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W_5-CURVATURE TENSOR IN THE SPACE-TIME OF GENERAL RELATIVITY

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Abstract. The W_5-curvature tensor has been studied in the space- time of general relativity. The space-time satisfying Einstein's field equations with cosmological term and vanishing W_5-curvature tensor has been considered and it has been shown that metric tensor is proportional to the energy-momentum tensor. The existence of Killing as well as conformal Killing vector fields have been shown. Further for a W_5-flat perfect fluid space-time satisfying Einstein's field equations, the isotropic pressure has been found to be the function of cosmological constant and non-zero gravitational constant.

1. Introduction

Consider an n-dimensional space V_n in which the curvature tensor W_5 has been defined as:

(1)

$$W_5(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}[g(X,Z)Ric(Y,T) - g(Y,T)Ric(X,Z)]$$

for vector fields X, Y, Z and T on V_n [4]. Here $R(X,Y,Z,T), Ric(X,Y)$ and $g(X,Y)$ denote the curvature tensor, Ricci tensor and metric tensor of V_n, respectively, for arbitrary vector fields X, Y, Z and T. From (1), it is noticed that $W_5(X,Y,Z,T) = W_5(Z,T,X,Y)$, which shows that it is symmetric with change of pairs of vectors. This tensor does not satisfy the cyclic property. Studying the geometric properties, Moindi et.al [6] have shown that W_5-symmetric and W_5-flat K-contact manifolds are flat manifolds having zero curvature. Breaking

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W_5 into two parts using X and Y, we have

$$\mu(X,Y,Z,T) = \frac{1}{2}[W_5(X,Y,Z,T) - W_5(Y,X,Z,T)]$$
$$= R(X,Y,Z,T) + \frac{1}{2n-2}[g(X,Z)Ric(Y,T) - g(Y,T)Ric(X,Z)$$
$$-g(Y,Z)Ric(X,T) + g(X,T)Ric(Y,Z)]$$

and

$$\begin{aligned} v(X,Y,Z,T) &= \frac{1}{2}[W_5(X,Y,Z,T) + W_5(Y,X,Z,T)] \\ &= \frac{1}{2n-2}[g(X,Z)Ric(Y,T) - g(Y,T)Ric(X,Z) \\ &+ g(Y,Z)Ric(X,T) - g(X,T)Ric(Y,Z)]. \end{aligned}$$

Again, breaking W_5 into two parts using Z and T, we get

$$\gamma(X,Y,Z,T) = \frac{1}{2}[W_5(X,Y,Z,T) - W_5(X,Y,T,Z)]$$
$$= R(X,Y,Z,T) + \frac{1}{2n-2}[g(X,Z)Ric(Y,T) - g(Y,T)Ric(X,Z)$$
$$-g(X,T)Ric(Y,Z) + g(Y,Z)Ric(X,T)]$$

and

$$\digamma(X,Y,Z,T) = \frac{1}{2}[W_5(X,Y,Z,T) + W_5(X,Y,T,Z)]$$
$$= \frac{1}{2n-2}[g(X,Z)Ric(Y,T) - g(Y,T)Ric(X,Z)$$
$$+g(X,T)Ric(Y,Z) - g(Y,Z)Ric(X,T)].$$

It was found that ([4] Pokhariyal,1982)

$$v(X,Y,Z,T) + v(X,Z,T,Y) + v(X,T,Y,Z) = 0$$

and

$$\digamma(X,Y,Z,T) + \digamma(Y,Z,X,T) + \digamma(Z,X,Y,T) = 0.$$

Also,

$$\begin{aligned} \mu(X,Y,Z,T) &= R(X,Y,Z,T) + \digamma(X,Y,Z,T), \\ \gamma(X,Y,Z,T) &= R(X,Y,Z,T) + v(X,Y,Z,T) \end{aligned}$$

and

$$W_5(X,Y,Z,T) + W_5(X,Y,T,Z) = R(X,Y,Z,T) + R(X,Y,T,Z).$$

2. W_5-Curvature Tensor

Consider V_4 for the four-dimensional space-time of general relativity, then we have

$$W_{5hijk} = R_{hijk} + \frac{1}{3}[g_{hj}R_{ik} - g_{ik}R_{hj}],$$

and

$$W_{5\,ijk}^{\;h} = R^h_{ijk} + \frac{1}{3}[g^h_j R_{ik} - g_{ik}R^h_j].$$

On contracting h and k, we get

$$W_{5ij} = R_{ij} + \frac{1}{3}[g^h_j R_{ih} - g_{ih}R^h_j].$$

$$W_{5ij} = R_{ij}$$

and

$$W_5 = R.$$

Rainich [3] has shown that the necessary and sufficient conditions for the existence of non-null elector-variance are:

(2) $$R = 0,$$

(3) $$R^i_j R^j_k = (1/4)\delta^i_k R_{ab}R^{ab}.$$

and

(4) $$\theta_{i,j} = \theta_{j,i},$$

where θ_i is complexion vector. Replacing the matter tensor R_{ij} by W_{5ij} in (2)-(4), we get the Rainich conditions with W_{5hijk}. In [7, 8], authors studied the similar properties of space-times.

3. W_5-flat space-time

Consider W_5-flat space-time, then from (2) we have

$$R^h_{ijk} = (1/3)[g_{ik}R^h_j - g^h_j R_{ik}].$$

On contracting h and k, we get

$$R_{ij} = (1/3)[g_{ih}R^h_j - g^h_j R_{ih}] = 0.$$

Thus, the W_5-flat space-time results in an empty space.

Consider Einstein's field equations with cosmological term Δ as

$$R_{ij} - (1/2)Rg_{ij} + \Delta g_{ij} = \kappa T_{ij},$$

where R_{ij} is Ricci tensor, R is the scalar curvature, Δ is the cosmological constant, κ is the non-zero gravitational constant and T_{ij} is the energy-momentum tensor.

Using the condition of W_5-flat space-time (with $R_{ij} = 0$), the Einstein's field equations become

$$g_{ij} = \frac{\kappa}{\Delta} T_{ij} \text{ or, } g_{ij} = \alpha T_{ij}. \tag{5}$$

Thus, we have:

Theorem 3.1. *For a W_5-flat space-time satisfying Einstein's field equations with cosmological term Δ, the metric tensor g_{ij} is proportional to the energy-momentum tensor T_{ij}.*

The gravitational field is adequately described by curvature tensor, which consist of matter part and gravitational part, whose interaction is depicted to be Bianchi identities. The focus of several studies has been the construction of gravitational potential satisfying the Einstein's equations for a given distribution of matter. This is accomplished by imposing symmetries on the geometry compatible with the dynamics of the selected distribution of the matter. For the space-time, the gravitational symmetries are given by the following equation [2]

$$\pounds_\xi A - 2\Omega A = 0,$$

where A represents a geometrical/physical quantity, $\pounds_\xi$ denotes the Lie derivative with respect to a vector field ξ and Ω is a scalar.

Taking Lie derivative of both sides of equation (5), we get

$$\pounds_\xi g_{ij} = \alpha \pounds_\xi T_{ij}. \tag{6}$$

Thus, we have the following theorem:

Theorem 3.2. *For W_5-flat space-time satisfying Einstein's field equations with cosmological term Δ, there exists a Killing vector field ξ, if and only if the energy-momentum tensor T_{ij} vanishes with respect to ξ.*

Definition 3.3. *A vector field ξ satisfying the equation*

$$\pounds_\xi g_{ij} = 2Q g_{ij}, \tag{7}$$

is called a conformal Killing vector field, where Q is a scalar. The space-time satisfying equation (7) is known to admit a conformal motion.

From (6) and (7), we get

$$2Q g_{ij} = \alpha \pounds_\xi T_{ij}. \tag{8}$$

Using (5) in (8) and on simplification, we get

$$\alpha \pounds_\xi T_{ij} = 2Q T_{ij}. \tag{9}$$

The energy-momentum tensor satisfying (9) is known to have symmetric inheritance property [1]. Thus, we have the following theorem:

Theorem 3.4. *The W_5-flat space-time satisfying Einstein's field equations with cosmological term Δ admits a conformal Killing vector field if and only if the energy-momentum tensor has the symmetric inheritance property.*

Remark 3.5. *Theorem 3.1 uniquely holds for W_5-curvature tensor and W_3-curvature tensor. The Theorem 3.2 and Theorem 3.4 hold for W_2, W_3 and W_5 curvature tensors* [5].

4. Perfect Fluid Space-Time

Consider a perfect fluid space-time with vanishing W_5-curvature tensor. The energy-momentum tensor T_{ij} for perfect fluid is given by:

$$T_{ij} = (\mu + p)u_i u_j + p g_{ij}, \tag{10}$$

where μ is the energy density, p is isotropic pressure, u_i is the velocity of fluid, such that $u_i u^i = -1$ and $g_{ij}u^i = u_j$. From equations (5) and (10), on simplification we get,

$$(1 - \alpha p)g_{ij} = \alpha(\mu + p)u_i u_j. \tag{11}$$

Multiplying (11) by g^{ij} and on simplification, we get

$$4(\alpha p - 1) = \alpha(\mu + p). \tag{12}$$

Again contracting (11) with $u^i u^j$ we get on simplification,

$$(\alpha p - 1) = \alpha(\mu + p). \tag{13}$$

Comparing (12) and (13), we get

$$p = (1/\alpha) = (\Delta/\kappa).$$

Thus, we have the following theorem:

Theorem 4.1. *For a W_5-flat perfect fluid space-time satisfying Einstein's field equations with a cosmological term Δ, the isotropic pressure is a constant, which is a function of non-zero gravitational constant and cosmological constant.*

Remark 4.2. *Theorem 4.1, is uniquely satisfied by W_5 and W_3 curvature tensors. Though W_3 tensor is skew-symmetric in vector fields Z and T and satisfies cyclic property with fixed X as well as Bianchi like differential identity with Ricci tensor being of Codazzi type. On the other hand, W_5 is symmetric with the change of vector field pairs X, Y and Z, T and does not satisfy any cyclic property.*

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References

[1] Z. Ahsan, *On a geometrical symmetry of the space-time of general relativity*, Bull. Cal. Math. Soc. **97** (2005), no. 3, 191–200.

[2] Z. Ahsan and M. Ali, *Curvature tensor for the space-time of general relativity*, Int. J. Geom. Methods Mod. Phys. **14** (2017), 1–14.

[3] G. Y. Rainich, *Electrodynamics in the general relativity theory*, Trans. Amer. Math. Soc. **27** (1952), 106–136.

[4] G. P. Pokhariyal, *Relativistic significance of curvature tensors*, Int. J. Math & Math. Sci. **5** (1982), no. 1, 133–139.

[5] G. P. Pokhariyal, *Study of W_3-curvature tensor in the space-time of general relativity*, ICASTOR Jour. of Mathematical Sci. **15** (2021), no. 2, 1–7.

[6] S. K. Moindi, F. Njui, and G. P. Pokhariyal, *A study of W_5-symmetric K-contact Riemannian manifold*, Int. Jour. of Trd. in R & D. **5** (2018), no. 3, 749–751.

[7] M. Vasiulla, M. Ali, and İnan Ünal, *A study of mixed super quasi-Einstein manifolds with applications to general relativity*, Int. J. Geom. Methods Mod. Phys. **21** (2024), no. 9, Paper No. 2450177.

[8] M. Vasiulla and M. Ali, *Pseudo generalized quasi-Einstein manifolds with applications to general relativity*, Filomat **38** (2024), no. 2, 553–567.

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