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# PERFECT POWERS AS DIFFERENCE OF PERRIN NUMBERS AND PADOVAN NUMBERS

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**Abstract.** In this paper, we investigate the perfect powers that are the difference between Perrin numbers $(R_k)_{k\geq 0}$ and Padovan numbers $(P_k)_{k\geq 0}$. Hence, we solve the equations $P_n = x^a$, $2P_n = x^a$, $P_n - R_m = x^a$, or $R_n - P_m = x^a$ such that $a \geq 1$ and $2 \leq x \leq 10$ are positive integers and $n$, $m$, and $k$ are non-negative integers.

## 1. Introduction

In recent years, researchers have studied generalized Fibonacci numbers and their multiples and Lucas numbers under certain conditions. They have used congruence, properties of the Legendre symbol, and linear forms of logarithms in their studies. Bugeaud et al. [6] found that the perfect powers among Lucas numbers are $L_n \in \{1, 4\}$ and among Fibonacci numbers are $F_n \in \{0, 1, 8, 144\}$. Bugeaud et al. [7] determined all non-negative integer solutions of the equations $F_n \pm 1 = y^p$ for $p \geq 2$. Farrokhi [11] solved the equation $F_n = kF_m$ and found that it has a solution $(n, m)$ where $k = F_{a_1} F_{a_2} \cdots F_{a_n}$. Bravo and Luca [4] showed that the perfect power of 2 can be expressed as a sum of two Fibonacci numbers. Tiebekabe and Diouf [24] discovered that the perfect power of 2 can be expressed as a difference of two Lucas numbers. Patel et al. [19] determined that the perfect power of an odd prime $p$ can be represented as either a sum or a difference of two Lucas numbers, where $p < 10^3$. Luca and Patel [17] showed that all perfect powers are sums of two Fibonacci numbers where $p \geq 2$ and $n \equiv m(\text{mod } 2)$. Erduvan and Keskin [10] proved that all non-negative integer solutions of the equation $F_n - F_m = 5^a$ are $(n, m, a) \in \{(4, 3, 1), (6, 4, 1), (7, 6, 1), (5, 0, 1), (1, 0, 0), (3, 1, 0), (2, 0, 0), (3, 2, 0)\}$. Şiar and Keskin [23] determined that all non-negative integer solutions of the equation $F_n - F_m = 2^a$ are $(n, m, a) \in \{(5, 2, 2), (5, 4, 1), (1, 0, 0), (2, 0, 0), (3, 0, 1), (3, 1, 0), (3, 2, 0), (4, 1, 1), (4, 2, 1), (4, 3, 0), (5, 1, 2), (8, 5, 4), (8, 7, 3), (9, 3, 5), (6, 0, 3), (7, 5, 3)\}$. Bitim and Keskin [3] demonstrated that all non-negative integer solutions of the equation $F_n - F_m = 3^a$ are $(n, m, a) \in \{(11, 6, 4), (6, 5, 1), (3, 1, 0),$

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$(3,2,0),(4,0,1),(4,3,0),(5,3,1),(1,0,0),\ (2,0,0)\}$.
Some authors have conducted similar research on generalized Lucas numbers $(V_n)$ and generalized Fibonacci numbers $(U_n)$ under certain assumptions. Keskin and Demirtürk [14] demonstrated that there is no solution to the equation $L_n = L_m L_r x^2$ under certain assumptions. Keskin and Şiar [15] investigated the equations $U_n = U_r U_m$, $V_n = V_r V_m x^2$, and $V_n = V_m V_r$. Some authors have examined equations such as $U_n = x^2, U_n = kx^2, V_n = x^2, U_n = kU_m x^2, V_n = kx^2$, and $V_n = kV_m x^2$ under certain assumptions involving $k$, $P$, and $Q$, where $U_n = U_n(P,Q)$ and $V_n = V_n(P,Q)$. Rihane and Togbé [22] demonstrated Padovan numbers which are Pell or Pell-Lucas numbers, and Perrin numbers which are Pell or Pell-Lucas numbers. For more details, see [11]-[12].

Assume that $\eta$ is an algebraic number of degree $d$ with minimal polynomial

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0\prod_{j=1}^{d}\left(x-\eta^{(j)}\right) \in \mathbb{Z}[x]$$

where the $\eta^{(j)}$'s are conjugates of $\eta$ and the $a_j$'s are relatively prime integers with $a_0 > 0$. Then,

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{j=1}^{d}\log\left(\max\left\{1, |\eta^{(j)}|\right\}\right)\right)$$

is called the logarithmic height of $\eta$. Moreover, if $\eta = a/b$ is a rational number with $b \geq 1$ and $\gcd(a,b) = 1$, then $h(\eta) = \log(\max\{|a|,|b|\})$. Here are the recalled inequalities based on the reference [8]:

$$h(\gamma \pm \eta) \leq \log 2 + h(\gamma) + h(\eta)$$

$$h(\gamma^{\pm m}\eta^{\pm r}) \leq |m|h(\gamma) + |r|h(\eta).$$

Let $(R_k)_{k\geq 0}$ be the sequence of Perrin numbers given by

$$R_0 = 3,\ R_1 = 0,\ R_2 = 2,\ R_k = R_{k-2} + R_{k-3}$$

and $(P_k)_{k\geq 0}$ be the sequence of Padovan numbers given by

$$P_0 = P_1 = P_2 = 1,\ P_k = P_{k-2} + P_{k-3}$$

for $k \geq 3$. Suppose that

$$\alpha = \left((9-\sqrt{69})/18\right)^{1/3} + \left((9+\sqrt{69})/18\right)^{1/3}$$

and

$$\overline{\gamma} = \beta = -\left(\left((27-3\sqrt{69})/2\right)^{1/3}(1+i\sqrt{3})/6\right) - \left(\left((9+\sqrt{69})/18\right)^{1/3}(1-i\sqrt{3})/2\right)$$

are the roots of the characteristic equation $x^3 - x - 1 = 0$. The Binet formulas for these numbers are:

$$P_k = t\alpha^k + s\beta^k + r\gamma^k$$
$$R_k = \alpha^k + \beta^k + \gamma^k$$

where

$$23t = \alpha^3 + 7\alpha^2 + 2, 23s = \beta^3 + 7\beta^2 + 2, r = \overline{s}.$$

and the minimal polynomial of $t$ over $\mathbb{Z}$ is given by $23x^3 - 23x^2 + 6x - 1$. Moreover, with simple calculation, for $k \geq 1$, it can be shown that

$$0.86 < |\beta| = |\gamma| = \alpha^{-1/2} < 0.87$$

$$1.32 < \alpha < 1.33$$

$$0.24 < |s| = |r| < 0.25$$

$$h(t) \leq \frac{1}{3}\log 23$$

$$|e(k)| := \left|P_k - t\alpha^k\right|$$
$$\leq |s|\alpha^{-k/2} + |r|\alpha^{-k/2}$$
$$< 0.5\alpha^{-k/2} \tag{1}$$

and

$$|e'(k)| := \left|R_k - \alpha^k\right|$$
$$\leq |\beta|^k + |\gamma|^k < 2\alpha^{-k/2}. \tag{2}$$

By induction method, it can be observed that

$$R_k = P_{k+1} + P_{k-10}$$
$$\alpha^{k-2} \leq P_k \leq \alpha^{k-1}, \text{ for } k \neq 3, \ k \geq 1$$

and

$$\alpha^{k-2} \leq R_k \leq \alpha^{k+1}, \text{ for } k \geq 2.$$

It is clear that

$$[\mathbb{Q}(\alpha,\beta):\mathbb{Q}] \simeq \{(1), (\alpha\beta), (\alpha\gamma), (\alpha\gamma\beta), (\beta\gamma), (\alpha\beta\gamma)\} \simeq S_3,$$
$$[\mathbb{Q}(\alpha,\beta):\mathbb{Q}] = 6,$$
$$[\mathbb{Q}(\alpha):\mathbb{Q}] = 3.$$

Let $a, n, x, k$, and $m$ be non-negative integers where $a \geq 1$ and $2 \leq x \leq 10$. In this study, we solve the Diophantine equation

$$bP_n = x^a \tag{3}$$

where $b = 1, 2$ and find as $(b, P_n) \in \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,7),(1,9),(1,16),(1,49),(2,1),(2,2),(2,3),(2,4),(2,5),(2,16)\}$. Later, we solve the Diophantine equation

$$P_n - R_m = x^a \tag{4}$$

and obtain as

$$P_n \in \{2,3,4,5,7,9,12,16,21,28,37,49,65,86,151,200,265,465\}.$$

Finally, we solve the Diophantine equation

$$R_n - P_m = x^a \tag{5}$$

and show as

$$R_n \in \{3,5,7,10,12,17,22,29,39,51,68,90,119,158,209,277,367,1130,3480\}.$$

## 2. Preliminaries

This section will give some lemmas necessary for the proof of theorems. These lemmas provide the basic inequalities and properties related to the logarithmic height of algebraic numbers, Diophantine approximations, and continued fractions. These results will be useful in determining the bounds and solutions for the equations considered.

**Lemma 2.1.** [26] *Let $a, x \in \mathbb{R}$. If $|x| < a$ and $0 < a < 1$ , then*

$$|\log(1+x)| < -|x| \frac{\log(1-a)}{a}$$

*and*

$$\frac{a}{1-e^{-a}} |e^x - 1| > |x|.$$

The following lemma is given, as found in Corollary 2.3 of Matveev [18] or Theorem 9.4 in [6].

**Lemma 2.2.** *Assume that $\gamma_1, \gamma_2, \cdots,$ and $\gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree $D$, $\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1 \neq 0$, and $b_1, b_2, \cdots b_t$ are nonzero integers. Then,*

$$|\Lambda| > \exp\left(-(1+\log B)D^2(1+\log D)1.4t^{4.5}30^{t+3}A_1A_2\cdots A_t\right)$$

*where $\max\{|b_i|\} \leq B$ and $\max\{0.16, Dh(\gamma_i), |\log \gamma_i|\}) \leq A_i$ for all $i \in \{1,2,\cdots,t\}$.*

**Lemma 2.3.** [16] *Let $\tau$ be an irrational number where $\tau = [a_0; a_1, a_2, a_3, \cdots]$, $p_0/q_0, p_1/q_1, \cdots$ be all the convergent of the continued fraction expansion of $\tau$ and $s, r,$ and $M$ be positive integers, and $N$ be a non-negative integer such that $q_N > M$. Then, the inequality*

$$\left|\tau - \frac{r}{s}\right| > \frac{1}{(a(M)+2)s^2}$$

*holds for all $(r, s)$ with $a(M) := \max\{a_i : i = 0,1,\cdots,N\}$ and $0 < s < M$.*

**Lemma 2.4.** [5] *Let $A > 0$, $B > 1$, and $\mu$ be some real numbers, $u, v, M$, and $w$ be positive integers and $p/q$ be a convergent of the continued fraction of the irrational number $\gamma$ such that $q > 6M$. Let $\epsilon := ||q\mu|| - M||q\gamma||$. If $\epsilon > 0$, then there exists no solution to the inequality*

$$0 < |\gamma u + \mu - v| < AB^{-w}$$

*with*

$$u \leq M \text{ and } w \geq \frac{\log(qA/\epsilon)}{\log B}.$$

## 3. Main Theorem

**Theorem 3.1.** *Let $a \geq 1$ and $2 \leq x \leq 10$ be positive integers, and $b \in \{1, 2\}$. If the equation $bP_n = x^a$ has a solution, then*

$$\begin{aligned}(b, P_n, n, x, a) \in \; & \{(1,2,3,2,1),(1,2,4,2,1),(1,3,5,3,1),(1,4,6,2,2),(1,4,6,4,1),\\ & (1,5,7,5,1),(1,7,8,7,1),(1,9,9,3,2),(1,9,9,9,1),(1,16,11,2,4),\\ & (1,16,11,4,2),(1,49,15,7,2),(2,1,0,2,1),(2,1,1,2,1),(2,1,2,2,1),\\ & (2,2,3,2,2),(2,2,3,4,1),(2,2,4,2,2),(2,2,4,4,1),(2,3,5,6,1),\\ & (2,4,6,2,3),(2,4,6,8,1),(2,5,7,10,1),(2,16,11,2,5)\}.\end{aligned}$$

*Proof.* The equation $bP_n = x^a$ holds. Suppose that $n < 185$. Then, it can be seen that the given solutions are provided with the help of a computer program. Suppose that $n \geq 185$. Since

$$bP_n = b(t\alpha^n + s\beta^n + r\gamma^n) = x^a,$$

we obtain

$$bt\alpha^n - x^a = -b(s\beta^n + r\gamma^n).$$

Then

$$\left|1 - \frac{1}{bt}\alpha^{-n}x^a\right| \leq \frac{b|s\beta^n + r\gamma^n|}{bt\alpha^n} < \frac{0.5}{t\alpha^n} \leq \frac{0.7}{\alpha^n}. < \tag{6}$$

To apply Lemma 2.2, we take

$$(\Lambda, \gamma_1, \gamma_2, \gamma_3, b_1, b_2, b_3) := \left(1 - \frac{\alpha^{-n}x^a}{bt}, \alpha, x, bt, -n, a, -1\right). \tag{7}$$

Then, it can be easily seen that $\Lambda \neq 0$. We can choose

$$(A_1, A_2, A_3) := (\log\alpha, \log x^3, 5.22). \tag{8}$$

Because,

$$h(x) := \log x, h(\alpha) := \frac{\log\alpha}{3}, h(bt) \leq \log b + \frac{1}{3}\log 23.$$

Since $B \geq \max\{a, |-1|, |-n|\}$ and $\alpha^{n-1} \geq P_n = x^a \geq 2^a > \alpha^{2a}$, we can take $B := n$. From Lemma 2.2 and (6)-(8), we have

$$\begin{aligned} n\log\alpha - \log 0.7 &< 30^5 2^{4.5} 5.22 \cdot 9 \cdot 1.4(1+\log 3)\log\alpha(1+\log n)\log x^3 \\ &< 1.48 \cdot 10^{11}(\log n + 1). \end{aligned}$$

Hence, we get $n < 1.66 \cdot 10^{13}$. Assume that $z := \log\alpha \cdot n - \log x \cdot a + \log(bt)$. Here, $|e^z - 1| < \frac{0.7}{\alpha^n} < 0.001$ and from Lemma 2.1, we write

$$|z| < \frac{\log(1000/999)}{0.001}\frac{0.7}{\alpha^n} < 0.71\alpha^{-n} \tag{9}$$

and find

$$0 < \left|\frac{\log\alpha}{\log x} \cdot n + \frac{\log(bt)}{\log x} - a\right| < \frac{1.10}{\alpha^n}.$$

Then, $\mu := \dfrac{\log(bt)}{\log x}, \gamma := \dfrac{\log\alpha}{\log x} \notin \mathbb{Q}$, $M := 1.66 \cdot 10^{13} > n$. Therefore, the denominator of the 37th convergent of $\gamma$ exceeds $6M$. From Lemma 2.4, if $\log(1.1q_{37}/\epsilon)/\log\alpha < 180.8 < n$, there is no solution to (9). Hence, we get $n \leq 180$. This is impossible since $n \geq 185$. $\square$

**Theorem 3.2.** *Let $a \geq 1$ and $2 \leq x \leq 10$ be positive integers. If the equation $P_n - R_m = x^a$ has a solution, then*

$$\begin{aligned} (P_n, n, m, x, a) \in\ &\{(2,3,1,2,1), (3,5,1,3,1), (2,4,1,2,1), (4,6,1,2,2), (4,6,1,4,1), \\ &(4,6,2,2,1), (5,7,1,5,1), (4,6,4,2,1), (5,7,0,2,1), (5,7,2,3,1), \\ &(5,7,3,2,1), (5,7,4,3,1), (7,8,0,2,2), (7,8,0,4,1), (7,8,1,7,1), \\ &(7,8,2,5,1), (7,8,3,2,2), (7,8,3,4,1), (7,8,4,5,1), (7,8,5,2,1), \\ &(7,8,6,2,1), (9,9,0,6,1), (9,9,1,3,2), (9,9,1,9,1), (9,9,2,7,1), \\ &(9,9,3,6,1), (9,9,4,7,1), (9,9,5,2,2), (9,9,5,4,1), (86,17,5,9,2), \\ &(9,9,6,4,1), (9,9,7,2,1), (12,10,0,3,2), (12,10,0,9,1), (9,9,6,2,2), \\ &(12,10,3,3,2), (12,10,3,9,1), (12,10,4,10,1), (12,10,5,7,1), \\ &(12,10,7,5,1), (12,10,8,2,1), (16,11,1,2,4), (16,11,1,4,2), \\ &(16,11,7,9,1), (16,11,8,6,1), (16,11,9,2,2), (16,11,9,4,1), \\ &(21,12,5,4,2), (21,12,6,2,4), (21,12,6,4,2), (21,12,9,3,2), \\ &(21,12,10,2,2), (21,12,10,4,1), (28,13,0,5,2), (28,13,3,5,2), \end{aligned}$$

$$(28,13,9,4,2),(28,13,11,6,1),(37,14,5,2,5),(37,14,6,2,5),$$
$$(37,14,9,5,2),(37,14,12,2,3),(37,14,12,8,1),(49,15,1,7,2),$$
$$(49,15,11,3,3),(49,15,13,10,1),(65,16,12,6,2),(86,17,5,3,4),$$
$$(86,17,6,3,4),(86,17,6,9,2),(86,17,11,2,6),(86,17,11,4,3),$$
$$(151,19,17,2,5),(21,12,5,2,4),(265,21,11,3,5),(465,23,19,2,8),$$
$$(12,10,6,7,1),(16,11,7,3,2),(21,12,9,9,1),(200,20,17,3,4),$$
$$(28,13,9,2,4),(37,14,8,3,3),(49,15,10,2,5),(12,10,2,10,1),$$
$$(86,17,11,8,2),(200,20,17,9,2),(151,19,14,10,2),(465,23,19,4,4)\}.$$

*Proof.* The equation $P_n - R_m = x^a$ holds. Assume that $n < 270$. It can be seen that the given solutions are provided with the help of a computer program. Suppose that $n \geq 270$ and $m = 1$. Then, $P_n = x^a$. From Theorem 3.1, this is impossible. Assume that $n \geq 270, (n-m) < 3$, and $m \neq 1$. Since $R_0 = R_3 = 3$ and $R_2 = R_4 = 2$, we can take $m \geq 3$ and since $\alpha^{n-1} \geq P_n = R_m + x^a \geq \alpha^{m-2} + 2^a > \alpha^{m-2}$, we find $n - m > -1$. Then,

$$2^a \leq x^a = P_n - R_n = P_n - P_{n+1} - P_{n-10} = -P_{n-4} - P_{n-10} < 0$$
$$2^a \leq x^a = P_n - R_{n-1} = -P_{n-11} < 0$$
$$x^a = P_n - R_{n-2} = P_n - P_{n-1} - P_{n-12} = P_{n-5} - P_{n-12} = 2P_{n-8}.$$

These are impossible, from Theorem 3.1. Suppose $n \geq 270$, $m \neq 1$, and $n - m \geq 3$. Since $R_0 = R_3 = 3$ and $R_2 = R_4 = 2$, we can take $m \geq 3$. We write

$$P_n = t\alpha^n + s\beta^n + r\gamma^n = \alpha^m + \beta^m + \gamma^m + x^a$$

from (4). We can write

$$t\alpha^n - x^a = -(s\beta^n + r\gamma^n) + R_m$$

and

$$t\alpha^n - \alpha^m - x^a = (\beta^m + \gamma^m) - (s\beta^n + r\gamma^n).$$

Then, from (1) and (2), for $n \geq 270$, $m \geq 3$, and $n - m \geq 3$,

$$\begin{aligned} \left|1 - \frac{1}{t}\alpha^{-n}x^a\right| &\leq \frac{|s\beta^n + r\gamma^n|}{t\alpha^n} + \frac{R_m}{t\alpha^n} \\ &\leq \frac{|e(n)|}{t\alpha^n} + \frac{\alpha^{m+1}}{t\alpha^n} \\ &\leq \frac{0.5}{t\alpha^{n+n/2}} + \frac{1}{t\alpha^{n-m-1}} < \frac{1.39}{\alpha^{n-m-1}} \end{aligned} \tag{10}$$

and

$$\left|1-\frac{x^a}{(t-\alpha^{m-n})\alpha^n}\right| \leq \frac{|\beta^m+\gamma^m|}{(t-\alpha^{m-n})\alpha^n}+\frac{|s\beta^n+r\gamma^n|}{(t-\alpha^{m-n})\alpha^n}$$
$$=\left|\frac{1}{(t-\alpha^{m-n})}\right|\left[\frac{|e'(m)|}{\alpha^n}+\frac{|e(n)|}{\alpha^n}\right]$$
$$<\left|\frac{1}{(t-\alpha^{-3})}\right|\left[\frac{2}{\alpha^n\alpha^{m/2}}+\frac{0.5}{\alpha^n\alpha^{n/2}}\right]$$
$$\leq \frac{4.5}{\alpha^n}. \tag{11}$$

By using Lemma 2.2, we take

$$(\Lambda_1,\gamma_1,\gamma_2,\gamma_3,b_1,b_2,b_3) := \left(1-\frac{x^a}{t\alpha^n},\alpha,x,t,-n,a,-1\right) \tag{12}$$

$$(\Lambda_2,\gamma_1',\gamma_2',\gamma_3') := \left(1-\frac{x^a}{\alpha^n(t-\alpha^{m-n})},\alpha,\ x,\frac{1}{t-\alpha^{m-n}}\right) \tag{13}$$

$$(b_1',b_2',b_3') := (-n,a,1). \tag{14}$$

Hence, it can be observed that $\Lambda_1 \neq 0$ and $\Lambda_2 \neq 0$. Since

$$h(x) \leq \log x, h(\alpha) \leq \frac{\log\alpha}{3}, h(t) \leq 23/3$$

$$h\left(\frac{1}{t-\alpha^{m-n}}\right)h(\alpha) \leq (n-m)+log2+h(t)+\frac{\log\alpha}{3}(n-m) \leq +1.74,$$

we can choose

$$(A_1,A_2,A_3) := (\log\alpha,\log x^3,3.14) \tag{15}$$

$$(A_1',A_2',A_3') := (\log\alpha,\log x^3,5.22+\log\alpha(n-m)) \tag{16}$$

and since

$$\alpha^{n-1} \geq P_n = R_m + x^a > 2^a > \alpha^{2a},$$

$B \geq \max\{|-1|,|-n|,a\}$ and $B' \geq \max\{1,n,a\}$, we can choose

$$B' := B := n. \tag{17}$$

From Lemma 2.2 and (10)-(17), we obtain

$$1.39\alpha^{m-n+1} > |\Lambda_1| > \exp\left(-30^6 3^{4.5} 3^3 1.4\cdot 3.14\log\alpha\log x(1+\log 3)(1+\log n)\right),$$

i.e.,

$$\log\alpha(n-m) < \log(1.39\alpha)+1.65\cdot 10^{13}(1+\log n),$$

and so

$$\begin{aligned}4.5\alpha^{-n} &> |\Lambda_2| \\ &> \exp\left(-30^6 3^{6.5} 1.4 \log\alpha \log x^3 (1+\log 3)(1+\log n)\left((n-m)\log\alpha + 5.22\right)\right).\end{aligned}$$

Hence,

$$\log\alpha n - \log 4.5 < \left(1.65\cdot 10^{13}(1+\log n) + \log(1.39\alpha) + 5.22\right) 5.26\cdot 10^{12}(1+\log n)$$

and $n < 1.54\cdot 10^{30}$. Assume that $z_1 := \log\alpha n - a\log x + \log t$. Then, $|e^{z_1} - 1| < \frac{1.39}{\alpha^{n-m-1}} < 0.8$ for $n - m \geq 3$. From Lemma 2.1, we obtain

$$|z_1| < \frac{\log(10/2)}{0.8}\frac{1.39}{\alpha^{n-m-1}} < 3.71\alpha^{-n+m},$$

i.e.,

$$0 < \left|\frac{\log\alpha}{\log x}n + \frac{\log t}{\log x} - a\right| < \frac{5.4}{\alpha^{n-m}}. \tag{18}$$

We take $M := 1.54\cdot 10^{30} > n, \gamma := \dfrac{\log\alpha}{\log x} \notin \mathbb{Q}$, and $\mu := \log t/\log x$. From Lemma 2.4, we find $\epsilon > 0$ and if

$$\log(5.4q_{71}/\epsilon)/\log\alpha < 350.2 < n - m,$$

then there is no solution to (18). Because the denominator of the 71st convergent of $\gamma$ exceeds $6M$. Hence, $n - m \leq 350$ and $n < 7.8\cdot 10^{16}$ from (10). Now, we will find a better bound for $n$ by using Lemma 2.4. Let

$$z_2 := \log\alpha n - \log xa - \log\left(\frac{1}{t-\alpha^{m-n}}\right).$$

We write

$$|e^{z_2} - 1| < \frac{4.5}{\alpha^n} < 0.01$$

for $n \geq 270$ from (11). From Lemma 2.1, we obtain

$$|z_2| < \frac{\log(100/99)}{0.01}\frac{4.5}{\alpha^n} < 4.53\alpha^{-n}$$

and

$$0 < \left|\frac{\log\alpha}{\log x}n - \frac{\log\left(\frac{1}{t-\alpha^{m-n}}\right)}{\log x} - a\right| < 6.54\alpha^{-n}. \tag{19}$$

We determine $M := 7.8\cdot 10^{16} > n$ and $\gamma = \dfrac{\log\alpha}{\log x} \notin \mathbb{Q}$. It can be shown that $q_{47}$ exceeds $6M$. Take $\mu := -\frac{\log\left(\frac{1}{t-\alpha^{m-n}}\right)}{\log x}, w := n, A := 6.54$, and $B := \alpha$ in Lemma 2.4. Then, we find $\epsilon > 0$ and we can observe that (19) has no solution, if

$$\frac{\log(6.54q_{47}/\epsilon)}{\log\alpha} < 261.1 < n$$

for each $3 \leq n - m \leq 350$. Hence, we get $n \leq 261$. This is impossible since $270 \leq n$. $\square$

**Theorem 3.3.** *Let $a \geq 1$ and $2 \leq x \leq 10$ be positive integers. If the equation $R_n - P_m = x^a$ has a solution, then*

$$\begin{aligned}(R_n, n, m, x, a) \in \ & \{(3,0,0,2,1),(5,5,5,2,1),(3,0,2,2,1),(3,3,0,2,1),(3,3,1,2,1),\\ & (3,3,2,2,1),(5,6,5,2,1),(3,0,1,2,1),(7,7,7,2,1),(39,13,14,2,1),\\ & (51,14,15,2,1),(5,5,0,2,2),(5,5,1,2,2),(5,5,2,2,2),(5,6,0,2,2),\\ & (5,6,1,2,2),(6,2,2,2),(7,7,5,2,2),(90,16,17,2,2),(10,8,3,2,3),\\ & (10,8,4,2,3),(12,9,6,2,3),(17,10,9,2,3),(29,12,12,2,3),(17,10,0,2,4),\\ & (17,10,1,2,4),(17,10,2,2,4),(367,21,22,2,4),(39,13,8,2,5),(68,15,6,2,6),\\ & (277,20,12,2,8),(3480,29,27,2,11),(5,5,3,3,1),(5,5,4,3,1),(5,6,3,3,1),\\ & (5,6,4,3,1),(7,7,6,3,1),(10,8,8,3,1),(12,9,9,3,1),(68,15,16,3,1),\\ & (10,8,0,3,2),(10,8,1,3,2),(10,8,2,3,2),(12,9,5,3,2),(209,19,20,3,2),\\ & (7,7,5,4,1),(90,16,17,4,1),(17,10,0,4,2),(17,10,1,4,2),(17,10,2,4,2),\\ & (367,21,22,4,2),(68,15,6,4,3),(277,20,12,4,4),(7,7,3,5,1),(17,10,10,5,1),\\ & (7,7,4,5,1),(10,8,7,5,1),(12,9,8,5,1),(90,16,16,5,2),(7,7,0,6,1),\\ & (119,17,18,5,1),(29,12,6,5,2),(7,7,2,6,1),(10,8,6,6,1),(22,11,11,6,1),\\ & (7,7,1,6,1),(39,13,5,6,2),(367,21,19,6,3),(10,8,5,7,1),(12,9,7,7,1),\\ & (158,18,19,7,1),(51,14,3,7,2),(51,14,4,7,2),(1130,25,26,7,2),(10,8,3,8,1),\\ & (10,8,4,8,1),(12,9,6,8,1),(17,10,9,8,1),(29,12,12,8,1),(68,15,6,8,1),\\ & (10,8,0,9,1),(10,8,1,9,1),(10,8,2,9,1),(12,9,5,9,1),(209,19,20,9,1),\\ & (29,12,3,3,3),(29,12,4,3,3),(39,13,10,3,3),(90,16,9,3,4),(5,5,0,4,1),\\ & (90,16,9,9,2),(12,9,3,10,1),(12,9,4,10,1),(17,10,8,10,1),(22,11,10,10,1),\\ & (5,5,1,4,1),(5,5,2,4,1),(5,6,0,4,1),(5,6,1,4,1),(5,6,2,4,1)\}.\end{aligned}$$

*Proof.* The equation $R_n - P_m = x^a$ holds. Let $n < 310$. Then, it can be seen that the given solutions are provided with the help of a computer program. Let $n \geq 310$. Since $P_0 = P_1 = P_2 = 1, P_3 = P_4 = 2$, and $\alpha^{n+1} \geq R_n = P_m + x^a > P_m \geq \alpha^{m-2}$, we can take $n - m > -3, m \geq 2$, and $m \neq 3$. On the other hand, we find

$$x^a = R_n - P_{n+1} = (P_{n+1} + P_{n-10}) - P_{n+1} = P_{n-10}$$

$$x^a = R_n - P_{n+2} = (P_{n+1} + P_{n-10}) - P_{n+2} = P_{n-10} - P_{n-3} = -2P_{n-6} < 0.$$

These are impossible, from Theorem 3.1. Thus, we can take $n - m \geq 0$. We write

$$R_n = \alpha^n + \beta^n + \gamma^n = t\alpha^m + s\beta^m + r\gamma^m + x^a.$$

Then

$$\alpha^n - x^a = -(\beta^n + \gamma^n) + P_m$$

and

$$\alpha^n - t\alpha^m - x^a = (s\beta^m + r\gamma^m) - (\beta^n + \gamma^n).$$

Hence, we obtain

$$\begin{aligned} \left|\alpha^{-n}x^a - 1\right| &\leq \frac{|\beta^n + \gamma^n|}{\alpha^n} + \frac{P_m}{\alpha^n} \\ &\leq \frac{|e'(n)|}{\alpha^n} + \frac{\alpha^{m-1}}{\alpha^n} \\ &< \frac{2}{\alpha^{n+n/2}} + \frac{1}{\alpha^{n-m+1}} < \frac{1.1}{\alpha^{n-m+1}} \end{aligned} \tag{20}$$

and

$$\begin{aligned} \left|1 - \frac{x^a}{\alpha^n(1 - t\alpha^{m-n})}\right| &\leq \frac{|s\beta^m + r\gamma^m|}{\alpha^n(1 - t\alpha^{m-n})} + \frac{|\beta^n + \gamma^n|}{\alpha^n(1 - t\alpha^{m-n})} \\ &= \left|\frac{1}{(1 - t\alpha^{m-n})}\right| \left[\frac{|e(m)|}{\alpha^n} + \frac{|e'(n)|}{\alpha^n}\right] \\ &< \left|\frac{1}{(1 - t\alpha^0)}\right| \left[\frac{0.5}{\alpha^n\alpha^{m/2}} + \frac{2}{\alpha^n\alpha^{n/2}}\right] \\ &\leq \frac{1.36}{\alpha^n} \end{aligned} \tag{21}$$

from (1) and (2). By Lemma 2.2, we take

$$(\Lambda_1, \gamma_1, \gamma_2, b_1, b_2) := \left(\frac{x^a}{\alpha^n} - 1, \alpha, x, -n, a\right) \tag{22}$$

$$(\Lambda_2, \gamma_1', \gamma_2', \gamma_3') := \left(1 - \frac{x^a}{\alpha^n(1 - t\alpha^{m-n})}, \alpha, x, \frac{1}{1 - t\alpha^{m-n}}\right) \tag{23}$$

$$(b_1', b_2', b_3') := (-n, a, 1). \tag{24}$$

Moreover, it can be observed that $\Lambda_1 \neq 0$ and $\Lambda_2 \neq 0$. Since

$$\begin{aligned} h\left(\frac{1}{1 - t\alpha^{m-n}}\right) &\leq \log 2 + h(\alpha)(n - m) + h(t) \\ &\leq 1.74 + \log\alpha(n - m)/3, \end{aligned}$$

we can choose

$$(A_1, A_2) := (\log\alpha, \log x^3) \tag{25}$$

$$(A'_1, A'_2, A'_3) := (\log\alpha, \log x^3, \log\alpha(n-m) + 5.22) \tag{26}$$

and since

$$\alpha^{n+1} \geq R_n = P_m + x^a > 2^a > \alpha^{2a}$$

$B \geq \max\{a, n\}$ and $B' \geq \max\{a, n, 1\}$, we can take

$$B = B' := n. \tag{27}$$

From Lemma 2.2 and (20)-(27), we find

$$1.1\alpha^{m-n-1} > |\Lambda_1| > \exp\left(-30^5 2^{4.5} 3^3 1.4 \log x(1+\log n)(1+\log 3)\log\alpha\right),$$

i.e.,

$$\log\alpha(n-m) < (1+\log n)2.83 \cdot 10^{10} + \log(1.1/\alpha) \tag{28}$$

and

$$\begin{aligned}1.36\alpha^{-n} &> |\Lambda_2| \\ &> \exp\left(-30^6 3^{6.5} 1.4 \log\alpha(1+\log 3)\log x^3(1+\log n)\left(5.22+\log\alpha(n-m)\right)\right),\end{aligned}$$

i.e.,

$$\log\alpha n - \log 1.36 < (1+\log n)5.26 \cdot 10^{12}\left(5.22 + \log\alpha(n-m)\right). \tag{29}$$

From (28) and (29), we have

$$\log\alpha n - \log 1.36 < (1+\log n)5.26 \cdot 10^{12}\left(5.22 + 2.83 \cdot 10^{10}(1+\log n) + \log(1.1/\alpha)\right).$$

Hence, it follows that $n < 2.17 \cdot 10^{27}$. Assume that

$$z_3 := n\log\alpha - a\log x$$

from (20). Then,

$$|e^{z_3} - 1| < \frac{1.1}{\alpha^{n-m+1}} < 0.85$$

for $n - m \geq 0$. From Lemma 2.1, we write

$$|z_3| < \frac{\log(100/15)}{0.85}\frac{1.1}{\alpha^{n-m+1}} < 1.86\alpha^{-n+m}$$

and get

$$0 < \left|\frac{\log\alpha}{\log x} - \frac{a}{n}\right| < \frac{2.7}{n\alpha^{n-m}}.$$

Moreover, we can choose $M := 2.17 \cdot 10^{27} > n$ and $\gamma := \dfrac{\log\alpha}{\log x} \notin \mathbb{Q}$. Then, $q_{63}$ exceeds $M$. Let $n - m \geq 255$. Hence, we have $\frac{\alpha^{n-m}}{5.4} > n$ and

$$\left|\frac{\log\alpha}{\log x} - \frac{a}{n}\right| < \frac{2.7}{n}\alpha^{m-n} < \frac{1}{2n^2}.$$

We can see that the rational number $\frac{a}{n}$ is a convergent of $\frac{\log\alpha}{\log x}$. Let $\frac{p_r}{q_r}$ be $r$-th convergent of $\frac{\log\alpha}{\log x}$ and $\frac{a}{n}=\frac{p_t}{q_t}$, for some $t$. Then, we find $q_{63}>2\cdot 10^{27}$ and also $a_M=\max\{a_i|i=0,1,\cdots,63\}=433$. From Lemma 2.3, we have $\left|\frac{\log\alpha}{\log x}-\frac{p_t}{q_t}\right|>\frac{1}{435n^2}$ and $\frac{2.7}{n}\alpha^{m-n}>\frac{1}{435n^2}$.
Then,

$$2\cdot 10^{-31}>\frac{2.7}{\alpha^{255}}\geq\frac{2.7}{\alpha^{n-m}}>\frac{1}{435n}>\frac{1}{435\cdot 2\cdot 10^{27}}>1.14\cdot 10^{-30}.$$

But, this is impossible. Thus, we have $n-m<255$ and $n<5.7\cdot 10^{16}$ from (29). Let

$$z_4:=\log\alpha n-\log xa-\log\left(\frac{1}{1-t\alpha^{m-n}}\right).$$

We obtain

$$|e^{z_4}-1|<\frac{1.36}{\alpha^n}<0.001$$

for $n\geq 310$. From Lemma 2.1, we have

$$|z_4|<\frac{\log(1000/999)}{0.001}\frac{1.36}{\alpha^n}<1.37\alpha^{-n}$$

and so

$$0<\left|\frac{\log\alpha}{\log x}n-\frac{\log\left(\frac{1}{1-t\alpha^{m-n}}\right)}{\log x}-a\right|<1.98\alpha^{-n}. \tag{30}$$

If we can choose $\gamma=\dfrac{\log\alpha}{\log x}\notin\mathbb{Q}$ and $M:=5.7\cdot 10^{16}>n$, then $q_{57}>6M$. We take $\mu:=-\frac{\log\left(\frac{1}{1-t\alpha^{m-n}}\right)}{\log x}, w:=n, A:=1.98$, and $B:=\alpha$. The inequality (30) has no solution, if

$$\frac{\log(1.98q_{57}/\epsilon)}{\log\alpha}<299.99<n$$

where $0\leq n-m\leq 255$. Hence, $n\leq 299$. This is a impossible since $310\leq n$. □

## 4. Conclusion

This study delves into solving certain Diophantine equations through the application of Baker's theory and reduction methods. Initially, we established the upper limit of $n$ by examining linear forms in logarithms. Subsequently, we further refined this upper bound using reduction techniques. Additionally, the same methods proved effective in addressing perfect powers that are either the sum or product of a Perrin number or a Padovan number, as well as combinations of two Perrin numbers or two Padovan numbers. Furthermore, the equation $P_n=bx^a$ can be extended by determining solutions for various values of $b$.

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## Author Contributions

The author read and approved the final version of the paper.

## Conflicts of Interest

The author declares no conflict of interest.

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