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A Note on Embedding Homology 3-Spheres in the 4-Sphere

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Abstract. Recently, Şavk introduced the notion of a generalized Mazur manifold, which is a contractible 4-manifold obtained by attaching a 2-handle on the complement of a ribbon disk, and observed that many classical examples of homology 3-spheres bounding contractible 4-manifolds actually bound generalized Mazur manifolds. In this note, we prove that homology 3-spheres bounding generalized Mazur manifolds smoothly embed in the 4-sphere by using 5-dimensional arguments. As a consequence, we show that any homology 3-sphere obtained from the 3-sphere by Dehn surgery on a ribbon link and certain plumbed 3-manifolds smoothly embed in the 4-sphere.

1. Introduction

One simple way to produce a smooth homotopy 4-sphere is to take the double of a smooth contractible 4-manifold. It follows that the boundary of a smooth contractible 4-manifold is a homology 3-sphere which can be smoothly embedded in a smooth homotopy 4-sphere. This raises the following question on smooth embeddings of homology 3-spheres.

Question 1.1. Suppose that a homology 3-sphere Y bounds a smooth contractible 4-manifold X. Does Y smoothly embed in S^4?

The special case of Question 1.1 when X has no 3-handles is implied by the Andrews-Curtis conjecture on balanced presentation of the trivial group [3] as follows: In this case, the double of X bounds a contractible, 5-dimensional 2-handlebody $X \times [0,1]$. If the Andrews-Curtis conjecture is true (for the corresponding balanced presentation), then the handle decomposition of $X \times [0,1]$ can be trivialized by handle slides and handle cancellations so that $X \times [0,1]$ is diffeomorphic to D^5 and hence the double of X is diffeomorphic to S^4.

The above argument (originally due to Mazur [11]) shows that Question 1.1 holds for a homology 3-sphere Y bounding a *Mazur manifold* X (that is, if X admits a

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handle decomposition consisting of one $0, 1, 2$-handles). There are several explicit examples of homology 3-spheres bounding contractible 4-manifolds including [10, 2, 4, 14, 7]. See [6, Section 2.2] for a more detailed discussion and references. Many (but not all) of these examples are Brieskorn homology 3-spheres bounding Mazur manifolds so that Question 1.1 is true for them.

Recently, Şavk introduced in [5] the notion of a *generalized Mazur manifold* which is a contractible 4-manifold obtained by attaching a 2-handle on the complement of a ribbon disk and observed that many classical examples of homology 3-spheres bounding contractible 4-manifolds actually bound generalized Mazur manifolds.

In this note, we prove that homology 3-spheres bounding generalized Mazur manifolds satisfy Question 1.1 by showing that the double of a generalized Mazur manifold is diffeomorphic to S^4 via 5-dimensional arguments. This might be well-known to experts but the author could not find this in the literature.

Theorem 1.2. *Let M be the 0-surgery manifold of a ribbon knot K. Suppose that a homology 3-sphere Y is obtained from M by integral surgery on a knot J where the longitude of J is freely homotopic to a meridian of K in M. Then Y smoothly embeds in S^4.*

By considering the special case when J is isotopic to a meridian of K, we have the following immediate corollary of Theorem 1.2.

Corollary 1.3. *Suppose that a homology 3-sphere Y is obtained from S^3 by doing Dehn surgery along a ribbon knot K. Then Y smoothly embeds in S^4.*

In [10], Gordon proved that any homology 3-sphere obtained from S^3 by doing Dehn surgery along a *slice* knot bounds a contractible 4-manifold. It seems to be an open question whether such a manifold smoothly embeds in S^4 or not. Corollary 1.3 shows that Question 1.4 below is true if the slice-ribbon conjecture is true for knots in the 3-sphere.

Question 1.4. Suppose that a homology 3-sphere Y is obtained from S^3 by doing Dehn surgery along a slice knot. Does Y smoothly embed in S^4?

Remark 1.5. The proofs of Theorem 1.2 and Corollary 1.3 in Section 2 can be extended to the multi-component version (that is, when K is a ribbon link) after making obvious changes.

In [1, Proof of Theorem A], Aguilar and Şavk showed that the plumbed 3-manifold Z_n corresponding to the plumbing diagram in Figure 1 is diffeomorphic to the (-1)-surgery manifold of the generalized square knot $T_{n+1,n+2}\# - T_{n+1,n+2}$ and hence bounds a contractible 4-manifold for all $n \geq 1$ where $T_{n+1,n+2}$ is the $(n+1, n+2)$-torus knot. Since $T_{n+1,n+2}\# - T_{n+1,n+2}$ is a ribbon knot for all n, we obtain the following corollary from Corollary 1.3 and [1, Proof of Theorem A].

Corollary 1.6. *The plumbed 3-manifold Z_n corresponding to the plumbing diagram in Figure 1 smoothly embeds in S^4 for all $n \geq 2$.*

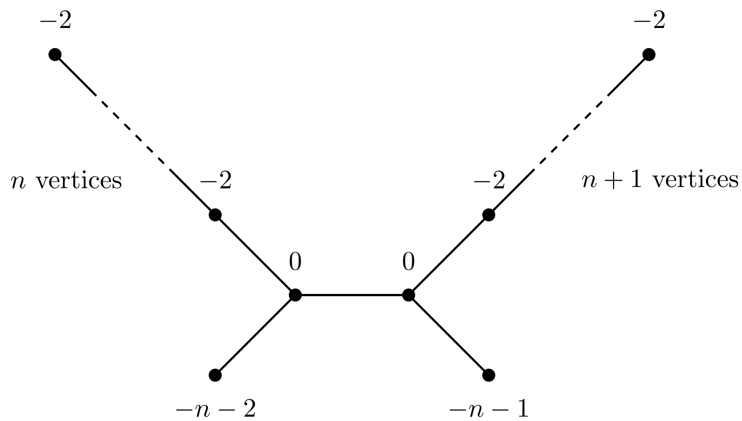

Figure 1. The plumbing diagram of Z_n.

2. Proofs of Theorem 1.2 and Corollary 1.3

In this section, we prove Theorem 1.2 and Corollary 1.3.

Proof of Theorem 1.2. Suppose that K is a ribbon knot with a chosen ribbon disk D in D^4. Then the complement $W = D^4 \setminus \nu(D)$ admits a handle decomposition consists of one 0-handle, n 1-handles and $(n-1)$ 2-handles and ∂W is diffeomorphic to M. The natural number n depends on the number of ribbon singularities of D. Regard J as a framed knot in M where the framing is given so that the corresponding surgery manifold of M along J is Y.

Consider a compact 4-manifold X obtained by attaching a 2-handle to W along the (framed) longitude of J. Since $\pi_1(W)$ is normally generated by the longitude of J, X is a contractible 4-manifold by Van Kampen theorem. By our construction, ∂X is obtained from $\partial W = M$ by doing surgery along J so ∂X is diffeomorphic to Y.

Since Y smoothly embeds in the double $DX = X \cup_Y -X$ of X which is the boundary of $X \times [0,1]$, it suffices to prove that $X \times [0,1]$ is diffeomorphic to D^5. Note that $X \times [0,1]$ is obtained from $W \times [0,1]$ by attaching the 5-dimensional 2-handle along $J \times \{0\}$. Since homotopy implies isotopy in dimension 4 and $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2$, the diffeomorphism type of $X \times [0,1]$ only depends on the free homotopy class of J and the mod 2 reduction of the framing of J. Without loss of generality, we assume that J is a meridian of K. Consider a presentation

$$\pi_1(X) = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_n \rangle$$

corresponding to the given handle decomposition of X consists of one 0-handle, n 1-handles and n 2-handles. Here we choose an ordering of the generators $x_1, x_2, \ldots, x_n$ and an ordering of the relators $r_1, r_2, \ldots, r_n$ in such a way that

$$\pi_1(W) = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_{n-1} \rangle$$

and each r_i is of the form $w_i x_i w_i^{-1} x_{i+1}^{-1}$ for some word w_i in $x_1, x_2, \ldots, x_n$ for all $i = 1, 2, \ldots, n-1$ (determined by a description of the chosen ribbon disk D). Hence the final relator r_n of $\pi_1(X)$ corresponds to the longitude of J which is a meridian of K, so we can assume that $r_n = x_1$. It is now straightforward to see that the corresponding balanced presentation

$$\pi_1(X) = \langle x_1, x_2, \ldots, x_n \mid w_1 x_1 w_1^{-1} x_2^{-1}, \ldots w_{n-1} x_{n-1} w_{n-1}^{-1} x_n^{-1}, x_1 \rangle$$

is Andrews-Curtis trivial. Since $X \times [0,1]$ is a contractible 5-dimensional 2-handlebody such that the corresponding balanced presentation of $\pi_1(X \times [0,1])$ is Andrews-Curtis trivial, a typical 5-dimensional Mazur type argument shows that $X \times [0,1]$ is diffeomorphic to the 5-ball D^5 (for example, see [8, Lemma 3.4]). □

Remark 2.1. In the proof of Theorem 1.2, we showed that $X \times [0,1]$ is diffeomorphic to D^5, or equivalently, the double DX of X is diffeomorphic to S^4. Here is an alternative way to see this which we explain briefly. As mentioned in the above, we can assume that J is a meridian of K. Then the double DX of X is either S^4 or is obtained from S^4 by doing Gluck twist on a 2-knot S where S is the double of the chosen ribbon disk D. Precisely speaking, we perform the Gluck twist on S when the framing of J is odd. It is well-known that the Gluck twist of S^4 along a ribbon 2-knot is diffeomorphic to S^4 (for example, see [12, 9, 13]). Since S is the double of a ribbon disk D in D^4, S is a ribbon 2-knot in S^4 and hence the double DX of X is diffeomorphic to S^4.

We noted that Corollary 1.3 is a special case of Theorem 1.2 that J is isotopic to a meridian of K. We give a detailed proof of this as follows.

Proof of Corollary 1.3. Suppose that a homology 3-sphere Y is obtained from S^3 by doing Dehn surgery along a ribbon knot K. Since Y is a homology 3-sphere, the surgery coefficient is $\frac{1}{m}$ for some integer m. Then Y is obtained from the 0-surgery manifold M of K by doing an integer surgery along a meridian of K with framing $-m$ (see [9, pp. 163–164]). By Theorem 1.2, Y smoothly embeds in S^4. □

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