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Problems in the Geometry of the Siegel-Jacobi Space

Jae-Hyun Yang
Department of Mathematics, Inha University, Incheon 22212, Republic of Korea
e-mail : jhyang@inha.ac.kr; jhyang8357@gmail.com

Abstract. The Siegel-Jacobi space is a non-symmetric homogeneous space which is very important geometrically and arithmetically. In this short paper, we propose the basic problems in the geometry of the Siegel-Jacobi space.

1. Introduction

For a given fixed positive integer n, we let

$$\mathbb{H}_n = \{\, \Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega, \quad \mathrm{Im}\,\Omega > 0 \,\}$$

be the Siegel upper half plane of degree n and let

$$Sp(n,\mathbb{R}) = \{M \in \mathbb{R}^{(2n,2n)} \mid {}^tMJ_nM = J_n\}$$

be the symplectic group of degree n, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^tM$ denotes the transposed matrix of a matrix M and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Then $Sp(n,\mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{1.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $\Omega \in \mathbb{H}_n$. Let

$$\Gamma_n = Sp(n,\mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}) \,\Big|\, A,B,C,D \text{ integral} \right\}$$

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be the Siegel modular group of degree n. This group acts on $\mathbb{H}_n$ properly discontinuously. C. L. Siegel investigated the geometry of $\mathbb{H}_n$ and automorphic forms on $\mathbb{H}_n$ systematically. Siegel [16] found a fundamental domain $\mathcal{F}_n$ for $\Gamma_n\backslash\mathbb{H}_n$ and described it explicitly. Moreover he calculated the volume of $\mathcal{F}_n$. We also refer to [13, 16] for some details on $\mathcal{F}_n$.

For two positive integers m and n, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \big\{ (\lambda,\mu;\kappa) \,|\, \lambda,\mu \in \mathbb{R}^{(m,n)},\ \kappa\in\mathbb{R}^{(m,m)},\ \kappa+\mu\,{}^t\lambda \text{ symmetric}\big\}$$

endowed with the following multiplication law

$$\big(\lambda,\mu;\kappa\big)\circ\big(\lambda',\mu';\kappa'\big) = \big(\lambda+\lambda',\mu+\mu';\kappa+\kappa'+\lambda\,{}^t\mu'-\mu\,{}^t\lambda'\big)$$

with $\big(\lambda,\mu;\kappa\big),\big(\lambda',\mu';\kappa'\big)\in H_{\mathbb{R}}^{(n,m)}$. We define the *Jacobi group* G^J of degree n and index m that is the semidirect product of $Sp(n,\mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$

$$G^J = Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

endowed with the following multiplication law

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot\big(M',(\lambda',\mu';\kappa'\,)\big) = \big(MM',(\tilde\lambda+\lambda',\tilde\mu+\mu';\kappa+\kappa'+\tilde\lambda\,{}^t\mu'-\tilde\mu\,{}^t\lambda'\,)\big)$$

with $M,M'\in Sp(n,\mathbb{R}),(\lambda,\mu;\kappa),\ (\lambda',\mu';\kappa')\in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde\lambda,\tilde\mu)=(\lambda,\mu)M'$. Then G^J acts on $\mathbb{H}_n\times\mathbb{C}^{(m,n)}$ transitively by

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot(\Omega,Z) = \Big(M\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}\Big), \tag{1.2}$$

where $M=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega,Z)\in\mathbb{H}_n\times\mathbb{C}^{(m,n)}$. We note that the Jacobi group G^J is *not* a reductive Lie group and the homogeneous space $\mathbb{H}_n\times\mathbb{C}^{(m,n)}$ is not a symmetric space. From now on, for brevity we write $\mathbb{H}_{n,m}=\mathbb{H}_n\times\mathbb{C}^{(m,n)}$. The homogeneous space $\mathbb{H}_{n,m}$ is called the *Siegel-Jacobi space* of degree n and index m.

In this short article, we propose the basic and natural problems in the geometry of the Siegel-Jacobi space.

Notations: We denote by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by $\mathbb{Z}$ the ring of integers. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k\times l$ matrices with entries in a commutative ring F. For a square matrix $A\in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For any $M\in F^{(k,l)}$, ${}^tM$ denotes the transpose of a matrix M. I_n denotes the identity matrix of degree n.

For a complex matrix A, $\overline{A}$ denotes the complex *conjugate* of A. For a number field F, we denote by $\mathbb{A}_F$ the ring of adeles of F. If $F = \mathbb{Q}$, the subscript will be omitted.

2. Brief Review on the Geometry of the Siegel Space

We let $G := Sp(n, \mathbb{R})$ and $K = U(n)$. The stabilizer of the action (1.1) at iI_n is

$$\left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \,\middle|\, A + iB \in U(n) \right\} \cong U(n).$$

Thus we get the biholomorphic map

$$G/K \longrightarrow \mathbb{H}_n, \qquad gK \mapsto g\cdot iI_n, \quad g \in G.$$

$\mathbb{H}_n$ is a Hermitian symmetric manifold.

For $\Omega = (\omega_{ij}) \in \mathbb{H}_n$, we write $\Omega = X + iY$ with $X = (x_{ij}),\ Y = (y_{ij})$ real. We put $d\Omega = (d\omega_{ij})$ and $d\overline{\Omega} = (d\overline{\omega}_{ij})$. We also put

$$\frac{\partial}{\partial \Omega} = \left(\frac{1+\delta_{ij}}{2}\, \frac{\partial}{\partial \omega_{ij}} \right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{\Omega}} = \left(\frac{1+\delta_{ij}}{2}\, \frac{\partial}{\partial \overline{\omega}_{ij}} \right).$$

C. L. Siegel [16] introduced the symplectic metric $ds^2_{n;A}$ on $\mathbb{H}_n$ invariant under the action (1.1) of $Sp(n, \mathbb{R})$ that is given by

$$ds^2_{n;A} = A\, \sigma(Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega}), \qquad A > 0. \tag{2.1}$$

It is known that the metric $ds^2_{n;A}$ is a Kähler-Einstein metric. H. Maass [12] proved that its Laplace operator $\Delta_{n;A}$ is given by

$$\Delta_{n;A} = \frac{4}{A}\, \sigma \left(Y\, {}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}} \right) \frac{\partial}{\partial \Omega} \right). \tag{2.2}$$

And

$$dv_n(\Omega) = (\det Y)^{-(n+1)} \prod_{1 \le i \le j \le n} dx_{ij} \prod_{1 \le i \le j \le n} dy_{ij} \tag{2.3}$$

is a $Sp(n, \mathbb{R})$-invariant volume element on $\mathbb{H}_n$ (cf. [17, p. 130]).

Siegel proved the following theorem for the Siegel space $(\mathbb{H}_n, ds^2_{n;1})$.

Theorem 2.1. (**Siegel** [16]). (1) *There exists exactly one geodesic joining two arbitrary points* Ω_0, Ω_1 *in* $\mathbb{H}_n$. *Let* $R(\Omega_0, \Omega_1)$ *be the cross-ratio defined by*

$$R(\Omega_0, \Omega_1) = (\Omega_0 - \Omega_1)(\Omega_0 - \overline{\Omega}_1)^{-1}(\overline{\Omega}_0 - \overline{\Omega}_1)(\overline{\Omega}_0 - \Omega_1)^{-1}.$$

For brevity, we put $R_* = R(\Omega_0, \Omega_1)$*. Then the symplectic length* $\rho(\Omega_0, \Omega_1)$ *of the geodesic joining* Ω_0 *and* Ω_1 *is given by*

$$\rho(\Omega_0, \Omega_1)^2 = \sigma\left(\left(\log \frac{1 + R_*^{\frac{1}{2}}}{1 - R_*^{\frac{1}{2}}}\right)^2\right),$$

where

$$\left(\log \frac{1 + R_*^{\frac{1}{2}}}{1 - R_*^{\frac{1}{2}}}\right)^2 = 4\, R_* \left(\sum_{k=0}^{\infty} \frac{R_*^k}{2k+1}\right)^2.$$

(2) *For* $M \in Sp(n, \mathbb{R})$*, we set*

$$\tilde{\Omega}_0 = M \cdot \Omega_0 \quad \textit{and} \quad \tilde{\Omega}_1 = M \cdot \Omega_1.$$

Then $R(\Omega_1, \Omega_0)$ *and* $R(\tilde{\Omega}_1, \tilde{\Omega}_0)$ *have the same eigenvalues.*

(3) *All geodesics are symplectic images of the special geodesics*

$$\alpha(t) = i\, diag(a_1^t, a_2^t, \cdots, a_n^t),$$

where $a_1, a_2, \cdots, a_n$ *are arbitrary positive real numbers satisfying the condition*

$$\sum_{k=1}^{n} (\log a_k)^2 = 1.$$

The proof of the above theorem can be found in [16, pp. 289-293].

Let $\mathbb{D}(\mathbb{H}_n)$ be the algebra of all differential operators on $\mathbb{H}_n$ invariant under the action (1.1). Then according to Harish-Chandra [5, 6],

$$\mathbb{D}(\mathbb{H}_n) = \mathbb{C}[D_1, \cdots, D_n],$$

where $D_1, \cdots, D_n$ are algebraically independent invariant differential operators on $\mathbb{H}_n$. That is, $\mathbb{D}(\mathbb{H}_n)$ is a commutative algebra that is finitely generated by n algebraically independent invariant differential operators on $\mathbb{H}_n$. Maass [13] found the explicit $D_1, \cdots, D_n$. Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of the Lie algebra of G. It is known that $\mathbb{D}(\mathbb{H}_n)$ is isomorphic to the center of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ (cf. [7]).

Example 2.2. We consider the simplest case $n = 1$ and $A = 1$. Let $\mathbb{H}$ be the Poincaré upper half plane. Let $\omega = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$ and $y > 0$. Then the Poincaré metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{d\omega\, d\overline{\omega}}{y^2}$$

is a $SL(2,\mathbb{R})$-invariant Kähler-Einstein metric on $\mathbb{H}$. The geodesics of $(\mathbb{H}, ds^2)$ are either straight vertical lines perpendicular to the x-axis or circular arcs perpendicular to the x-axis (half-circles whose origin is on the x-axis). The Laplace operator Δ of $(\mathbb{H}, ds^2)$ is given by

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

and

$$dv = \frac{dx \wedge dy}{y^2}$$

is a $SL(2,\mathbb{R})$-invariant volume element. The scalar curvature, i.e., the Gaussian curvature is -1. The algebra $\mathbb{D}(\mathbb{H})$ of all $SL(2,\mathbb{R})$-invariant differential operators on $\mathbb{H}$ is given by

$$\mathbb{D}(\mathbb{H}) = \mathbb{C}[\Delta].$$

The distance between two points $\omega_1 = x_1 + iy_1$ and $\omega_2 = x_2 + iy_2$ in $(\mathbb{H}, ds^2)$ is given by

$$\begin{aligned} \rho(\omega_1, \omega_2) &= 2 \ln \frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \sqrt{(x_2 - x_1)^2 + (y_2 + y_1)^2}}{2\sqrt{y_1 y_2}} \\ &= \cosh^{-1} \left(1 + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{2 y_1 y_2} \right) \\ &= 2 \sinh^{-1} \frac{1}{2} \sqrt{\frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{y_1 y_2}}. \end{aligned}$$

3. Basic Problems in the Geometry of the Siegel-Jacobi Space

For a coordinate $(\Omega, Z) \in \mathbb{H}_{n,m}$ with $\Omega = (\omega_{\mu\nu})$ and $Z = (z_{kl})$, we put $d\Omega$, $d\overline{\Omega}$, $\frac{\partial}{\partial \Omega}$, $\frac{\partial}{\partial \overline{\Omega}}$ as before and set

$$\begin{aligned} Z &= U + iV, \qquad U = (u_{kl}), \qquad V = (v_{kl}) \ \text{ real}, \\ dZ &= (dz_{kl}), \qquad d\overline{Z} = (d\overline{z}_{kl}), \end{aligned}$$

$$\frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_{1n}} & \cdots & \frac{\partial}{\partial z_{mn}} \end{pmatrix}, \quad \frac{\partial}{\partial \overline{Z}} = \begin{pmatrix} \frac{\partial}{\partial \overline{z}_{11}} & \cdots & \frac{\partial}{\partial \overline{z}_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \overline{z}_{1n}} & \cdots & \frac{\partial}{\partial \overline{z}_{mn}} \end{pmatrix}.$$

The author proved the following theorems in [18].

Theorem 3.1. *For any two positive real numbers A and B,*

$$\begin{aligned} ds^2_{n,m;A,B} &= A\,\sigma\Big(Y^{-1}d\Omega\,Y^{-1}d\overline{\Omega}\Big) \\ &\quad + B\,\Big\{\sigma\Big(Y^{-1}\,{}^tV\,V\,Y^{-1}d\Omega\,Y^{-1}d\overline{\Omega}\Big) + \sigma\Big(Y^{-1}\,{}^t(dZ)\,d\overline{Z}\Big) \\ &\quad -\sigma\Big(V\,Y^{-1}d\Omega\,Y^{-1}\,{}^t(d\overline{Z})\Big) - \sigma\Big(V\,Y^{-1}d\overline{\Omega}\,Y^{-1}\,{}^t(dZ)\Big)\Big\} \end{aligned}$$

is a Riemannian metric on $\mathbb{H}_{n,m}$ which is invariant under the action (1.2) of G^J.

Proof. See [18, Theorem 1.1]. □

Theorem 3.2. *The Laplace operator $\Delta_{m,m;A,B}$ of the G^J-invariant metric $ds^2_{n,m;A,B}$ is given by*

$$\Delta_{n,m;A,B} = \frac{4}{A}\,\mathbb{M}_1 + \frac{4}{B}\mathbb{M}_2, \tag{3.1}$$

where

$$\begin{aligned} \mathbb{M}_1 &= \sigma\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right) + \sigma\left(VY^{-1}\,{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial Z}\right) \\ &\quad + \sigma\left(V\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial Z}\right) + \sigma\left({}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial\Omega}\right) \end{aligned}$$

and

$$\mathbb{M}_2 = \sigma\left(Y\,\frac{\partial}{\partial Z}\,{}^t\!\left(\frac{\partial}{\partial\overline{Z}}\right)\right).$$

Furthermore $\mathbb{M}_1$ and $\mathbb{M}_2$ are differential operators on $\mathbb{H}_{n,m}$ invariant under the action (1.2) of G^J.

Proof. See [18, Theorem 1.2]. □

Remark 3.3. Erik Balslev [2] developed the spectral theory of $\Delta_{1,1;1,1}$ on $H_{1,1}$ for certain arithmetic subgroups of the Jacobi modular group to prove that the set of all eigenvalues of $\Delta_{1,1;1,1}$ satisfies the Weyl law.

Remark 3.4. The scalar curvature of $(\mathbb{H}_{1,1}, ds^2_{1,1;A,B})$ is $-\frac{3}{A}$ and hence is independent of the parameter B. We refer to [21] for more detail.

Remark 3.5. Yang and Yin [22] showed that $ds^2_{n,m;A,B}$ is a Kähler metric. For some applications of the invariant metric $ds^2_{n,m;A,B}$ we refer to [22].

Now we propose the basic and natural problems.

Problem 1. Find all the geodesics of $(\mathbb{H}_{n,m}, ds^2_{n,m;A,B})$ explicitly.

Problem 2. Compute the distance between two points (Ω_1, Z_1) and (Ω_2, Z_2) of $\mathbb{H}_{n,m}$ explicitly.

Problem 3. Compute the Ricci curvature tensor and the scalar curvature of $(\mathbb{H}_n, ds^2_{n,m;A,B})$.

Problem 4. Find all the eigenfunctions of the Laplace operator $\Delta_{n,m;A,B}$.

Problem 5. Develop the spectral theory of $\Delta_{n,m;A,B}$.

Problem 6. Describe the algebra of all G^J-invariant differential operators on $\mathbb{H}_{n,m}$ explicitly. We refer to [19, 20, 22] for some details.

Problem 7. Find the trace formula for the Jacobi group $G^J(\mathbb{A})$.

Problem 8. Discuss the behaviour of the analytic torsion of the Siegel-Jacobi space $\mathbb{H}_{n,m}$ or the arithmetic quotients of $\mathbb{H}_{n,m}$.

We make some remarks on the above problems.

Remark 3.6. Problem 1 reduces to trying to solve a system of ordinary differential equations explicitly. If Problem 2 is solved, the distance formula would be a very beautiful one that generalizes the distance formula $\rho(\Omega_0, \Omega_1)$ given by Theorem 2.1 (the Siegel space case).

Remark 3.7. Problem 3 was recently solved in the case that $n = 1$ and m is arbitrary. Precisely the scalar and Ricci curvatures of the Siegel-Jacobi space $(\mathbb{H}_{1,m}, ds^2_{1,m;A,B})\,(m \geq 1)$ were completely computed by G. Khan and J. Zhang [8, Proposition 8, pp. 825–826]. Furthermore Khan and Zhang proved that $(\mathbb{H}_{1,m}, ds^2_{1,m;A,B})\,(m \geq 1)$ has non-negative orthogonal anti-bisectional curvature (cf. [8, Proposition 9, p. 826]).

Remark 3.8. Concerning Problem 4 and Problem 5, computing eigenfunctions explicitly is a tall order, but if this can be done it will shed a lot of light onto the geometry of this space. And understanding the spectral geometry seems to be a central question which will likely have applications in number theory and other areas.

Remark 3.9. The algebra $\mathbb{D}(\mathbb{H}_{n,m})$ of all G^J-invariant differential operators on $\mathbb{H}_{n,m}$ is not commutative. Concerning Problem 6, the case $n = m = 1$ was completely solved by M. Itoh, H. Ochiai and J.-H. Yang in 2013. They proved that the noncommutative algebra $\mathbb{D}(\mathbb{H}_{1,1})$ is generated by four explicit generators D_1, D_2, D_3, D_4, and found the relations among those $D_i\,(1 \leq i \leq 4)$. For more precise statements, we refer to [19, pp. 56–58] and [20, pp. 285–290]. We note that the above four generators $D_i\,(1 \leq i \leq 4)$ are not algebraically independent.

Remark 3.10. The solution of Problem 7 will provide lots of arithmetic properties of the Siegel-Jacobi space.

4. Final Remarks

Let $\Gamma_n(N)$ be the principal congruence subgroup of the Siegel modular group Γ_n. Let $\mathfrak{X}_n(N) := \Gamma_n(N)\backslash\mathbb{H}_n$ be the moduli of n-dimensional principally polarized abelian varieties with level N-structure. The Mumford school [1] found toroidal compactifications of $\mathfrak{X}_n(N)$ which are usefully applied in the study of the geometry and arithmetic of $\mathfrak{X}_n(N)$. D. Mumford [14] proved the Hirzebruch's Proportionality Theorem in the non-compact case introducing a *good singular* Hermitian metric on an automorphic vector bundle on a smooth toroidal compactification of $\mathfrak{X}_n(N)$ with $N \geq 3$.

We set

$$\Gamma_{n,m}(N) := \Gamma_n \ltimes H_{\mathbb{Z}}^{(n,m)},$$

where

$$H_{\mathbb{Z}}^{(n,m)} = \left\{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)} \mid \lambda, \mu, \kappa \textit{ integral} \right\}.$$

Let

$$\mathfrak{X}_{n,m}(N) := \Gamma_{n,m}(N)\backslash\mathbb{H}_{n,m}$$

be the universal abelian variety. An arithmetic toroidal compactification of $\mathfrak{X}_{n,m}(N)$ was intensively investigated by R. Pink [15]. D. Mumford described very nicely a toroidal compactification of the universal elliptic curve $\mathfrak{X}_{1,1}(N)$ (cf. [1, pp. 14–25]). The geometry of $\mathfrak{X}_{n,m}(N)$ is closely related to the theory of Jacobi forms (cf. [3, 9, 10, 11]). Jacobi forms play an important role in the study of the geometric and arithmetic of $\mathfrak{X}_{n,m}(N)$. We refer to [4, 23] for the theory of Jacobi forms.

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