varepsilon_2\kappa+\varepsilon_2\tau v\mathcal{A}+\varepsilon_1\varepsilon_2\mathcal{B}\big),\\ K &= \frac{-\tau^2}{4(\varepsilon_1(\tau v)^2+\varepsilon_2)}, \end{aligned}$$

where

$$\begin{aligned} \alpha &= \varepsilon_1\kappa(\tau v)^2+\varepsilon_2\kappa \quad \text{and} \quad \beta=\varepsilon_2\tau v\mathcal{A}+\varepsilon_2\varepsilon_3\mathcal{B}\\ \mathcal{A} &= \varepsilon_1\big(cT_1+\epsilon bT_2+(a+cf)T_3\big) \quad \text{and} \quad \mathcal{B}=\varepsilon_2\big(cN_1+\epsilon bN_2+(a+cf)N_3\big) \end{aligned}$$

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Conflict of interests

The authors declare that they made an equal contribution to this work and there are no conflicts of interest to be disclosed.

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