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UNIQUENESS OF ENTIRE FUNCTION SHARING TWO VALUES JOINTLY WITH ITS DIFFERENTIAL POLYNOMIALS

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Abstract. In this paper, we continue to investigate the uniqueness problem when an entire function f and its linear differential polynomial $L(f)$ share two distinct complex values CMW (counting multiplicities in the weak sense) jointly. Also, We investigate the same problem when f and its differential monomial $M(f)$ share two distinct complex values CMW, which is introduced by Lahiri in [Comput. Methods Funct. Theory, 21, 379–397 (2021)]. Our results generalize the recent result of Lahiri [Comput. Methods Funct. Theory, 21, 379–397 (2021)] to some extent.

1. Introduction, Definitions, and Results

A function analytic in the open complex plane $\mathbb{C}$ except possibly for poles is called meromorphic in $\mathbb{C}$. If no poles occur, then the function is called entire. For a non-constant meromorphic function f defined in $\mathbb{C}$ and for $a \in \mathbb{C}\cup\{\infty\}$, we denote by $E(a, f)$ the set of a-points of f counted multiplicities and $\overline{E}(a, f)$ the set of all a-points ignoring multiplicities. If for two non-constant meromorphic functions f and g, $E(a, f) = E(a, g)$, we say that f and g share the value a CM (counting multiplicities). If $\overline{E}(a, f) = \overline{E}(a, g)$, then we say that f and g are said to share the value a IM (ignoring multiplicities). Throughout the paper, the standard notations of Nevanlinna's value distribution theory of meromorphic functions [5, 16] have been adopted. A meromorphic function $a(z)$ is said to be small with respect to f provided that $T(r, a) = S(r, f)$, that is $T(r, a) = o(T(r, f))$ as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure.

In 1976, it was shown by Rubel and Yang [14] that if an entire function f and its derivative f' share two values a, b CM, then $f = f'$. After that Gundersen [4] improved the result by considering two IM shared Values. Yang [15] also extended the result of Rubel and Yang [14] by replacing f' with the

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k-th derivative $f^{(k)}$. Since then the subject of sharing values between a meromorphic function and its derivatives has become one of the most prominent branches of the uniqueness theory. Mues and Steinmetz [13] showed that if a meromorphic function f shares three finite values IM with f', then $f = f'$. Frank and Schwick [1] improved this result by replacing f' with $f^{(k)}$, where k is a positive integer. After that many mathematicians spent their times towards the improvements of this result (see [2, 3, 8, 12]). In 2000, Li and Yang [9] improved the result of Yang [15] in the following.

Theorem A. [9] *Let f be a non-constant entire function, k be a positive integer and a, b be distinct finite numbers. If f and $f^{(k)}$ share a and b IM, then $f = f^{(k)}$.*

We now recall the notion of set sharing as follows: Let S be a subset of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a\in S} E(a, f)$ and $\overline{E}_f(S) = \bigcup_{a\in S} \overline{E}(a, f)$. We say that two meromorphic functions f and g share the set S CM or IM if $E_f(S) = E_g(S)$ or $\overline{E}_f(S) = \overline{E}_g(S)$, respectively.

Using the notion of set sharing instead of value sharing, Li and Yang [10] proved the following theorem.

Theorem B. [10] *Let f be a non-constant entire function and a_1, a_2 be two distinct finite complex numbers. If f and $f^{(1)}$ share the set $\{a_1, a_2\}$ CM, then one and only one of the following holds:*

(i) $f = f^{(1)}$
(ii) $f + f^{(1)} = a_1 + a_2$
(iii) $f = c_1e^{cz} + c_2e^{-cz}$ *with* $a_1 + a_2 = 0$, *where* c, c_1 *and* c_2 *are non-zero constants satisfying* $c^2 \neq 1$ *and* $4c^2c_1c_2 = a_1^2(c^2 - 1)$.

In 2020, Lahiri [6] introduced a new type of set sharing notion called CMW (counting multiplicities in the weak sense)as follows:

Let f and g be two non-constant meromorphic functions in $\mathbb{C}$ and $a \in \mathbb{C} \cup \{\infty\}$ and $B \subset \mathbb{C} \cup \{\infty\}$. We denote by $E_B(a; f, g)$ the set of those distinct a-points of f which are the b-points of g having the same multiplicity for some $b \in B$. For $A \subset \mathbb{C} \cup \{\infty\}$, we put $E_B(A; f, g) = \bigcup_{a\in A} E_B(a; f, g)$. Clearly $E_B(A; f, g) = E_B(A; g, f)$ for $A = B$. For $S \subset \mathbb{C} \cup \{\infty\}$ we define

$$Y = \{\overline{E}(S, f) \cup \overline{E}(S, g)\} \setminus \overline{E}_S(S, ; f, g).$$

We say that f and g share the set S with counting multiplicities in the weak sense (CMW) if $N_Y(r, a; f) = S(r, f)$ and $N_Y(r, a; g) = S(r, g)$ for every $a \in S$, where $N_Y(r, a; f)$ denotes the counting function, counted with multiplicities of those a-points of f which lie in the set Y.

We note that f and g share the set S with counting multiplicities if and only if $Y = \emptyset$.

Lahiri [6] greatly improved Theorem B by considering the higher order derivative $f^{(k)}$ and CMW in place of CM set sharing and proved the following theorem.

Theorem C. [6] *Let f be a non-constant entire function and k be a positive integer such that*

$$\overline{N}\left(r, \frac{f^{(k)}}{f^{(1)}}\right) = S(r, f). \tag{1.1}$$

Suppose that a_1 and a_2 are two distinct finite complex numbers. If f and $f^{(k)}$ share the set $\{a_1, a_2\}$ CMW, then only one of the following holds:

(i) $f = f^{(k)}$
(ii) $f + f^{(k)} = a_1 + a_2$
(iii) $f = c_1 e^{cz} + c_2 e^{-cz}$ *with* $a_1 + a_2 = 0$, *where* c, c_1 *and* c_2 *are non-zero constants satisfying* $c^{2k} \neq 1$ *and* $4c^{2k}c_1c_2 = a_1^2(c^{2k} - 1)$ *and* k *is an odd positive integer.*

For further investigation of the above theorem, we now define a linear differential polynomial $L(f)$ and a differential monomial $M(f)$ of an entire function f as follows:

$$L(f) = b_1 f^{(1)} + b_2 f^{(2)} + \cdots + b_k f^{(k)} = \sum_{j=1}^{k} b_j f^{(j)}, \tag{1.2}$$

where $b_1, b_2, \ldots, b_k (\neq 0)$ are complex constants, and

$$M(f) = (f^{(1)})^{n_1} (f^{(2)})^{n_2} \cdots (f^{(k)})^{n_k}, \tag{1.3}$$

where k ia a positive integer and $n_1, n_2, \ldots, n_k$ are non-negative integers, not all of them are zero. We call k and $\lambda = \sum_{j=1}^{k} n_j$, respectively the order and the degree of the monomial $M(f)$.

From the above discussion it is natural to ask the following questions.

Question 1.1. *What can be said about the uniqueness when an entire function f share two values jointly CMW with its linear differential polynomial $L(f)$?*

Question 1.2. *What can be said about the uniqueness of an entire function f when f share two values jointly CMW with its differential monomial $M(f)$?*

In the present paper, we prove the following results which will answer the above questions positively. We use a methodology which is similar to [6] but with some modifications.

2. Main results

Theorem 2.1. *Let f is a non-constant entire function and $L(f)$ be a linear differential polynomial defined as in* (1.2) *such that*

$$\overline{N}\left(r, \frac{L(f)}{f^{(1)}}\right) = S(r, f). \tag{2.1}$$

Suppose that a_1 and a_2 are two distinct finite complex numbers. If f and $L(f)$ share the set $\{a_1, a_2\}$ CMW, then only one of the following holds:

(i) $f = L(f)$

(ii) $f + L(f) = a_1 + a_2$

(iii) $f = c_1e^{cz} + c_2e^{-cz}$ *with* $a_1 + a_2 = 0$, *where* c, c_1 *and* c_2 *are non-zero constants satisfying* $(b_1c + b_3c^3 + \cdots + b_kc^k)^2 \neq 1$ *and* $4(b_1c + b_3c^3 + \cdots + b_kc^k)^2c_1c_2 = a_1^2((b_1c + b_3c^3 + \cdots + b_kc^k)^2 - 1)$ *and k is an odd positive integer.*

Theorem 2.2. *Let f is a non-constant entire function and $M(f)$ be a differential monomial defined as in* (1.3) *such that*

$$\overline{N}\left(r, \frac{M(f)}{(f^{\lambda})^{(1)}}\right) = S(r, f). \tag{2.2}$$

Suppose that a_1 and a_2 are two distinct finite complex numbers. If f^{λ} and $M(f)$ share the set $\{a_1, a_2\}$ CMW, then only one of the following holds:

(i) $f^{\lambda} = M(f)$

(ii) $f^{\lambda} + M(f) = a_1 + a_2$

(iii) $f^{\lambda} = c_1e^{cz} + c_2e^{-cz}$, $M(f) = \sqrt{A}(c_1e^{2cz} - c_2)/e^{cz}$ *with* $a_1 + a_2 = 0$, *where* A, c, c_1 *and* c_2 *are non-zero constants and* $\lambda = \sum_{j=1}^{k} n_j$.

We give the following examples in the support of the main theorems.

Example 2.1. *Let* $f = e^{\omega z} + a_1 + a_2$, *where* $\omega^k = -1$, *k is a positive integer and* a_1, a_2 *are any two finite distinct complex constants. and* $L(f) = M(f) = f^{(k)}$. *Then all the conditions of Theorems* 2.1 *and* 2.2 *are satisfied. Here conclusion* (ii) *of Theorems* 2.1 *and* 2.2 *holds.*

Example 2.2. *Let* $f = e^{\lambda z}$, *where* $\lambda^5 = 1$ *and* $L(f) = M(f) = f^{(5)}$. *Then all the conditions of the above two theorems are satisfied and conclusion* (i) *of the above two theorems holds.*

Remark 2.1. *By taking* $L(f) = f^{(k)}$ *in Theorem* 2.1, *we get Theorem C, which is a particular case of our result.*

3. Key lemmas

In this section, we present some necessary lemmas which will be required to prove the main results.

Lemma 3.1. *Let f be a non-constant entire function and a_1, a_2 be two distinct finite complex numbers. If f and $L(f)$ share the set $\{a_1, a_2\}$ CMW, then $S(r, L(f)) = S(r, f)$.*

Proof. Since f is entire, we have

$$T(r, L(f)) = m(r, L(f)) \leq m\left(r, \frac{L(f)}{f}\right) + m(r, f)$$
$$\leq T(r, f) + S(r, f). \tag{3.1}$$

Again since f and $L(f)$ share the set $\{a_1, a_2\}$ CMW, we get by second fundamental theorem

$$T(r, f) \leq \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - a_2}\right) + S(r, f)$$
$$\leq 2T(r, L(f)) + S(r, f). \tag{3.2}$$

From 3.1 and 3.2, we conclude that $S(r, L(f)) = S(r, f)$. This proves the lemma. □

Lemma 3.2. *Let f be a non-constant entire function and a_1, a_2 be two distinct finite complex numbers. If f^λ and $M(f)$, where $\lambda = \sum_{j=1}^{k} n_j$ and $M(f)$ is defined as in* (1.3) *share the set $\{a_1, a_2\}$ CMW, then $S(r, M(f)) = S(r, f)$.*

Proof. The proof of the lemma can be carried out in the line of the proof of Lemma 3.1. So, we omit the details. □

Lemma 3.3. [11, 16] *Let f be a non-constant meromorphic function and $R(f) = P(f)/Q(f)$, where $P(f) = \sum_{k=0}^{p} a_k f^k$ and $Q(f) = \sum_{j=0}^{q} b_j f^j$ are two mutually prime polynomials in f. If $T(r, a_k) = S(r, f)$ and $T(r, b_j) = S(r, f)$ for $k = 0, 1, 2, \ldots, p$ and $j = 0, 1, 2, \ldots, q$ and $a_p \not\equiv 0$, $b_q \not\equiv 0$, then $T(r, R(f)) = \max\{p, q\}T(r, f) + S(r, f)$.*

Lemma 3.4. [7] *The coefficients $a_0(\not\equiv 0), a_1, \ldots, a_{n-1}$ of the differential equation $f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_1 f^{(1)} + a_0 f = 0$ are polynomials if and only if all solutions of it are entire functions of finite order .*

Lemma 3.5. *Let f be a non-constant entire function and a_1, a_2 be two non-zero distinct finite numbers. If f and $L(f)$ $(k \geq 1)$ share the set $\{a_1, a_2\}$ CMW and $T(r, h) \neq S(r, f)$, where*

$$h = \frac{(L(f) - a_1)(L(f) - a_2)}{(f - a_1)(f - a_2)}, \tag{3.3}$$

then the following fold:

(i) *$\Psi \not\equiv 0$ and $T(r, \Psi) = S(r, f)$, where*

$$\Psi = \frac{(f^{(1)}h - L^{(1)}(f))(f^{(1)}h + L^{(1)}(f))}{(L(f) - a_1)(L(f) - a_2)}. \tag{3.4}$$

(ii) $T(r,L(f)) = N\left(r, \dfrac{1}{L(f)-a_j}\right) + S(r,f)$ *for* $j=1,2$.

(iii) $m\left(r, \dfrac{1}{f-c}\right) = S(r,f)$, *where* $c \neq a_1, a_2 \in \mathbb{C}$.

(iv)

$$\begin{aligned} T(r,h) &= m\left(r, \frac{1}{f-a_1}\right) + m\left(r, \frac{1}{f-a_2}\right) + S(r,f) \\ &= m\left(r, \frac{1}{f^{(1)}}\right) + S(r,f) \leq m\left(r, \frac{1}{L(f)}\right) + S(r,f). \end{aligned}$$

(v) $2T(r,f) - 2T(r,L(f)) = m\left(r, \dfrac{1}{h}\right) + S(r,f)$.

Proof. Since f and $L(f)$ share the set $\{a_1,a_2\}$ CMW, $N(r,h)+N(r,1/h) = S(r,f)$. Now if $\Psi \equiv 0$, then $h = \pm L^{(1)}(f)/f^{(1)}$. This implies that $T(r,h) = S(r,f)$, which contradicts to our assumption. Therefore $\Psi \not\equiv 0$.

Let z_0 be a zero of $(L(f)-a_1)(L(f)-a_2)$ and $(f-a_1)(f-a_2)$ of multiplicity $p(\geq 2)$. Then z_0 is a zero of $(f^{(1)}h - L^{(1)})(f^{(1)}h + L^{(1)})$ with multiplicity $2(p-1) \geq p$. So, z_0 is not a pole of Ψ.

From (3.3), we get

$$(L(f)-a_1)(L(f)-a_2) = h(f-a_1)(f-a_2). \tag{3.5}$$

Differentiating (3.5), we obtain

$$\begin{aligned} &L^{(1)}(f)(2L(f)-a_1-a_2) \\ &= h^{(1)}(f-a_1)(f-a_2) + hf^{(1)}(2f-a_1-a_2). \end{aligned} \tag{3.6}$$

Let z_0 be a simple zero of $(L(f)-a_1)(L(f)-a_2)$ and $(f-a_1)(f-a_2)$. Then

$$2L(f)(z_1) - a_1 - a_2 = \pm(2f(z_1) - a_1 - a_2).$$

So from (3.6), we get

$$(h(z_1)f^{(1)}(z_1) - (L(f)(z_1))^2)(h(z_1)f^{(1)}(z_1) + (L(f)(z_1))^2) = 0.$$

Hence from (3.4), we see that z_1 is not a pole of Ψ. Since f and $L(f)$ share the set $\{a_1,a_2\}$ CMW, we obtain $N(r,\Psi) = S(r,f)$.

By (3.3), we get

$$\begin{aligned} &\frac{f^{(1)}h - L^{(1)}(f)}{L(f)-a_1} \\ &= \frac{f^{(1)}L(f)}{(f-a_1)(f-a_2)} - \frac{a_2 f^{(1)}}{(f-a_1)(f-a_2)} - \frac{L^{(1)}(f)}{L(f)-a_1}. \end{aligned} \tag{3.7}$$

Since

$$\frac{a_2 f^{(1)}}{(f-a_1)(f-a_2)} = \frac{1}{a_1-a_2}\left(\frac{f^{(1)}}{f-a_1} - \frac{f^{(1)}}{f-a_2}\right),$$

we get from (3.7) that

$$m\left(r, \frac{f^{(1)}h - L^{(1)}(f)}{L(f)-a_1}\right) = S(r,f).$$

Similarly,

$$m\left(r, \frac{f^{(1)}h + L^{(1)}(f)}{L(f)-a_2}\right) = S(r,f).$$

Therefore, from (3.4) we obtain $m(r,\Psi) = S(r,f)$ and hence $T(r,\Psi) = S(r,f)$, which is (i).

Now in view of (3.3), we get from (3.4) that

$$\frac{1}{f^{(1)}h - L^{(1)}(f)} = \frac{1}{\Psi}\left(\frac{f^{(1)}}{(f-a_1)(f-a_2)} + \frac{L^{(1)}(f)}{(L(f)-a_1)(L(f)-a_2)}\right).$$

Therefore,

$$m\left(r, \frac{1}{f^{(1)}h - L^{(1)}(f)}\right) = S(r,f).$$

Similarly, we get

$$m\left(r, \frac{1}{f^{(1)}h + L^{(1)}(f)}\right) = S(r,f).$$

So we obtain

$$\begin{aligned} m\left(r, \frac{1}{L(f)-a_1}\right) &\le m\left(r, \frac{f^{(1)}h - L^{(1)}(f)}{L(f)-a_1}\right) + m\left(r, \frac{1}{f^{(1)}h - L^{(1)}(f)}\right) \\ &= S(r,f) \end{aligned}$$

and $m\left(r, \frac{1}{L(f)-a_2}\right) = S(r,f)$. Therefore,

$$T(r, L(f)) = N\left(r, \frac{1}{L(f)-a_j}\right) + S(r,f),$$

for $j = 1, 2$, which is (ii).

for $c \neq a_1, a_2$, we get from (3.7)

$$\begin{aligned} \frac{f^{(1)}h - L^{(1)}(f)}{(L(f)-a_1)(f-c)} &= \frac{f^{(1)}L(f)}{(f-c)(f-a_1)(f-a_2)} - \frac{a_2 f^{(1)}}{(f-c)(f-a_1)(f-a_2)} \\ &\quad - \frac{L^{(1)}(f)}{(L(f)-a_1)(f-c)}. \end{aligned}$$

We note that

$$\frac{a_2 f^{(1)}}{(f-c)(f-a_1)(f-a_2)} = \alpha\frac{f^{(1)}}{f-c} + \beta\frac{f^{(1)}}{f-a_1} + \gamma\frac{f^{(1)}}{f-a_2},$$

where $\alpha = \dfrac{a_2}{(a_1-c)(a_2-c)}$, $\beta = \dfrac{a_2}{(c-a_1)(a_2-a_1)}$ and $\gamma = \dfrac{a_2}{(c-a_2)(a_1-a_2)}$.
Therefore, we get

$$m\left(r, \frac{f^{(1)}h - L^{(1)}(f)}{(f-c)(L(f)-a_1)}\right) = S(r,f).$$

Since by (3.4),

$$\frac{1}{f-c} = \frac{1}{\Psi}\frac{f^{(1)}h - L^{(1)}(f)}{(f-c)(L(f)-a_1)}\frac{f^{(1)}h + L^{(1)}(f)}{(L(f)-a_2)},$$

we get

$$m\left(r, \frac{1}{f-c}\right) = S(r,f),$$

which is (iii). Since

$$h = \frac{(L(f))^2 - (a_1+a_2)L(f) + a_1a_2}{(f-a_1)(f-a_2)}, \text{ we have}$$

$$\begin{aligned} T(r,h) &= m(r,h) + S(r,f) \le m\left(r, \frac{1}{(f-a_1)(f-a_2)}\right) + S(r,f) \\ &\le m\left(r, \frac{1}{f^{(1)}}\right) + m\left(r, \frac{f^{(1)}}{(f-a_1)(f-a_2)}\right) + S(r,f) \\ &\le m\left(r, \frac{1}{f^{(1)}}\right) + S(r,f). \end{aligned} \tag{3.8}$$

Since

$$\begin{aligned} \frac{\Psi}{f^{(1)}} &= \frac{f^{(1)}}{(f-a_1)(f-a_2)}\frac{(L(f))^2 - (a_1+a_2)L(f) + a_1a_2}{(f-a_1)(f-a_2)} \\ &\quad - \frac{L^{(1)}(f)}{f^{(1)}}\frac{L^{(1)}(f)}{(L(f)-a_1)(L(f)-a_2)}, \end{aligned}$$

we get by (i) that

$$\begin{aligned} m\left(r, \frac{1}{f^{(1)}}\right) &\le m\left(r, \frac{\Psi}{f^{(1)}}\right) + S(r,f) \\ &\le m\left(r, \frac{1}{(f-a_1)(f-a_2)}\right) + S(r,f). \end{aligned} \tag{3.9}$$

Since $\dfrac{1}{(f-a_1)(f-a_2)} = \dfrac{h}{(L(f)-a_1)(L(f)-a_2)}$, we have by (ii) that

$$m\left(r, \frac{1}{(f-a_1)(f-a_2)}\right) \le T(r,h) + S(r,f). \tag{3.10}$$

From (3.8), (3.9) and (3.10), we have

$$T(r,h) = m\left(r, \frac{1}{f-a_1}\right) + m\left(r, \frac{1}{f-a_2}\right) + S(r,f) \le m\left(r, \frac{1}{L(f)}\right) + S(r,f),$$

which is (iv).

Keeping in view of (3.3), we get from (ii) and (iv) that

$$\begin{aligned}
2T(r,L(f)) &= N\left(r,\frac{1}{L(f)-a_1}\right)+N\left(r,\frac{1}{L(f)-a_2}\right)+S(r,f)\\
&= N\left(r,\frac{1}{(L(f)-a_1)(L(f)-a_2)}\right)+S(r,f)\\
&= N\left(r,\frac{1}{h(f-a_1)(f-a_2)}\right)+S(r,f)\\
&= 2T(r,f)-m\left(r,\frac{1}{f-a_1}\right)-m\left(r,\frac{1}{f-a_2}\right)+N\left(r,\frac{1}{h}\right)\\
&+ S(r,f)\\
&= 2T(r,f)-T(r,h)+N\left(r,\frac{1}{h}\right)+S(r,f).
\end{aligned}$$

Therefore, $2T(r,f)-2T(r,L(f)) = m\left(r,\frac{1}{h}\right)+S(r,f)$, which is (v). This completes the proof of the lemma. □

Lemma 3.6. *Let f be a non-constant entire function and a_1, a_2 be two distinct finite complex numbers. If f and $L(f)$ share the set $\{a_1,a_2\}$ CMW, then $T(r,h)=S(r,f)$, where h is defined in Lemma 3.5.*

Proof. Since f and $L(f)$ share the set $\{a_1,a_2\}$ CMW, we must have $N(r,h)=S(r,f)$ and $N(r,1/h)=S(r,f)$. Assume on the contrary that $T(r,h)\not\equiv S(r,f)$. By Lemma 3.5, we know that $\Psi\not\equiv 0$ and $T(r,\Psi)=S(r,f)$.

Differentiating (3.3), we get

$$\begin{aligned}
&2L(f)L^{(1)}(f)-(a_1+a_2)L^{(1)}(f)\\
&=(2ff^{(1)}-(a_1+a_2)f^{(1)})h+h^{(1)}(f-a_1)(f-a_2).
\end{aligned}\tag{3.11}$$

From (3.3) and (3.11), we obtain

$$\frac{(2L(f)-(a_1+a_2))L^{(1)}(f)}{(L(f)-a_1)(L(f)-a_2)}=\frac{(2f-(a_1+a_2))f^{(1)}}{(f-a_1)(f-a_2)}+\frac{h^{(1)}}{h}.$$

Squaring the above equation, we get

$$\begin{aligned}
&\frac{((2L(f)-(a_1+a_2))^2(L^{(1)}(f))^2}{(L(f)-a_1)^2(L(f)-a_2)^2}\\
&=\frac{((2f-(a_1+a_2)))^2(f^{(1)})^2}{(f-a_1)^2(f-a_2)^2}+\beta^2+\frac{2\beta(2f-(a_1+a_2))f^{(1)}}{(f-a_1)(f-a_2)},
\end{aligned}$$

where $\beta=h^{(1)}/h$.

Eliminating $L^{(1)}(f)$ from (3.3), (3.4) and the above equation, we get

$$\begin{aligned}\frac{(2L(f)-(a_1+a_2))^2\Psi}{(L(f)-a_1)(L(f)-a_2)} &= \frac{4(L(f)+f-(a_1+a_2))(L(f)-f)(f^{(1)})^2}{(f-a_1)^2(f-a_2)^2}-\beta^2\\ &\quad -\frac{2\beta(2f-(a_1+a_2))f^{(1)}}{(f-a_1)(f-a_2)}. \end{aligned} \tag{3.12}$$

Let z_0 be a zero of $(f-a_1)(f-a_2)$ which is also a zero of $(L(f)-a_1)(L(f)-a_2)$. Since f and $L(f)$ share the set $\{a_1, a_2\}$ CMW and $T(r, \beta) = S(, f)$, almost all the poles of right hand side of (3.12) are simple, and hence it follows from the same equation that "almost all" the zeros of $(L(f)-a_1)(L(f)-a_2)$ are simple as long as they are not the zeros of Ψ. Thus

$$N\left(r, \frac{1}{L(f)-a_j}\right) = \overline{N}\left(r, \frac{1}{L(f)-a_j}\right) + S(r, f),\ \text{j=1, 2.} \tag{3.13}$$

Differentiating (3.4), we get

$$\begin{aligned}2h^2f^{(1)}(f^{(2)}+\beta f^{(1)})-2L^{(1)}(f)L^{(2)}(f) &= \Psi^{(1)}(L(f)-a_1)(L(f)-a_2)\\ &\quad +\Psi(2L(f)-(a_1+a_2))L^{(1)}(f).\end{aligned}$$

Now eliminating h from the above equation by using (3.4), we get

$$\begin{aligned}&\left[2\Psi(f^{(2)}+\beta f^{(1)})-f^{(1)}\Psi^{(1)}\right](L(f)-a_1)(L(f)-a_2) =\\ &L^{(1)}\left[2f^{(1)}\Psi L-(a_1+a_2)f^{(1)}\Psi-2(\beta f^{(1)}+f^{(2)})L^{(1)}+2f^{(1)}L^{(2)}\right].\end{aligned} \tag{3.14}$$

From the above equation, we see that any simple zeros of $(L(f)-a_1)(L(f)-a_2)$ must be the zeros of $2f^{(1)}\Psi L(f)-(a_1+a_2)f^{(1)}\Psi-2(\beta f^{(1)}+f^{(2)})L^{(1)}(f)+2f^{(1)}L^{(2)}(f)$.

Let

$$\Psi_1 = \frac{2f^{(1)}\Psi L(f)-(a_1+a_2)f^{(1)}\Psi-2(\beta f^{(1)}+f^{(2)})L^{(1)}(f)+2f^{(1)}L^{(2)}(f)}{(f-a_1)(f-a_2)}. \tag{3.15}$$

Since f and $L(f)$ share the set $\{a_1, a_2\}$ CMW and "almost all" the zeros of $(L(f)-a_1)(L(f)-a_2)$ are simple, we must have $N(r, \Psi_1) = S(r, f)$.

On the hand, by the lemma of logarithmic derivative, it can be easily seen that $m(r, \Psi_1) = S(r, f)$. Hence, $T(r, \Psi_1) = S(r, f)$.

We now consider the following two cases:

Case 1: $\Psi_1 \not\equiv 0$. Then it follows from (3.15) that

$$\begin{aligned}
2T(r,f) &= T(r,(f-a_1)(f-a_2)) + S(r,f) \\
&= m(r,(f-a_1)(f-a_2)) + S(r,f) \\
&\leq m(r,(f-a_1)(f-a_2)\Psi_1) + m\left(r,\frac{1}{\Psi_1}\right) + S(r,f) \\
&\leq m(r,f^{(1)}) + m(r,L(f)) + T(r,\Psi_1) + S(r,f) \\
&\leq T(r,f) + T(r,L(f)) + S(r,f).
\end{aligned}$$

Therefore, $T(r,f) \leq T(r,L(f)) + S(r,f)$.

Since $L(f)$ is a linear differential polynomial in f, we get

$$T(r,L(f)) \leq T(r,f) + S(r,f).$$

Combining the above two we have $T(r,f) = T(r,L(f)) + S(r,f)$.

By Lemma 3.5 (ii) and (3.13), we get

$$2T(r,f) = \overline{N}\left(r,\frac{1}{f-a_1}\right) + \overline{N}\left(r,\frac{1}{f-a_2}\right) + S(r,f),$$

which implies that

$$m\left(r,\frac{1}{f-a_1}\right) + m\left(r,\frac{1}{f-a_2}\right) = S(r,f).$$

Thus from (3.3), the lemma of logarithmic derivative and the above observation, we get

$$\begin{aligned}
T(r,h) &= N(r,h) + m(r,h) \\
&\leq m\left(r,\frac{(L(f))^2}{(f-a_1)(f-a_2)}\right) + m\left(r,\frac{a_1a_2}{(f-a_1)(f-a_2)}\right) \\
&\quad +2m\left(r,\frac{L(f)}{(f-a_1)(f-a_2)}\right) + S(r,f) = S(r,f).
\end{aligned}$$

i.e., $T(r,h) = S(r,f)$, which contradicts to our assumption.

Case 2: $\Psi_1 \equiv 0$. Then from (3.14) and (3.15), we obtain

$$\frac{\Psi^{(1)}}{\Psi} = 2\left(\frac{h^{(1)}}{h} + \frac{f^{(2)}}{f^{(1)}}\right).$$

Integrating above, we get

$$(hf^{(1)})^2 = c\Psi, \tag{3.16}$$

where c is a non-zero constant.

It follows from (3.4) and (3.16) that

$$\begin{aligned}
(L^{(1)}(f))^2 &= -((L(f))^2 - (a_1+a_2)L(f) + (a_1a_2 - c))\Psi \\
&= -(L(f)-d_1)(L(f)-d_2)\Psi,
\end{aligned}$$

where d_1 and d_2 are two complex constants. If $d_1 \neq d_2$, then by the lemma of logarithmic derivative, we get

$$m\left(r, \frac{1}{L^{(1)}(f)}\right) = m\left(r, \frac{-L^{(1)}(f)}{(L(f)-d_1)(L(f)-d_2)}\right) = S(r,f).$$

Therefore,

$$\begin{aligned} m\left(r, \frac{1}{(f-a_1)(f-a_2)}\right) &\leq m\left(r, \frac{L^{(1)}(f)}{(f-a_1)(f-a_2)}\right) + m\left(r, \frac{1}{L^{(1)}(f)}\right) \\ &= S(r,f). \end{aligned}$$

Hence, in view of the above, we get from (3.3) and the lemma of logarithmic derivative

$$\begin{aligned} T(r,h) &= N(r,h) + m(r,h) \\ &\leq m\left(r, \frac{(L(f))^2}{(f-a_1)(f-a_2)}\right) + m\left(r, \frac{L(f)}{(f-a_1)(f-a_2)}\right) \\ &\quad + m\left(r, \frac{1}{(f-a_1)(f-a_2)}\right) + S(r,f) \\ &= S(r,f), \end{aligned}$$

which contradicts to our assumption.

Therefore, $d_1 = d_2 = (a_1+a_2)/2 = d$, say. Hence,

$$(L^{(1)}(f))^2 = -\Psi(L(f)-d)^2. \tag{3.17}$$

From (3.3), (3.16) and (3.17), we get

$$(L(f)-d)(L(f)-a_1)(L(f)-a_2) = c_2(f-a_1)(f-a_2)\Psi_2, \tag{3.18}$$

where c_2 is a non-zero constant satisfying $c_2^2 = -c$ and $\Psi_2 = L^{(1)}(f)/f^{(1)}$.

From (3.16), it can be easily seen that $N(r, 1/f^{(1)}) = S(r,f)$. Therefore, $N(r,\Psi_2) = S(r,f)$. On the other hand, by the lemma of logarithmic derivative, we have $m(r,\Psi_2) = S(r,f)$, and hence $T(r,\Psi_2) = S(r,f)$.

Since $\Psi_2 \not\equiv 0$, it follow from (3.18) that

$$3T(r, L(f)) = 2T(r,f) + S(r,f). \tag{3.19}$$

Let

$$\Psi_3 = \frac{L^{(1)}(f)}{L(f)-d}. \tag{3.20}$$

Then from (3.17), we get $\Psi_3^2 = -\Psi$. Hence, $T(r,\Psi_3) = S(r,f)$ and $\Psi_3 \equiv 0$.

Now from (3.3) and (3.11), we get

$$(f-d)hf^{(1)} = (L(f)-d)^2\Psi_3 - \frac{1}{2}\beta(L(f)-a_1)(L(f)-a_2). \tag{3.21}$$

By (3.16), we get

$$T(r, hf^{(1)}) = S(r,f).$$

Therefore, by (3.21), we obtain

$$T(r,f)=T(r,L(f)),\ \text{or}\ T(r,f)=2T(r,L(f))+S(r,f) \tag{3.22}$$

according as when $\Psi_3=\beta/2$ or not.

Combining (3.19) and (3.22), we get $T(r,f)=S(r,f)$, which is a contradiction.

Hence $T(r,h)=S(r,f)$. This completes the proof of the lemma. □

Lemma 3.7. *Let f be a non-constant entire function and a_1,a_2 be two non-zero distinct finite numbers. If f^λ and $M(f)$ ($k\geq 1$) share the set $\{a_1,a_2\}$ CMW and $T(r,h_1)\not\equiv S(r,f)$, where*

$$h_1=\frac{(M(f)-a_1)(M(f)-a_2)}{(f^\lambda-a_1)(f^\lambda-a_2)}, \tag{3.23}$$

then the following fold:

(i) $\Phi\not\equiv 0$ *and* $T(r,\Phi)=S(r,f)$, *where*

$$\Phi=\frac{((f^\lambda)^{(1)}h_1-(M(f))^{(1)})((f^\lambda)^{(1)}h_1+(M(f))^{(1)})}{(M(f)-a_1)(M(f)-a_2)}. \tag{3.24}$$

(ii) $T(r,M(f))=N\left(r,\dfrac{1}{M(f)-a_j}\right)+S(r,f)$ *for* $j=1,2$.

(iii) $m\left(r,\dfrac{1}{f^\lambda-c}\right)=S(r,f)$, *where* $c\neq a_1,a_2\in\mathbb{C}$.

(iv)

$$\begin{aligned}T(r,h_1)&=m\left(r,\frac{1}{f^\lambda-a_1}\right)+m\left(r,\frac{1}{f^\lambda-a_2}\right)+S(r,f)\\&=m\left(r,\frac{1}{(f^\lambda)^{(1)}}\right)+S(r,f)\leq m\left(r,\frac{1}{M(f)}\right)+S(r,f).\end{aligned}$$

(v) $2\lambda T(r,f)-2T(r,M(f))=m\left(r,\dfrac{1}{h_1}\right)+S(r,f)$.

Proof. The proof of this lemma can be carried out in a similar manner as done in the proof of Lemma 3.5. So, we omit the details. □

Lemma 3.8. *Let f be a non-constant entire function and a_1, a_2 be two distinct finite complex numbers. If f^λ and $M(f)$ share the set $\{a_1,a_2\}$ CMW, then $T(r,h_1)=S(r,f)$, where h_1 is defined in Lemma 3.7.*

Proof. The proof of this lemma is essentially can be done in a similar manner as Lemma 3.6. So, we omit the details. □

4. Proof of the main results

Proof of Theorem 2.1. Let 2η be the principal branch of $\log h$, where h is defined as in Lemma 3.5. Then by Lemma 3.6, we obtain

$$T(r,e^{\eta})=\frac{1}{2}T(r,h)+S(r,f)=S(r,f).$$

Also (3.3) can be written as

$$(L(f)-a_1)(L(f)-a_2)=e^{2\eta}(f-a_1)(f-a_2). \tag{4.1}$$

And so

$$GH=\left(\frac{a_1-a_2}{2}\right)^2(e^{2\eta}-1), \tag{4.2}$$

where

$$G=e^{\eta}f-\frac{a_1+a_2}{2}e^{\eta}+L(f)-\frac{a_1+a_2}{2}$$

and

$$H=e^{\eta}f-\frac{a_1+a_2}{2}e^{\eta}-L(f)+\frac{a_1+a_2}{2}.$$

If $e^{2\eta}\equiv 1$, then from (4.1), we get

$$(f-L(f))(f+L(f)-a_1-a_2)=0,$$

which implies that either $f=L(f)$, or $f+L(f)=a_1+a_2$.

Now suppose that $e^{2\eta}\not\equiv 1$. Since f is entire we get $N(r,G)+N(r,H)=S(r,f)$, and so, from (4.2), we get $N(r,1/H)+N(r,1/G)=S(r,f)$. Therefore,

$$T\left(r,\frac{G^{(j)}}{G}\right)+T\left(r,\frac{H^{(j)}}{H}\right)=S(r,f), \tag{4.3}$$

where $j=1,2,\ldots,k$.

Suppose $f^{(1)}=bL(f)$. Then using the condition (1.1), the lemma of logarithmic derivative, and the first fundamental theorem of Nevalinna, it is easily seen that $T(r,b)=S(r,f)$.

From the definition of G and H it follows that

$$G+H=e^{\eta}(2f-a_1-a_2) \tag{4.4}$$

and

$$G-H=2L(f)-a_1-a_2=2\lambda f^{(1)}-a_1-a_2, \tag{4.5}$$

where $b\lambda=1$ and $T(r,\lambda)=S(r,f)$ as $T(r,b)=S(r,f)$.

Eliminating f and $f^{(1)}$, from (4.4) and (4.5), we get

$$\left(e^{\eta}+\lambda\eta^{(1)}-\lambda\frac{G^{(1)}}{G}\right)G+\left(\lambda\eta^{(1)}-e^{\eta}-\lambda\frac{H^{(1)}}{H}\right)H$$
$$+b(a_1+a_2)=0. \tag{4.6}$$

Now eliminating H from (4.2) and (4.6), we obtain

$$\Phi_1 G^2 + \Phi_2 G + \Phi_3 = 0, \tag{4.7}$$

where

$$\Phi_1 = e^{\eta} + \lambda\eta^{(1)} - \lambda\frac{G^{(1)}}{G}, \tag{4.8}$$

$$\Phi_2 = \lambda\eta^{(1)} - e^{\eta} - \lambda\frac{H^{(1)}}{H}\left(\frac{a_1 - a_2}{2}\right)^2 (e^{2\eta} - 1), \tag{4.9}$$

$$\Phi_3 = \lambda(a_1 + a_2). \tag{4.10}$$

If $\Phi_1 \not\equiv 0$ or $\Phi_2 \not\equiv 0$, then by Lemma 3.2, we see from (4.7) that $T(r, G) = S(r, f)$, and therefore from (4.4), we get $T(r, f) = S(r, f)$, which is a contradiction. Therefore, $\Phi_1 = \Phi_2 = 0$. Then from (4.7), we get $\Phi_3 = 0$. This implies that

$$e^{\eta} + \lambda\eta^{(1)} - \lambda\frac{G^{(1)}}{G} = 0, \tag{4.11}$$

$$\lambda\eta^{(1)} - e^{\eta} - \lambda\frac{H^{(1)}}{H} = 0, \tag{4.12}$$

$$a_1 + a_2 = 0. \tag{4.13}$$

Adding (4.11) and (4.12), we get

$$\frac{G^{(1)}}{G} + \frac{H^{(1)}}{H} = 2\eta^{(1)},$$

and so by integration , we have

$$GH = c_0 e^{2\eta}, \tag{4.14}$$

where c_0 is a non-zero constant.

Now from (4.2), (4.13) and (4.14), we get $e^{2\eta} = A$, where A is a constant.

From (4.4), (4.5) and (4.13), we get

$$\left(\sqrt{A} - \sum_{j=1}^{k} b_j \frac{G^{(j)}}{G}\right) G^2 = \left(\sqrt{A} + \sum_{j=1}^{k} b_j \frac{H^{(j)}}{H}\right) B, \tag{4.15}$$

where $B = (a_1 - a_2)^2/4(A - 1)$, constant.

If $\sqrt{A} - \sum_{j=1}^{k} b_j G^{(j)}/G \not\equiv 0$, then from (4.3) and (4.15), we get $T(r, G) = S(r, f)$ and so from (4.14), we get $T(r, F) = S(r, f)$. Therefore, from (4.4), we get $T(r, f) = S(r, f)$, which is a contradiction. hence we have $\sum_{j=1}^{k} b_j G^{(j)} - \sqrt{A}G = 0$ and $\sum_{j=1}^{k} b_j H^{(j)} + \sqrt{A}H = 0$. This implies by Lemma 3.4 that G and H are of finite order. Also from (4.14), we see that G and H do not assume the value 0.

Therefore, let us assume that $G = e^P$ and $H = e^Q$, where $P,\ Q$ are polynomials of degree p and q, respectively. Differentiating j times, we obtain $G^{(j)} = P_j e^P$ and $H^{(j)} = Q_j e^Q$, where P_j and Q_j are polynomials of degree $(p-1)j$ and $(q-1)j$, respectively. Since $\sum_{j=1}^{k} b_j G^{(j)} = \sqrt{A}G$ and $\sum_{j=1}^{k} b_j H^{(j)} = \sqrt{A}H$, we have $p = q = 1$. Hence in view of (4.14), we may write $G = 2d_1 e^{cz}$ and $H = 2d_2 e^{-cz}$, where $c,\ d_1,\ d_2$ are non-zero constants.

Now from (4.4) and (4.13), we get

$$f = c_1 e^{cz} + c_2 e^{-cz}, \tag{4.16}$$

where $c_1 = d_1/\sqrt{A}$ and $c_2 = d_2/\sqrt{A}$.

Differentiating (4.16), we have

$$f^{(j)} = \frac{c_1 c^j e^{2cz} + c_2(-c)^j}{e^{cz}},$$

where $j = 1, 2, \ldots, k$. Therefore,

$$L(f) = \frac{\sum_{j=1}^{k} b_j (c_j c^j e^{2cz} + c_2(-c)^j)}{e^{cz}}. \tag{4.17}$$

Again from (4.5) and (4.13), we get

$$L(f) = \frac{\sqrt{A}(c_1 e^{2cz} - c_2)}{e^{cz}}. \tag{4.18}$$

Comparing (4.17) and (4.18), we obtain

$$b_1 c + b_2 c^2 + \cdots + b_k c^k = \sqrt{A} \tag{4.19}$$

and

$$-b_1 c + b_2 c^2 - \cdots + (-1)^k b_k c^k = -\sqrt{A}. \tag{4.20}$$

from (4.19) and (4.20), it is clear that $A = (b_1 c + b_3 c^3 + \cdots + b_k c^k)^2$, where k is an odd positive integer.

Now from (4.2) and (4.13), we see that $4d_1 d_2 = a_1^2(A-1)$ and so

$$4c_1 c_2 A = a_1^2(A-1),$$

where $A = (b_1 c + b_3 c^3 + \cdots + b_k c^k)^2$, where k is an odd positive integer. This completes the proof of the Theorem 2.1. □

Proof of Theorem 2.2. Let 2ξ be the principal branch of $\log h_1$, where h_1 is defined as in 3.23. Then by Lemma 3.8, we obtain

$$T(r, e^{\xi}) = \frac{1}{2}T(r, h_1) + S(r, f) = S(r, f).$$

Also (3.23) can be written as

$$(M(f) - a_1)(M(f) - a_2) = e^{2\xi}(f - a_1)(f - a_2), \tag{4.21}$$

and so

$$G_1H_1 = \left(\frac{a_1 - a_2}{2}\right)^2 (e^{2\xi} - 1), \tag{4.22}$$

where

$$G_1 = e^{\xi} f^{\lambda} - \frac{a_1 + a_2}{2} e^{\xi} + M(f) - \frac{a_1 + a_2}{2}$$

and

$$H_1 = e^{\xi} f - \frac{a_1 + a_2}{2} e^{\xi} - M(f) + \frac{a_1 + a_2}{2}.$$

If $e^{2\xi} \equiv 1$, then from (4.21), we get

$$(f^{\lambda} - M(f))(f^{\lambda} + M(f) - a_1 - a_2) = 0,$$

which implies that either $f^{\lambda} = M(f)$, or $f^{\lambda} + M(f) = a_1 + a_2$.

Now suppose that $e^{2\xi} \not\equiv 1$. Since f is entire we get $N(r, G_1) + N(r, H_1) = S(r, f)$, and so, from (4.22), we get $N(r, 1/H_1) + N(r, 1/G_1) = S(r, f)$. Therefore,

$$T\left(r, \frac{G_1^{(j)}}{G_1}\right) + T\left(r, \frac{H_1^{(j)}}{H_1}\right) = S(r, f), \tag{4.23}$$

where $j = 1, 2, \ldots, k$.

Suppose $(f^{\lambda})^{(1)} = b_1 M(f)$. Then using the condition (2.2), the Lemma of logarithmic derivative, and the first fundamental theorem of Nevalinna, it is easily seen that $T(r, b_1) = S(r, f)$.

From the definition of G_1 and H_1 it follows that

$$G_1 + H_1 = e^{\xi}(2f^{\lambda} - a_1 - a_2) \tag{4.24}$$

and

$$G_1 - H_1 = 2M(f) - a_1 - a_2 = 2\mu(f^{\lambda})^{(1)} - a_1 - a_2, \tag{4.25}$$

where $b\mu = 1$ and so $T(r, \mu) = S(r, f)$ as $T(r, b) = S(r, f)$.

Eliminating f^{λ} and $(f^{\lambda})^{(1)}$ from (4.24) and (4.25), we obtain

$$\begin{aligned} &\left(e^{\xi} + \mu\xi^{(1)} - \mu\frac{G_1^{(1)}}{G_1}\right) G_1 + \left(\mu\xi^{(1)} - e^{\xi} - \mu\frac{H_1^{(1)}}{H_1}\right) H_1 \\ &+ b_1(a_1 + a_2) = 0. \end{aligned} \tag{4.26}$$

Now eliminating H_1 from (4.22) and (4.26), we obtain

$$\chi_1 G^2 + \chi_2 G + \chi_3 = 0, \tag{4.27}$$

where

$$\chi_1 = e^{\xi} + \mu\xi^{(1)} - \mu\frac{G_1^{(1)}}{G_1}, \tag{4.28}$$

$$\chi_2 = \mu\xi^{(1)} - e^{\xi} - \mu\frac{H_1^{(1)}}{H_1}\left(\frac{a_1 - a_2}{2}\right)^2 (e^{2\xi} - 1), \tag{4.29}$$

$$\chi_3 = \mu(a_1 + a_2). \tag{4.30}$$

If $\chi_1 \not\equiv 0$ or $\chi_2 \not\equiv 0$, then by Lemma 3.2, we get from (4.27) that $T(r, G_1) = S(r, f)$, and so from (4.22), we get $T(r, H_1) = S(r, f)$. So, from (4.24), we get $T(r, f) = S(r, f)$, which is a contradiction. Therefore, $\chi_1 = \chi_2 = 0$. Then from (4.27), we get $\chi_3 = 0$. This implies that

$$e^{\xi} + \mu\xi^{(1)} - \mu\frac{G_1^{(1)}}{G_1} = 0, \tag{4.31}$$

$$\mu\xi^{(1)} - e^{\xi} - \mu\frac{H_1^{(1)}}{H_1} = 0, \tag{4.32}$$

$$a_1 + a_2 = 0. \tag{4.33}$$

Adding (4.31) and (4.32), we get

$$\frac{G_1^{(1)}}{G_1} + \frac{H_1^{(1)}}{H_1} = 2\xi^{(1)},$$

and so by integration , we have

$$G_1 H_1 = c_0^* e^{2\xi}, \tag{4.34}$$

where c_0^* is a non-zero constant.

Now from (4.22), (4.33) and (4.34), we get $e^{2\xi} = A$, where A is a constant.

From (4.24), (4.25) and (4.33), we get

$$\left(\frac{\sqrt{A}}{\mu} - \frac{G_1^{(1)}}{G_1}\right) G_1^2 = -\left(\frac{\sqrt{A}}{\mu} - \frac{H_1^{(1)}}{H_1}\right) B, \tag{4.35}$$

where $B = (a_1 - a_2)^2/4(A - 1)$, constant.

If $\sqrt{A}/\mu - G_1^{(1)}/G_1 \not\equiv 0$, then from (4.23) and (4.35), we get $T(r, G_1) = S(r, f)$ and so from (4.34), we get $T(r, H_1) = S(r, f)$. Therefore, from (4.24), we get $T(r, f) = S(r, f)$, which is a contradiction. Hence we must have $\mu G_1^{(1)} - \sqrt{A}G_1 = 0$ and $\mu H_1^{(1)} - \sqrt{A}H_1 = 0$. This implies by Lemma 3.4 that G_1 and H_1 are of finite order. Also from (4.34), we see that G_1 and H_1 do not assume the value 0.

Therefore, we may assume that $G_1 = e^P$ and $H_1 = e^Q$, where P, Q are polynomials of degree p and q, respectively.

Differentiating once, we get $G_1^{(1)} = P^{(1)}e^P$ and $H_1^{(1)} = Q^{(1)}e^Q$. Therefore, $P^{(1)}$ and $Q^{(1)}$ are polynomials of degree $(p-1)$ and $(q-1)$, respectively. Since $\mu G_1^{(1)} = \sqrt{A}G_1$ and $\mu H_1^{(1)} = \sqrt{A}H_1$, we have $p = q = 1$. Hence in view of

(4.34), we may write $G_1 = 2d_1^* e^{cz}$ and $H = 2d_2^* e^{-cz}$, where c, d_1^*, d_2^* are non-zero constants.

Now from (4.24), (4.25) and (4.33), we get

$$f^\lambda = c_1 e^{cz} + c_2 e^{-cz},\ M(f) = \frac{\sqrt{A}(c_1 e^{2cz} - c_2)}{e^{cz}},$$

where $c_1 = d_1^*/\sqrt{A}$ and $c_2 = d_2^*/\sqrt{A}$. This completes the proof of the theorem. □

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