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Gegenbauer Polynomials For a New Subclass of Bi-univalent Functions

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Abstract. In this study, we introduce and investigate a novel subclass of analytic bi-univalent functions, which we define using Gegenbauer polynomials. We derive the initial

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coefficient bounds for $|a_2|, |a_3|$, and $|a_4|$, and establish Fekete-Szegö inequalities for this class. In addition, we confirm that Brannan and Clunie's conjecture, $|a_2| \leq \sqrt{2}$, is valid for this subclass. To facilitate better understanding, we provide visualizations of the functions, using appropriately chosen parameters.

1. Introduction

Let us denote the class of all normalized analytic functions as $\mathcal{A}$. These functions, represented as $\mathtt{f}(z)$, have the form

$$\mathtt{f}(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad (z \in \mathbb{D}), \tag{1.1}$$

where $\mathbb{D}$ is the set of complex numbers z such that $|z| < 1$. We define $\mathcal{S}$ as the subclass of $\mathcal{A}$ that consists of univalent functions.

If $\mathtt{h}_1(z)$ and $\mathtt{h}_2(z)$ belong to $\mathcal{A}$, we say that $\mathtt{h}_1(z)$ is subordinate to $\mathtt{h}_2(z)$ if there exists a function $\zeta(z)$ with $\zeta(0) = 0$ and $|\zeta(z)| < 1$ in $\mathbb{D}$ such that $\mathtt{h}_1(z) = \mathtt{h}_2(\zeta(z))$. We denote this as $\mathtt{h}_1(z) \prec \mathtt{h}_2(z)$.

A function $\mathtt{f}(z)$ in $\mathcal{S}$ is considered bi-univalent if its inverse, $\mathtt{f}^{-1}(w)$, has an analytic continuation to $|w| < 1$ in the w-plane. We denote σ as the class of all bi-univalent functions in $\mathbb{D}$.

If $\mathtt{f}^{-1}(w) = \mathtt{g}(w)$ is of the form

$$\mathtt{g}(w) = w + \sum_{j=2}^{\infty} b_j w^j, \quad (w \in \mathbb{D}),$$

then we have

$$\mathtt{g}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots, \quad (w \in \mathbb{D}).$$

The class of bi-univalent functions was introduced by Lewin [4] in 1967, who provided an estimate for the second coefficient for functions in this class as $|a_2| < 1.51$. This result was later improved by Brannan and Clunie [2] to $|a_2| \leq \sqrt{2}$. There is extensive literature on the estimates of the initial coefficients of bi-univalent functions (see [1, 6, 7]).

The Fekete-Szegö problem involves finding sharp bounds for $|a_3 - \varrho a_2^2|$ of any compact family of functions. When $\varrho = 1$, the functional represents the Schwarzian derivative, which plays a significant role in the theory of Geometric functions.

For $\eta \in \mathbb{R}$ and $\eta \neq 0$, the generating function of Gegenbauer polynomials is defined as

$$\mathcal{G}_\eta(r, z) = \frac{1}{(1 - 2rz + z^2)^\eta},$$

where $r \in [-1, 1]$ and $z \in \mathbb{D}$. The function $\mathcal{G}_\eta$, for a fixed r, is analytic in $\mathbb{D}$ and can be written as

$$\mathcal{G}_\eta(r, z) = \sum_{k=0}^{\infty} \mathrm{Q}_k^\eta(r) z^k,$$

where $\mathrm{Q}_k^\eta(r)$ is the Gegenbauer polynomial of degree k. When $\eta = 0$, $\mathcal{G}_\eta$ does not exist. Thus, the Gegenbauer polynomial for $\eta = 0$ is generated by the following function [3, 5]

$$\mathcal{G}_0(r, z) = 1 - \log(1 - 2rz + z^2) = \sum_{k=0}^{\infty} \mathrm{Q}_k^0(r) z^k.$$

The function $\mathcal{G}_\eta(r, z)$ gets normalized when $\eta > -1/2$. The Gegenbauer polynomials are defined by the following recurrence relation [1]

$$\mathrm{Q}_k^\eta(r) = \frac{1}{k}[2r(k + \eta - 1)\mathrm{Q}_{k-1}^\eta(r) - (k + 2\eta - 2)\mathrm{Q}_{k-2}^\eta(r)], \quad (k = 2, 3, ...) \tag{1.2}$$

with initial conditions

$$\mathrm{Q}_0^\eta(r) = 1, \quad \mathrm{Q}_1^\eta(r) = 2\eta r.$$

For $k = 2$, we have

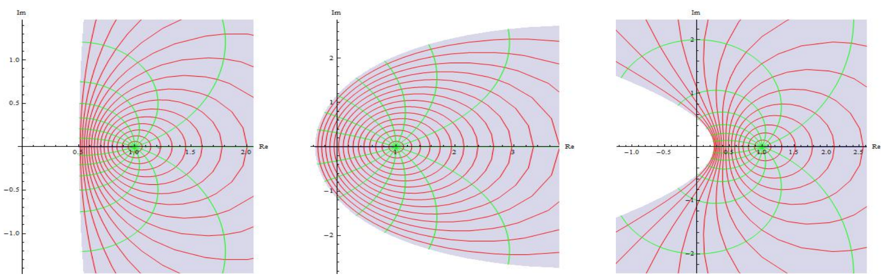
$$\mathrm{Q}_2^\eta(r) = 2\eta(1 + \eta)r^2 - \eta.$$

Table 1 shows the special cases of $\mathrm{Q}_k^\eta(r)$.

S. No.	*Parameter*	*Special Cases*
1	$\eta = 0.5$	Legendre polynomials
2	$\eta = 1$	Chebyshev polynomials

Table 1: Special cases of the Gegenbauer polynomials

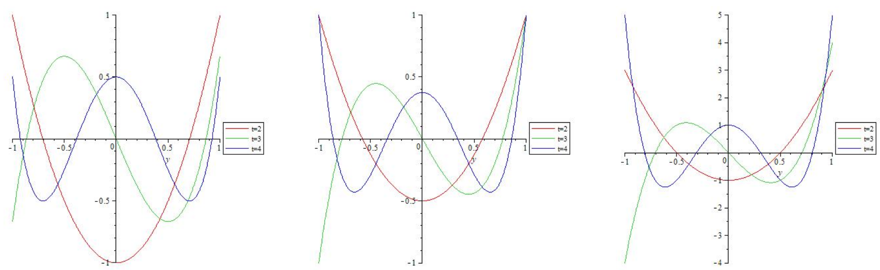
Figure 1 shows the images of $\mathbb{D}$ under $\mathcal{G}_\eta(r, z)$.



$r = -1$ and $\eta = 0.5$ $\quad$ $r = -1$ and $\eta = 0$ $\quad$ $r = 1$ and $\eta = 1$

Figure 1: Image of $\mathbb{D}$ under $\mathcal{G}_\eta(r, z)$.

Figure 2 shows the Graphs of $\mathrm{Q}_k^\eta(r)$.



$\eta = 0$ $\quad$ $\eta = 0.5$ $\quad$ $\eta = 1$

Figure 2: Graph of $\mathrm{Q}_k^\eta(r)$.

Unless otherwise mentioned, we assume in this paper that

$$\delta_n := 1 + (n-1)(\mathtt{v} + \mathtt{t}) + \left(n^2 + 1\right)\mathtt{vt}, \quad n \in \mathbb{N}. \tag{1.3}$$

where $\mathtt{v} \geq 0$ and $\mathtt{t} \in [0, 1]$. It is evident that δ_n is a real number and $\delta_n \geq 1$, and

$$\delta_{n+1} - \delta_n = \big(1 + (2n+1)\,\mathtt{t}\big)\mathtt{v} + \mathtt{t} \geq 0.$$

For every $\mathtt{h} \in \mathcal{A}$, we define

$$\Xi_{\mathtt{v},\mathtt{t}}\left(\mathtt{h}(z)\right) := (1-\mathtt{v})(1-\mathtt{t})\frac{\mathtt{h}(z)}{z} + (\mathtt{t} + \mathtt{v}(1+\mathtt{t}))\,\mathtt{h}'(z) + \mathtt{vt}(z\mathtt{h}''(z) - 2). \tag{1.4}$$

If $\mathtt{h} \in \mathcal{A}$ is of the form $\mathtt{h}(z) = z + \sum_{j=2}^{\infty} u_j z^j$, we have

$$\Xi_{\mathtt{v},\mathtt{t}}\left(\mathtt{h}(z)\right) = 1 + \sum_{j=2}^{\infty} \delta_j u_j z^{j-1}. \tag{1.5}$$

With the aid of Gegenbauer polynomials, we define subclasses of σ using the notion of subordination.

Definition 1.1. A function $\mathtt{f} \in \sigma$ is said to be in the class $\mathcal{A}_\sigma(\mathtt{v}, \mathtt{t}; r)$, if

$$\Xi_{\mathtt{v},\mathtt{t}}\left(\mathtt{f}(z)\right) \prec \mathcal{G}_\eta(r, z), \quad (z \in \mathbb{D}),$$

and

$$\Xi_{\mathtt{v},\mathtt{t}}\left(\mathtt{g}(w)\right) \prec \mathcal{G}_\eta(r, w), \quad (w \in \mathbb{D})$$

where $\mathtt{g}(w) = \mathtt{f}^{-1}(w) = w + \sum\limits_{j=2}^{\infty} b_j w^j$.

Example 1.1. For $\mathtt{t} = 0$ and $\mathtt{v} \geq 1$, $\mathtt{f} \in \sigma$ is in the class $\mathcal{A}_\sigma(\mathtt{v}, 0; r) = \mathcal{D}_\sigma(\mathtt{v}; r)$, if

$$\Xi_{\mathtt{v},0}\left(\mathtt{f}(z)\right) \prec \mathcal{G}_\eta(r, z),$$

and

$$\Xi_{\mathtt{v},0}\left(\mathtt{g}(w)\right) \prec \mathcal{G}_\eta(r, w)$$

where $\mathtt{g} = \mathtt{f}^{-1}$.

Example 1.2. For $\mathtt{t} = 0$ and $\mathtt{v} = 1$, $\mathtt{f} \in \sigma$ is in the class $\mathcal{A}_\sigma(1, 0; r) = \mathcal{D}_\sigma(1; r) = \mathcal{H}_\sigma(r)$, if

$$\Xi_{1,0}\left(\mathtt{f}(z)\right) \prec \mathcal{G}_\eta(r, z),$$

and

$$\Xi_{1,0}\left(\mathtt{g}(w)\right) \prec \mathcal{G}_\eta(r, w)$$

where $\mathtt{g} = \mathtt{f}^{-1}$.

2. The Coefficient Bounds

Theorem 2.1. *If $\mathtt{f}(z)$, given by (1.1), is in $\mathcal{A}_\sigma(\mathtt{v}, \mathtt{t}; r)$, then*

$$|a_2| \leq \frac{2|\eta r|\sqrt{2|\eta r|}}{\sqrt{|4\delta_3\eta^2 r^2 - \delta_2^2(2\eta(1+\eta)r^2 - \eta)|}} \tag{2.1}$$

$$|a_3| \leq \frac{2|\eta r|}{\delta_3} + \frac{4\eta^2 r^2}{\delta_2^2} \tag{2.2}$$

and

$$|a_4| \leq \frac{20\eta^2 r^2}{\delta_2\delta_3} + \frac{|\mathtt{D}|}{2\delta_4} \tag{2.3}$$

where $\mathtt{D} = \frac{8}{3}\eta^2\left(1+\eta\right)r^3 + 8\eta(1+\eta)r^2 + \frac{2}{3}\left(\eta - 6\eta^2 + 4\right)r - 4\eta.$

Proof. Since $\mathtt{f} \in \mathcal{A}_\sigma(\mathtt{v},\mathtt{t};r)$, there exist two analytic functions $\alpha, \beta : \mathbb{D} \to \mathbb{D}$ given by

$$\alpha(z) = \sum_{j=1}^{\infty} \alpha_j z^j \tag{2.4}$$

and

$$\beta(w) = \sum_{j=1}^{\infty} \beta_j w^j \tag{2.5}$$

with $\alpha(0) = \beta(0) = 0, |\alpha(z)| < 1, |\beta(w)| < 1$ for all $z, w \in \mathbb{D}$ such that

$$\Xi_{\mathtt{v},\mathtt{t}}\left(\mathtt{f}(z)\right) = \mathcal{G}_\eta(r, \alpha(z))$$

and

$$\Xi_{\mathtt{v},\mathtt{t}}\left(\mathtt{g}(w)\right) = \mathcal{G}_\eta(r, \beta(w)).$$

Or equivalently

$$1 + \sum_{j=2}^{\infty} \delta_j a_j z^{j-1} = 1 + \mathrm{Q}_1^\eta(r)\alpha_1 z + [\mathrm{Q}_1^\eta(r)\alpha_2 + \mathrm{Q}_2^\eta(r)\alpha_1^2]z^2 + \cdots \tag{2.6}$$

and

$$1 + \sum_{j=2}^{\infty} \delta_j b_j w^{j-1} = 1 + \mathrm{Q}_1^\eta(r)\beta_1 w + [\mathrm{Q}_1^\eta(r)\beta_2 + \mathrm{Q}_2^\eta(r)\beta_1^2]w^2 + \cdots . \tag{2.7}$$

Since $|\alpha(z)| < 1$ and $|\beta(w)| < 1$, it is clear that

$$|\alpha_j| \leq 1, \tag{2.8}$$

$$|\beta_j| \leq 1 \tag{2.9}$$

for $j = 1, 2, ...$
From (2.6) and (2.7), we have

$$\delta_2 a_2 = \mathrm{Q}_1^\eta(r)\alpha_1 \tag{2.10}$$

$$\delta_3 a_3 = \mathrm{Q}_1^\eta(r)\alpha_2 + \mathrm{Q}_2^\eta(r)\alpha_1^2 \tag{2.11}$$

$$-\delta_2 a_2 = \mathrm{Q}_1^\eta(r)\beta_1 \tag{2.12}$$

$$\delta_3(2a_2^2 - a_3) = \mathrm{Q}_1^\eta(r)\beta_2 + \mathrm{Q}_2^\eta(r)\beta_1^2 \tag{2.13}$$

$$\delta_4 a_4 = \mathrm{Q}_1^{\eta}(r)\alpha_3 + 2\mathrm{Q}_2^{\eta}(r)\alpha_1\alpha_2 + \mathrm{Q}_3^{\eta}(r)\alpha_1^3 \tag{2.14}$$

and

$$\delta_4(5a_2a_3 - a_4 - 5a_2^3) = \mathrm{Q}_1^{\eta}(r)\beta_3 + 2\mathrm{Q}_2^{\eta}(r)\beta_1\beta_2 + \mathrm{Q}_3^{\eta}(r)\beta_1^3. \tag{2.15}$$

From (2.10) and (2.12), we can easily see that

$$\alpha_1 = -\beta_1 \tag{2.16}$$

and

$$2\delta_2^2 a_2^2 = [\mathrm{Q}_1^{\eta}(r)]^2[\alpha_1^2 + \beta_1^2]. \tag{2.17}$$

Upon adding (2.11) and (2.13), we get

$$2\delta_3 a_2^2 = \mathrm{Q}_1^{\eta}(r)(\alpha_2 + \beta_2) + \mathrm{Q}_2^{\eta}(r)(\alpha_1^2 + \beta_1^2). \tag{2.18}$$

By using (2.17) in (2.18), we have

$$2\left[\delta_3(\mathrm{Q}_1^{\eta}(r))^2 - \delta_2^2\mathrm{Q}_2^{\eta}(r)\right] a_2^2 = [\mathrm{Q}_1^{\eta}(r)]^3(\alpha_2 + \beta_2) \tag{2.19}$$

which implies

$$|a_2| \leq \frac{2|\eta r|\sqrt{2|\eta r|}}{\sqrt{|4\delta_3\eta^2 r^2 - \delta_2^2(2\eta(1+\eta)r^2 - \eta)|}}.$$

Upon subtracting (2.13) from (2.11) and using (2.16), we get

$$a_3 - a_2^2 = \frac{\mathrm{Q}_1^{\eta}(r)(\alpha_2 - \beta_2)}{2\delta_3}. \tag{2.20}$$

Then, in aid of (2.17), we get

$$a_3 = \frac{\mathrm{Q}_1^{\eta}(r)(\alpha_2 - \beta_2)}{2\delta_3} + \frac{[\mathrm{Q}_1^{\eta}(r)]^2(\alpha_1^2 + \beta_1^2)}{2\delta_2^2}. \tag{2.21}$$

Thus

$$|a_3| \leq \frac{2|\eta r|}{\delta_3} + \frac{4\eta^2 r^2}{\delta_2^2}.$$

By using (2.10), (2.11), (2.12) and (2.14), we get

$$\begin{aligned} a_4 = & \frac{5\mathrm{Q}_1^{\eta}(r)\alpha_1(\alpha_2-\beta_2)}{\delta_2\delta_3} + \frac{5(\mathrm{Q}_1^{\eta}(r))^3\alpha_1(\alpha_1^2+\beta_1^2)}{\delta_2^3} \\ & - \frac{10(\mathrm{Q}_1^{\eta}(r))^3\alpha_1^3}{\delta_2^3} + \frac{2\mathrm{Q}_2^{\eta}(r)(\alpha_1\alpha_2-\beta_1\beta_2)}{2\delta_4} + \frac{\mathrm{Q}_1^{\eta}(r)(\alpha_3-\beta_3)}{2\delta_4} + \frac{\mathrm{Q}_3^{\eta}(r)(\alpha_1^3-\beta_1^3)}{2\delta_4} \end{aligned}$$

which implies

$$|a_4| \leq \frac{20\eta^2 r^2}{\delta_2\delta_3} + \frac{|\mathtt{D}|}{2\delta_4}.$$

□

Corollary 2.1. If $\mathtt{f}(z)$, given by (1.1), is in $\mathcal{D}_\sigma(\mathtt{v}; r)$, then

$$|a_2| \leq \frac{2|\eta r|\sqrt{2|\eta r|}}{\sqrt{|\left(1+2\mathtt{v}\right)4\eta^2 r^2 - \left(1+\mathtt{v}\right)^2\left(2\eta(1+\eta)r^2 - \eta\right)|}}$$

$$|a_3| \leq \frac{2|\eta r|}{1+2\mathtt{v}} + \frac{4\eta^2 r^2}{\left(1+\mathtt{v}\right)^2}$$

and

$$|a_4| \leq \frac{20\eta^2 r^2}{(1+2\mathtt{v})(1+\mathtt{v})} + \frac{|\mathtt{D}|}{2(1+3\mathtt{v})}$$

where $\mathtt{D}$ is as in Theorem 2.1.

Corollary 2.2. If $\mathtt{f}(z)$, given by (1.1), is in $\mathcal{H}_\sigma(r)$, then

$$|a_2| \leq \frac{|\eta r|\sqrt{2|\eta r|}}{\sqrt{|3\eta^2 r^2 - (2\eta(1+\eta)r^2 - \eta)|}}$$

$$|a_3| \leq \frac{2|\eta r|}{3} + \eta^2 r^2$$

and

$$|a_4| \leq 10\eta^2 r^2 + \frac{|\mathtt{D}|}{8}$$

where $\mathtt{D}$ is as in Theorem 2.1.

Corollary 2.3. If $\mathtt{f}(z)$, given by (1.1), is in $\mathcal{A}_\sigma(\mathtt{v}, \mathtt{t}; r)$ with $\eta = 1$, then

$$|a_2| \leq \frac{2|r|\sqrt{2|r|}}{\sqrt{|4\delta_3 r^2 - (4r^2-1)\delta_2^2|}}$$

$$|a_3| \leq \frac{2|r|}{\delta_3} + \frac{4r^2}{\delta_2^2}$$

and

$$|a_4| \leq \frac{20r^2}{\delta_2\delta_3} + \frac{\left|8r^3 + 24r^2 + r - 6\right|}{3\delta_3}.$$

Corollary 2.4. If $\mathtt{f}(z)$, given by (1.1), is in $\mathcal{A}_\sigma(\mathtt{v}, \mathtt{t}; r)$ with $\eta = \frac{1}{2}$, then

$$|a_2| \leq \frac{|r|\sqrt{2|r|}}{\sqrt{\left|2\delta_3 r^2 - (3r^2-1)\,\delta_2^2\right|}}$$

$$|a_3| \leq \frac{|r|}{\delta_3} + \frac{r^2}{\delta_2^2}$$

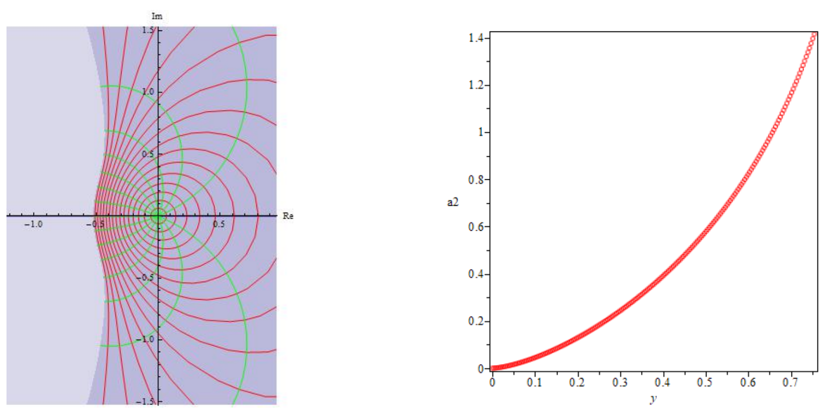
and

$$|a_4| \leq \frac{5r^2}{\delta_2\delta_3} + \frac{|r^3+6r^2+2r-2|}{2\delta_4}.$$

Remark 2.1. When $\mathtt{v}=0, \mathtt{t}=1, \eta=1$ and $|r| \leq 0.75490...$, we obtain Brannan and Cliunie's [2] conjecture $|a_2| \leq \sqrt{2}$.

2.1. Visualization of Certain Class-Specific Functions and Co-efficient bounds

Results need to be clear and recognized. Geometrical visualization is the use of visualization to comprehend and investigate mathematical processes. Some of the diagrams that assist us in seeing, comprehending, and analyzing the nature of the functions and co-efficient bounds for a_2 which holds Brannan and Clunie's conjecture in $\mathcal{A}_\sigma(\mathtt{v},\mathtt{t};r)$ with a suitable choice of parameters, are presented in this article.



(a) $\frac{9z}{9-9z-z^2}$ (b) Co-efficient bounds for a_2

Figure 3: Example for class and co-efficient bounds a_2 for class $\mathcal{A}_\sigma(\mathtt{v},\mathtt{t};r)$ with a suitable choice of parameter.

3. Fekete-Szegö Inequalities

Theorem 3.1. *If* $\mathtt{f}(z)$*, given by* (1.1)*, is in* $\mathcal{A}_\sigma(\mathtt{v},\mathtt{t};r)$ *and* $\varrho \in \mathbb{R}$*, then*

$$|a_3-\varrho a_2^2| \leq \begin{cases} \dfrac{2|\eta r|}{\delta_3}, & |\varrho-1| \leq \left|1-\dfrac{\delta_2^2(2\eta(1+\eta)r^2-\eta)}{4\delta_3\eta^2r^2}\right| \\ \dfrac{8|\varrho-1||\eta^3r^3|}{|4\delta_3\eta^2r^2-\delta_2^2(2\eta(1+\eta)r^2-\eta)|}, & |\varrho-1| \geq \left|1-\dfrac{\delta_2^2(2\eta(1+\eta)r^2-\eta)}{4\delta_3\eta^2r^2}\right|. \end{cases}$$

Proof. For $\varrho \in \mathbb{R}$ and from (2.20), we have

$$a_3 - \varrho a_2^2 = \frac{\mathrm{Q}_1^{\eta}(r)(\alpha_2 - \beta_2)}{2\delta_3} + (1-\varrho)a_2^2. \tag{3.1}$$

By using (2.19), we get

$$\begin{aligned} a_3 - \varrho a_2^2 &= \frac{\mathrm{Q}_1^{\eta}(r)(\alpha_2 - \beta_2)}{2\delta_3} + (1-\varrho)\left(\frac{[\mathrm{Q}_1^{\eta}(r)]^3(\alpha_2+\beta_2)}{2[\delta_3[\mathrm{Q}_1^{\eta}(r)]^2 - \delta_2^2\mathrm{Q}_2^{\eta}(r)]}\right) \\ &= \mathrm{Q}_1^{\eta}(r)\left[\left(\frac{1}{2\delta_3} + \Upsilon(\mathtt{v},\mathtt{t})\right)\alpha_2 + \left(\frac{-1}{2\delta_3} + \Upsilon(\mathtt{v},\mathtt{t})\right)\beta_2\right] \end{aligned}$$

where $\Upsilon(\mathtt{v},\mathtt{t}) = \dfrac{(1-\varrho)[\mathrm{Q}_1^{\eta}(r)]^2}{2\left(\delta_3[\mathrm{Q}_1^{\eta}(r)]^2 - \delta_2^2\mathrm{Q}_2^{\eta}(r)\right)}$.

Thus

$$|a_3 - \varrho a_2^2| \leq \begin{cases} \frac{|\mathrm{Q}_1^{\eta}(r)|}{\delta_3}, & 0 \leq |\Upsilon(\mathtt{v},\mathtt{t})| \leq \frac{1}{2\delta_3} \\ 2|\mathrm{Q}_1^{\eta}(r)\Upsilon(\mathtt{v},\mathtt{t})|, & |\Upsilon(\mathtt{v},\mathtt{t})| \geq \frac{1}{2\delta_3}. \end{cases}$$

□

Corollary 3.1. If $\mathtt{f}(z)$, given by (1.1), is in $\mathcal{D}_\sigma(\mathtt{v};r)$ and $\varrho \in \mathbb{R}$, then

$$|a_3 - \varrho a_2^2| \leq \begin{cases} \frac{2|\eta r|}{|1+2\mathtt{v}|}, & |\varrho - 1| \leq |1 - \frac{(1+\mathtt{v})^2(2\eta(1+\eta)r^2-\eta)}{4(1+2\mathtt{v})\eta^2r^2}| \\ \frac{8|\varrho-1||\eta^3r^3|}{|4(1+2\mathtt{v})\eta^2r^2-(1+\mathtt{v})^2(2\eta(1+\eta)r^2-\eta)|}, & |\varrho - 1| \geq |1 - \frac{(1+\mathtt{v})^2(2\eta(1+\eta)r^2-\eta)}{4(1+2\mathtt{v})\eta^2r^2}|. \end{cases}$$

Corollary 3.2. If $\mathtt{f}(z)$, given by (1.1), is in $\mathcal{H}_\sigma(r)$ and $\varrho \in \mathbb{R}$, then

$$|a_3 - \varrho a_2^2| \leq \begin{cases} \frac{2|\eta r|}{3}, & |\varrho - 1| \leq |1 - \frac{2\eta(1+\eta)r^2-\eta}{3\eta^2r^2}| \\ \frac{2|\varrho-1||\eta^3r^3|}{|3\eta^2r^2-(2\eta(1+\eta)r^2-\eta)|}, & |\varrho - 1| \geq |1 - \frac{2\eta(1+\eta)r^2-\eta}{3\eta^2r^2}|. \end{cases}$$

Corollary 3.3. If $\mathtt{f}(z)$, given by (1.1), is in $\mathcal{A}_\sigma(\mathtt{v},\mathtt{t};r)$ with $\eta = 1$ and $\varrho \in \mathbb{R}$, then

$$|a_3 - \varrho a_2^2| \leq \begin{cases} \frac{2|r|}{\delta_3}, & |\varrho - 1| \leq |1 - \frac{\delta_2^2(4r^2-1)}{4\delta_3r^2}| \\ \frac{8|\varrho-1||r^3|}{|4\delta_3r^2-\delta_2^2(4r^2-1)|}, & |\varrho - 1| \geq |1 - \frac{\delta_2^2(4r^2-1)}{4\delta_3r^2}|. \end{cases}$$

Corollary 3.4. If $\mathtt{f}(z)$, given by (1.1), is in $\mathcal{A}_\sigma(\mathtt{v},\mathtt{t};r)$ with $\eta = 1/2$ and $\varrho \in \mathbb{R}$, then

$$|a_3 - \varrho a_2^2| \leq \begin{cases} \frac{|r|}{\delta_3}, & |\varrho - 1| \leq |1 - \frac{\delta_2^2(3r^2-1)}{2\delta_3r^2}| \\ \frac{2|\varrho-1||r^3|}{|2\eta r^2-\delta_2^2(3r^2-1)|}, & |\varrho - 1| \geq |1 - \frac{\delta_2^2(3r^2-1)}{2\delta_3r^2}|. \end{cases}$$

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