KYUNGPOOK Math. J. 64(2024), 511-518
https://doi.org/10.5666/KMJ.2024.64.3.511
pISSN 1225-6951 eISSN 0454-8124

$2n$-Moves and the Γ-Polynomial for Knots

Hideo Takioka
School of Mechanical Engineering,
College of Science and Engineering, Kanazawa University,
Kakuma-Machi, Kanazawa, 920-1192, Japan
e-mail: takioka@se.kanazawa-u.ac.jp

Abstract. A $2n$-move is a local change for knots and links which changes $2n$-half twists to 0-half twists or vice versa for a natural number n. In 1979, Yasutaka Nakanishi conjectured that the 4-move is an unknotting operation. This is still an open problem. It is known that the Γ-polynomial is an invariant for oriented links which is the common zeroth coefficient polynomial of the HOMFLYPT and Kauffman polynomials. In this paper, we show that the $4k$-move is not an unknotting operation for any integer $k(\geq 2)$ by using the Γ-polynomial, and if $\Gamma(K;-1) \equiv 9 \pmod{16}$ then the knot K cannot be deformed into the unknot by a single 4-move. Moreover, we give a one-to-one correspondence between the value $\Gamma(K;-1) \pmod{16}$ and the pair $(a_2(K), a_4(K)) \pmod{2}$ of the second and fourth coefficients of the Alexander-Conway polynomial for a knot K.

1. Introduction

As mentioned in the papers [1, 8, 12, 13, 15], a local change for oriented knots K and K' in the left-hand side of Fig. 1 is called a t_{2n}*-move* and that in the right-hand side of Fig. 1 is called a $\bar{t}_{2n}$*-move*, both of which change $2n$-half twists to 0-half twists or vice versa for a natural number n, where dotted arcs in Fig. 1 can be knotted and linked. Moreover, the unoriented version of t_{2n}, $\bar{t}_{2n}$-moves is called a $2n$*-move*. In 1979, Yasutaka Nakanishi conjectured that the 4-move is an unknotting operation. This is still an open problem. It is known that the Γ-polynomial is an invariant for oriented links which is the common zeroth coefficient polynomial of the HOMFLYPT and Kauffman polynomials. (See Sect. 2.) In this paper, we study $2n$-moves and the Γ-polynomial for knots. By applying the skein relation (1), we have the following theorem.

Received August 22, 2020; revised July 6, 2021; accepted July 12, 2021.
2020 Mathematics Subject Classification: 57K10, 57K14.
Key words and phrases: $2n$-move, Alexander-Conway polynomial, Γ-polynomial.
This work was supported by JSPS KAKENHI Grant Number JP22K13911.

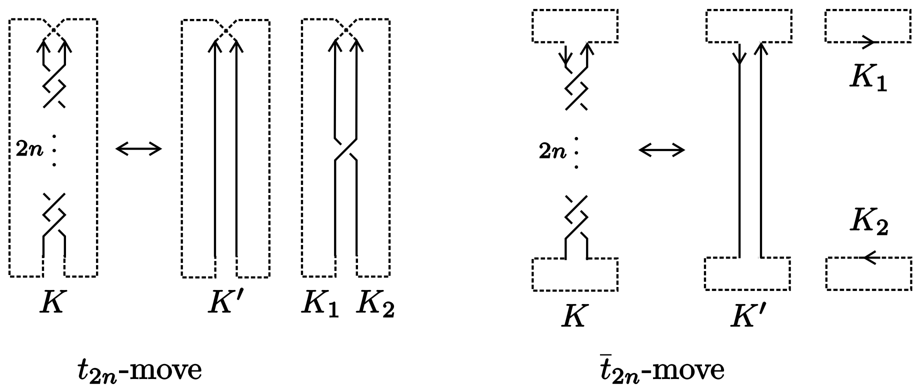

FIGURE 1. t_{2n}, $\bar{t}_{2n}$-moves.

Theorem 1.1. *Let n be a natural number. Let K, K', K_1, K_2 be oriented knots as shown in Fig. 1. Let ℓ be the linking number of K_1 and K_2. Then we have*

$$\Gamma(K) = \begin{cases} x^{-n}\Gamma(K') - nx^{-(\ell+n)}(1-x)\Gamma(K_1)\Gamma(K_2) & (t_{2n}\text{-move}), \\ x^n\Gamma(K') + (1-x^n)x^{-\ell}\Gamma(K_1)\Gamma(K_2) & (\bar{t}_{2n}\text{-move}). \end{cases}$$

From this, we have the following corollary.

Corollary 1.2.

$$\begin{cases} \Gamma(K) = x^{-n}\Gamma(K') \text{ in } \mathbb{Z}_n[x^{\pm 1}] & (t_{2n}\text{-move}), \\ \Gamma(K;q) = \Gamma(K';q) \text{ for any } n\text{-th root of unity } q & (\bar{t}_{2n}\text{-move}). \end{cases}$$

It is known that the Γ-polynomial is characterized as follows.

Lemma 1.3. ([5]) *Let $\mathcal{K}$ be the set of oriented knots. The image of $\mathcal{K}$ under Γ is the following:*

$$\Gamma(\mathcal{K}) = \left\{1 + (1-x)^2 f(x) \mid f(x) \in \mathbb{Z}[x^{\pm 1}]\right\}.$$

Lemma 1.4. ([3]) *Let $\mathcal{K}'$ be the set of 2-bridge knots with unknotting number one. Then we have*

$$\Gamma(\mathcal{K}') = \Gamma(\mathcal{K}).$$

In particular, for any $K \in \mathcal{K}$, there exists an integer i such that $\Gamma(K;-1) = 1 + 4i$, and for any integer j, there exists $K \in \mathcal{K}'$ such that $\Gamma(K;-1) = 1+4j$. Since 2-bridge knots can be deformed into the unknot by 4-moves, we cannot show the 4-move is not an unknotting operation by using the Γ-polynomial. Immediately, we have the following corollary.

Corollary 1.5.

$$\Gamma(K;-1) \equiv \begin{cases} -\Gamma(K';-1) + (4k-2)(-1)^{\ell} \pmod{16k-8} & (t_{4k-2}\text{-move, } k \in \mathbb{N}),\\ \Gamma(K';-1) - 4k(-1)^{\ell} \pmod{16k} & (t_{4k}\text{-move, } k \in \mathbb{N}),\\ -\Gamma(K';-1) + 2(-1)^{\ell} \pmod{8} & (\bar{t}_{4k-2}\text{-move, } k \in \mathbb{N}),\\ \Gamma(K';-1) & (\bar{t}_{4k}\text{-move, } k \in \mathbb{N}). \end{cases}$$

Therefore, by applying Lemma 1.3, we see that the $4k$-move is not an unknotting operation for any integer $k(\geq 2)$, and if $\Gamma(K;-1) \equiv 9 \pmod{16}$ then the knot K cannot be deformed into the unknot by a single 4-move. By Lemma 1.3, we have $\Gamma(K;-1) \equiv 1, 5, 9, 13 \pmod{16}$ for any knot K. Here, we show a one-to-one correspondence between the value $\Gamma(K;-1) \pmod{16}$ and the pair $(a_2(K), a_4(K))$ of the second and fourth coefficients modulo 2 of the Alexander-Conway polynomial for a knot K as follows.

Theorem 1.6. *Let $(a_2(K), a_4(K))$ be the pair of the second and fourth coefficients of the Alexander-Conway polynomial for a knot K. Then we have the following correspondence:*

$$\begin{aligned} (a_2(K), a_4(K)) &\equiv (0,0) \pmod 2 \Longleftrightarrow \Gamma(K;-1) \equiv 1 \pmod{16},\\ (a_2(K), a_4(K)) &\equiv (0,1) \pmod 2 \Longleftrightarrow \Gamma(K;-1) \equiv 9 \pmod{16},\\ (a_2(K), a_4(K)) &\equiv (1,0) \pmod 2 \Longleftrightarrow \Gamma(K;-1) \equiv 13 \pmod{16},\\ (a_2(K), a_4(K)) &\equiv (1,1) \pmod 2 \Longleftrightarrow \Gamma(K;-1) \equiv 5 \pmod{16}. \end{aligned}$$

Remark 1.7. *The value* $\Gamma(K;-1) \pmod{16}$ *is also considered in* [16] *to study clasp disks with two clasp singularities.*

2. The Γ-Polynomial for Oriented Links

The HOMFLYPT polynomial $P(L;y,z) \in \mathbb{Z}[y^{\pm 1}, z^{\pm 1}]$ and the Kauffman polynomial $F(L;a,b) \in \mathbb{Z}[a^{\pm 1}, b^{\pm 1}]$ of an oriented link L are computed by the following recursive formulas [2, 4, 14]:
For the unknot U, we have

$$P(U) = F(U) = 1.$$

For a triple (L_+, L_-, L_0) of oriented links which are identical except near one point as shown in Fig. 2, we have

$$yP(L_+) + y^{-1}P(L_-) = zP(L_0).$$

Therefore, we have

$$P(L; \sqrt{-1}, \sqrt{-1}z) = \nabla(L;z),$$

where $\nabla(L;z)$ is the Alexander-Conway polynomial.

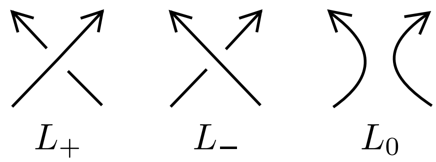

Figure 2. Skein triple.

For a quadruple $(D_+, D_-, D_0, D_\infty)$ of oriented link diagrams which are identical except near one point as shown in Fig. 3, we have

$$aF(D_+) + a^{-1}F(D_-) = b(F(D_0) + a^{-2\nu}F(D_\infty)),$$

where $2\nu = w(D_+) - w(D_\infty) - 1$ and $w(D_+)$, $w(D_\infty)$ are the writhes of D_+, D_∞, respectively. In particular, we call (L_+, L_-, L_0), $(D_+, D_-, D_0, D_\infty)$ a skein triple, a skein quadruple, respectively. The HOMFLYPT and Kauffman polynomials of an

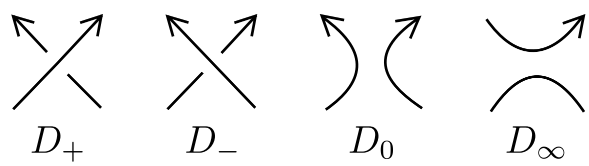

Figure 3. Skein quadruple.

oriented r-component link L are presented by the following:

$$P(L; y, z) = (yz)^{-r+1} \sum_{n \geq 0} p_n(L; y) z^{2n},$$

$$F(L; a, b) = (ab)^{-r+1} \sum_{n \geq 0} f_n(L; a) b^n,$$

where $p_n(L; y) \in \mathbb{Z}[y^{\pm 1}]$ and $f_n(L; a) \in \mathbb{Z}[a^{\pm 1}]$. In particular, $p_n(L; y)$ is a Laurent polynomial in the variable $-y^2$ for $n (\geq 0)$. Therefore, putting $-y^2 = x$, we denote $p_n(L; y)$ by $c_n(L; x) \in \mathbb{Z}[x^{\pm 1}]$, that is, $c_n(L; -y^2) = p_n(L; y)$. We call $c_n(L; x)$ and $f_n(L; a)$ the n-th coefficient HOMFLYPT and Kauffman polynomials of an oriented link L, respectively. As mentioned in the paper [10], we have

$$p_0(L; y) = f_0(L; y).$$

In particular, we call the zeroth coefficient HOMFLYPT polynomial $c_0(L; x)$ of an oriented link L the Γ-polynomial of L and denote it by $\Gamma(L; x) \in \mathbb{Z}[x^{\pm 1}]$. It is known that the following holds [5, 6, 7].

Proposition 2.1. (i) *For the unknot U, we have*

$$\Gamma(U) = 1.$$

For a skein triple (L_+, L_-, L_0)*, we have*

$$-x\Gamma(L_+) + \Gamma(L_-) = \begin{cases} \Gamma(L_0) & \text{if } \delta = 0, \\ 0 & \text{if } \delta = 1, \end{cases}$$

where $\delta = (r_+ - r_0 + 1)/2$ $(= 0, 1)$ *for the numbers* r_+*,* r_0 *of components of* L_+*,* L_0*, respectively.*

(ii) *Let* $L \sqcup L'$ *and* $L\#L'$ *be the split union and a connected sum of oriented links* L *and* L'*, respectively. Then we have*

$$\Gamma(L \sqcup L') = (1 - x)\Gamma(L\#L').$$

(iii) *Let* $L\#L'$ *be a connected sum of oriented links* L *and* L'*. Then we have*

$$\Gamma(L\#L') = \Gamma(L)\Gamma(L').$$

(iv) *Let* L *be an oriented* r*-component link with the components* $K_1, \ldots, K_r$ *and* $\mathrm{lk}(L)$ *the total linking number of* L*. Then we have*

$$\Gamma(L) = (1 - x)^{r-1} x^{-\mathrm{lk}(L)} \Gamma(K_1) \cdots \Gamma(K_r).$$

(v) *For a skein triple* $(K_+, K_-, K_1 \cup K_2)$ *with oriented knots* K_+*,* K_-*,* K_1*,* K_2*, we have*

$$-x\Gamma(K_+) + \Gamma(K_-) = (1 - x)x^{-\mathrm{lk}(K_1 \cup K_2)}\Gamma(K_1)\Gamma(K_2). \tag{1}$$

(vi) *Let* K *be an oriented knot. Then we have*

$$\Gamma(K; 1) = 1.$$

(vii) *Let* $-L$ *be the inverse of an oriented link* L*, that is, the link obtained by reversing the orientation of each component of* L*. Then we have*

$$\Gamma(-L) = \Gamma(L).$$

(viii) *Let* L^* *be the mirror image of an oriented* r*-component link* L*. Then we have*

$$\Gamma(L^*; x) = (-x)^{r-1}\Gamma(L; x^{-1}).$$

3. Proof of Theorem 1.1

First, we consider the case of the t_{2n}-move for a natural number n. By applying the skein relation (1) to the parallel $2n$-half twists in the left-hand side of Fig. 1 as follows:

$$\begin{aligned} &\Gamma(K) \\ &= x^{-n}\Gamma(K') - x^{-1}(1 - x)x^{-(\ell+n-1)}\Gamma(K_1)\Gamma(K_2) - \cdots - x^{-n}(1 - x)x^{-\ell}\Gamma(K_1)\Gamma(K_2) \\ &= x^{-n}\Gamma(K') - nx^{-(\ell+n)}(1 - x)\Gamma(K_1)\Gamma(K_2). \end{aligned}$$

Therefore, by applying Lemma 1.3, we have

$$\begin{aligned}
&\Gamma(K;-1)\\
&= (-1)^n\Gamma(K';-1) - 2n(-1)^{\ell+n}\Gamma(K_1;-1)\Gamma(K_2;-1)\\
&= \begin{cases} -\Gamma(K';-1) + 2n(-1)^{\ell}\Gamma(K_1;-1)\Gamma(K_2;-1) & \text{if } n \text{ is odd},\\ \Gamma(K';-1) - 2n(-1)^{\ell}\Gamma(K_1;-1)\Gamma(K_2;-1) & \text{if } n \text{ is even}\end{cases}\\
&= \begin{cases} -\Gamma(K';-1) + (4k-2)(-1)^{\ell}\Gamma(K_1;-1)\Gamma(K_2;-1) & \text{if } n = 2k-1\ (k \in \mathbb{N}),\\ \Gamma(K';-1) - 4k(-1)^{\ell}\Gamma(K_1;-1)\Gamma(K_2;-1) & \text{if } n = 2k\ (k \in \mathbb{N})\end{cases}\\
&= \begin{cases} -\Gamma(K';-1) + (4k-2)(-1)^{\ell}(1+4i+4j+16ij) & \text{if } n = 2k-1\ (k \in \mathbb{N}),\\ \Gamma(K';-1) - 4k(-1)^{\ell}(1+4i+4j+16ij) & \text{if } n = 2k\ (k \in \mathbb{N})\end{cases}\\
&\equiv \begin{cases} -\Gamma(K';-1) + (4k-2)(-1)^{\ell} \pmod{16k-8} & \text{if } n = 2k-1\ (k \in \mathbb{N}),\\ \Gamma(K';-1) - 4k(-1)^{\ell} \pmod{16k} & \text{if } n = 2k\ (k \in \mathbb{N}).\end{cases}
\end{aligned}$$

Next, we consider the case of the $\bar{t}_{2n}$-move for a natural number n. By applying the skein relation (1) to the antiparallel $2n$-half twists in the right-hand side of Fig. 1 as follows:

$$\begin{aligned}
&\Gamma(K)\\
&= x^n\Gamma(K') + (1-x)x^{-\ell}\Gamma(K_1)\Gamma(K_2) + \cdots + x^{n-1}(1-x)x^{-\ell}\Gamma(K_1)\Gamma(K_2)\\
&= x^n\Gamma(K') + (1+\cdots+x^{n-1})(1-x)x^{-\ell}\Gamma(K_1)\Gamma(K_2)\\
&= x^n\Gamma(K') + ((1-x^n)/(1-x))(1-x)x^{-\ell}\Gamma(K_1)\Gamma(K_2)\\
&= x^n\Gamma(K') + (1-x^n)x^{-\ell}\Gamma(K_1)\Gamma(K_2).
\end{aligned}$$

Therefore, by applying Lemma 1.3, we have

$$\begin{aligned}
&\Gamma(K;-1)\\
&= (-1)^n\Gamma(K';-1) + (1-(-1)^n)(-1)^{\ell}\Gamma(K_1;-1)\Gamma(K_2;-1)\\
&= \begin{cases} -\Gamma(K';-1) + 2(-1)^{\ell}\Gamma(K_1;-1)\Gamma(K_2;-1) & \text{if } n \text{ is odd},\\ \Gamma(K';-1) & \text{if } n \text{ is even}\end{cases}\\
&= \begin{cases} -\Gamma(K';-1) + 2(-1)^{\ell}(1+4i+4j+16ij) & \text{if } n \text{ is odd},\\ \Gamma(K';-1) & \text{if } n \text{ is even}\end{cases}\\
&\equiv \begin{cases} -\Gamma(K';-1) + 2(-1)^{\ell} \pmod{8} & \text{if } n \text{ is odd},\\ \Gamma(K';-1) & \text{if } n \text{ is even}.\end{cases}
\end{aligned}$$

This completes the proofs of Theorem 1.1 and Corollary 1.5.

4. Proof of Theorem 1.6

It is known that the following lemmas hold.

Lemma 4.1. ([11]) *Let $a_2(K)$ be the second coefficient of the Alexander-Conway polynomial of a knot K.*

$$\sum_{n\geq 0} 2^n c_n(K;-1) = (-1)^{a_2(K)}. \tag{2}$$

Lemma 4.2. ([9])

$$\sum_{n\geq 0} 4^n c_n(K;-1) = 1. \tag{3}$$

By (2) $\times 2 +$ (3), we have the following lemma.

Lemma 4.3.

$$3\Gamma(K;-1) + 8c_1(K;-1) + 8c_2(K;-1) \equiv 2(-1)^{a_2(K)} + 1 \pmod{16}.$$

We can see the following lemma easily.

Lemma 4.4. *Let $a_{2n}(K)$ be the $2n$-th coefficient of the Alexander-Conway polynomial of a knot K for $n \geq 0$.*

$$c_n(K;-1) \equiv c_n(K;1) \equiv a_{2n}(K) \pmod 2.$$

Here, we denote $a_2(K)$, $a_4(K)$, $\Gamma(K;-1)$ by a_2, a_4, Γ, respectively. We prove Theorem 1.6 as follows:

(1) We start with $(a_2, a_4) \equiv (0,0) \pmod 2$. By Lemmas 4.3, 4.4, $\Gamma \equiv 1 \pmod{16}$. Conversely, we start with $\Gamma \equiv 1 \pmod{16}$. Assume $a_2 \equiv 1 \pmod 2$. By Lemmas 4.3, 4.4, $8c_2(K;-1) \equiv 4 \pmod{16}$. This contradicts. Therefore, $a_2 \equiv 0 \pmod 2$. By Lemmas 4.3, 4.4, $a_4 \equiv 0 \pmod 2$.

(2) We start with $(a_2, a_4) \equiv (0,1) \pmod 2$. By Lemmas 4.3, 4.4, $\Gamma \equiv 9 \pmod{16}$. Conversely, we start with $\Gamma \equiv 9 \pmod{16}$. Assume $a_2 \equiv 1 \pmod 2$. By Lemmas 4.3, 4.4, $8c_2(K;-1) \equiv -4 \pmod{16}$. This contradicts. Therefore, $a_2 \equiv 0 \pmod 2$. By Lemmas 4.3, 4.4, $a_4 \equiv 1 \pmod 2$.

(3) We start with $(a_2, a_4) \equiv (1,0) \pmod 2$. By Lemmas 4.3, 4.4, $\Gamma \equiv 13 \pmod{16}$. Conversely, we start with $\Gamma \equiv 13 \pmod{16}$. Assume $a_2 \equiv 0 \pmod 2$. By Lemmas 4.3, 4.4, $8c_2(K;-1) \equiv -4 \pmod{16}$. This contradicts. Therefore, $a_2 \equiv 1 \pmod 2$. By Lemmas 4.3, 4.4, $a_4 \equiv 0 \pmod 2$.

(4) We start with $(a_2, a_4) \equiv (1,1) \pmod 2$. By Lemmas 4.3, 4.4, $\Gamma \equiv 5 \pmod{16}$. Conversely, we start with $\Gamma \equiv 5 \pmod{16}$. Assume $a_2 \equiv 0 \pmod 2$. By Lemmas 4.3, 4.4, $8c_2(K;-1) \equiv 4 \pmod{16}$. This contradicts. Therefore, $a_2 \equiv 1 \pmod 2$. By Lemmas 4.3, 4.4, $a_4 \equiv 1 \pmod 2$.

This completes the proof of Theorem 1.6.

References

[1] R. H. Fox, *Congruence classes of knots*, Osaka Math. J., **10**(1958), 37–41.

[2] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett and A. Ocneanu, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc., **12**(1985), 239–246.

[3] H. Fujii, *First common terms of the HOMFLY and Kauffman polynomials, and the Conway polynomial of a knot*, J. Knot Theory Ramifications, **8(4)**(1999), 447–462.

[4] L. H. Kauffman, *An invariant of regular isotopy*, Trans. Amer. Math. Soc., **318**(1990), 417–471.

[5] A. Kawauchi, *On coefficient polynomials of the skein polynomial of an oriented link*, Kobe J. Math., **11**(1994), 49–68.

[6] A. Kawauchi, *A Survey of Knot Theory*, Birkhäuser, Basel, (1996).

[7] A. Kawauchi, *Lectures on knot theory* (in Japanese), Kyoritsu Shuppan, (2007).

[8] S. Kinoshita, *On Wendt's theorem of knots*, Osaka Math. J., **9**(1957), 61–66.

[9] W. B. R. Lickorish and K. C. Millett, *A polynomial invariant of oriented links*, Topology, **26**(1987), 107–141.

[10] W. B. R. Lickorish, *Polynomials for links*, Bull. London Math. Soc., **20**(1988), 558–588.

[11] H. Murakami, *A recursive calculation of the Arf invariant of a link*, J. Math. Soc. Japan, **38(2)**(1986), 335–338.

[12] Y. Nakanishi, *Fox's congruence modulo* $(2,1)$, Sūrikaisekikenkyūsho Kōkyūroku **518**(1984), 96–101.

[13] Y. Nakanishi and S. Suzuki, *On Fox's congruence classes of knots*, Osaka J. Math., **24**(1987), 217–225.

[14] J. H. Przytycki and P. Traczyk, *Invariants of links of Conway type*, Kobe J. Math., **4**(1987), 115–139.

[15] J. H. Przytycki, t_k *moves on links*, Braids (Santa Cruz, CA, 1986), 615–656, Contemp. Math., **78**, Amer. Math. Soc., Providence, RI, 1988.

[16] H. Takioka, *A characterization of the* Γ*-polynomials of knots with clasp number at most two*, J. Knot Theory Ramifications, **26(4)**(2017), 1750013.