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The Exponential Representations of Pell and Its Generalized Matrix Sequences

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Abstract. In this paper we define a matrix sequence called the Pell matrix sequence whose elements consist of Pell numbers. Using a positive parameter k, we generalize the Pell matrix sequence to a k-Pell matrix sequence and using two parameters s, t we generalize them to (s, t)-Pell matrix sequences. We give the basic properties of these matrix sequences. Then, using these properties we obtain exponential representations of the Pell matrix sequence and its generalizations in different ways.

1. Introduction and Preliminaries

Sequences of positive integers have long been studied and many special integer sequences are known to have applications in different areas of science. Many researchers devote their attention to special sequences, such a Pell, Pell-Lucas, and Modified Pell sequences, which satisfy a second-order recurrence relation. Horadam studied various properties of Pell numbers and Pell polynomials. Ercolano found generating matrices for Pell sequences. Many mathematicians have looked at generalizations of Pell sequences one gets by adding one or two parameters to the recursion relation but not altering the initial conditions. Identities and *generalting* functions for the k-Pell numbers were established in [3]. The authors of [2] investigated (s, t)-Pell and (s, t)-Pell-Lucas sequences and their matrix representations. In [4], (s, t)-Pell and Pell-Lucas numbers are studied using matrix methods. In [5], the exponential representations of the Jacobsthal matrix sequences were found. In this paper we give the definitions of Pell sequence and its parametrized generalizations. Using the elements of the sequence, we establish matrix sequences for the integer sequences. We demonstrate the exponential matrices for the Pell matrix sequence

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and its generalizations by various methods.

As seen [1, 6], the recurrence relation with initial conditions for the Pell sequence is given as

$$p_n = 2p_{n-1} + p_{n-2}, \quad p_0 = 0 \text{ and } p_1 = 1, \quad n \geq 2.$$

The characteristic equation for the recurrence relation of the Pell sequence is

$$x^2 - 2x - 1 = 0$$

with roots $r_1 = 1 + \sqrt{2}$ and $r_2 = 1 - \sqrt{2}$. It is easily seen that $r_1 + r_2 = 2$, $r_1 r_2 = -1$ and $r_1 - r_2 = 2\sqrt{2}$. The Binet formula for showing Pell numbers as a function of the roots r_1, r_2 is established as

$$p_n = \frac{r_1^n - r_2^n}{r_1 - r_2}. \tag{1.1}$$

The sequence can be generalized using one parameter k, which is any positive integer. The k-Pell sequence $\{p_{k,n}\}_{n\in\mathbb{N}}$ in [3] is demonstrated by

$$p_{k,n} = 2p_{k,n-1} + kp_{k,n-2}, \quad p_{k,0} = 0 \text{ and } p_{k,1} = 1, \quad n \geq 2.$$

It has the characteristic equation

$$x^2 - 2x - k = 0$$

with roots

$$r_{k,1} = 1 + \sqrt{1+k} \text{ and } r_{k,2} = 1 - \sqrt{1+k}.$$

So, the following properties are established

$$r_{k,1} r_{k,2} = -k, \quad r_{k,1} + r_{k,2} = 2, \quad r_{k,1} - r_{k,2} = 2\sqrt{1+k}. \tag{1.2}$$

The Binet formula for the k-Pell sequence with roots $r_{k,1}$ and $r_{k,2}$ is given by

$$p_{k,n} = \frac{r_{k,1}^n - r_{k,2}^n}{r_{k,1} - r_{k,2}}. \tag{1.3}$$

As established in [4, 5], the two-parameter Pell sequence (s,t) -Pell sequence $\{p_n(s,t)\}_{n\in\mathbb{N}}$ is obtained by the following recurrence relation

$$p_n(s,t) = 2sp_{n-1}(s,t) + tp_{n-2}(s,t), \quad p_0(s,t) = 0, \ p_1(s,t) = 1, n \geq 2,$$

where s, t are real numbers such that $s > 0, t \neq 0$ and $s^2 + t > 0$. The characteristic equation of this recurrence relation is

$$x^2 - 2sx - t = 0$$

with roots $r_1(s,t) = s + \sqrt{s^2+t}$ and $r_2(s,t) = s - \sqrt{s^2+t}$. The roots satisfy the relations:

(1.4)
$$r_1(s,t)r_2(s,t) = -t, \quad r_1(s,t) + r_2(s,t) = 2s\ , \quad r_1(s,t) - r_2(s,t) = 2\sqrt{s^2+t}.$$

The Binet formula for (s,t)-Pell numbers with the roots $r_1(s,t)$, $r_2(s,t)$ is given by

$$p_n\left(s,t\right) = \frac{r_1^n(s,t) - r_2^n(s,t)}{r_1(s,t) - r_2(s,t)}. \tag{1.5}$$

2. Pell and Its Generalized Matrix Sequences

The Pell matrix sequence $\{P_n\}_{n\in\mathbb{N}}$ is defined in [2] by the recurrence relation

$$P_{n+1} = 2P_n + P_{n-1}, \quad P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, P_1 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.1}$$

The elements of Pell matrix sequence are the elements of Pell sequence such that

$$P_n = \begin{pmatrix} p_{n+1} & p_n \\ p_n & p_{n-1} \end{pmatrix}.$$

The k-Pell matrix sequence $\{P_{k,n}\}_{n\in\mathbb{N}}$ is established by

$$P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}, \qquad P_{k,0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_{k,1} = \begin{pmatrix} 2 & 1 \\ k & 0 \end{pmatrix}. \tag{2.2}$$

The elements of k-Pell matrix sequence are the elements of k-Pell sequence such that

$$P_{k,n} = \begin{pmatrix} p_{k,n+1} & p_{k,n} \\ kp_{k,n} & kp_{k,n-1} \end{pmatrix}.$$

The (s,t)-Pell matrix sequence $\{P_n(s,t)\}_{n\in\mathbb{N}}$ is defined in [4, 5] by
(2.3)
$$P_{n+1}\left(s,t\right) = 2sP_n\left(s,t\right) + tP_{n-1}\left(s,t\right), \quad P_0\left(s,t\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ P_1\left(s,t\right) = \begin{pmatrix} 2s & 1 \\ t & 0 \end{pmatrix}.$$

The elements of (s,t)-Pell matrix sequence are the elements of (s,t)-Pell sequence such that

$$P_n\left(s,t\right) = \begin{pmatrix} p_{n+1}\left(s,t\right) & p_n\left(s,t\right) \\ tp_n\left(s,t\right) & tp_{n-1}\left(s,t\right) \end{pmatrix}.$$

Lemma 2.1. Assume s,t are real numbers such that $s,t > 0$, $t \neq 0$, $n \geq 1$ an integer, k any positive integer, the following identities hold:

$$P_n = P_1^n, \quad P_{k,n} = P_{k,1}^n, \quad P_n\left(s,t\right) = P_1^n\left(s,t\right).$$

Proof. The proof is made by induction method. We want to prove the last equality that $P_n(s,t) = P_1^n(s,t)$. For $n = 1$, it is easily seen that the assumption is true. Assume that $P_k(s,t) = P_1^k(s,t)$ is true for $k \leq n$. We want to seek for the assumption is valid for $k = n+1$:

$$\begin{aligned} P_1^{n+1}(s,t) &= P_1^n(s,t)\,P_1(s,t) = P_n(s,t)\,P_1(s,t) \\ &= \begin{pmatrix} p_{n+1}(s,t) & p_n(s,t) \\ tp_n(s,t) & tp_{n-1}(s,t) \end{pmatrix} \begin{pmatrix} 2s & 1 \\ t & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2sp_{n+1}(s,t) + tp_n(s,t) & p_{n+1}(s,t) \\ tp_{n+1}(s,t) & tp_n(s,t) \end{pmatrix} \\ &= P_{n+1}(s,t). \end{aligned}$$

If we choose $s = t = 1$ in this equality, we get the first equality.

Similarly, If we choose $s = 1$, $t = k$, we get the second equality. □

Lemma 2.2. Assume s,t are real numbers such that $s,t > 0$, $t \neq 0$ and $n \geq 1$ an integer, k any positive integer, the following identities hold:

$$P_{m+n} = P_m P_n, \quad P_{k,m+n} = P_{k,m} P_{k,n}, \quad P_{m+n}(s,t) = P_m(s,t)\,P_n(s,t).$$

Proof. The proof is made by induction method. We want to prove the second equality that $P_{k,m+n} = P_{k,m}P_{k,n}$. For $n = 0$, it is easily seen that the assumption is true. Assume that $P_{k,m+i} = P_{k,m}P_{k,i}$ is true for $i \leq n$. We want to seek for the assumption is valid for $i = n+1$:

$$\begin{aligned} P_{k,m+n+1} &= 2P_{k,m+n} + kP_{k,m+n-1} = 2P_{k,m}P_{k,n} + kP_{k,m}P_{k,n-1} \\ &= P_{k,m}(2P_{k,n} + kP_{k,n-1}) = P_{k,m}P_{k,n+1}. \end{aligned}$$

The other proofs are made by using the same procedure.

3. The Exponential Representations of Pell Matrix Sequences

In this section, we want to present the exponential representations of the nth element of Pell matrix sequence and the nth element of generalized Pell matrix sequences. If a function $f(z)$ of a complex variable z has a Maclaurin series expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$ which converges for $|z| \prec R$, then the matrix series $\sum_{k=0}^{\infty} a_k A^k$ converges, provided A is square and each of its eigenvalues has absolute value less than R. In such a case, $f(A)$ is defined as $f(A) = \sum_{k=0}^{\infty} a_k A^k$.

Theorem 3.1. *For any integer $n \geq 0$, the exponential representation of the nth element of (s,t)-Pell matrix sequence is in the following form:*

$$e^{P_n(s,t)} = -(-t)^n \sum_{k=0}^{\infty} \frac{p_{nk-n}(s,t)}{p_n(s,t)k!} I_2 + \sum_{k=0}^{\infty} \frac{p_{nk}(s,t)}{p_n(s,t)k!} P_1^n(s,t)$$

where I_2 is the identity matrix and $p_n(s,t)$ is defined in [4]. The theorem gives us the opportunity finding the exponential representations of the nth element of (s,t)-Pell matrix sequence using the nth power of first element of (s,t)-Pell matrix sequence. Similarly, the exponential representation of the nth element of k-Pell matrix sequence is

$$e^{P_{k,n}} = -(-k)^n \sum_{i=0}^{\infty} \frac{p_{k,ni-n}}{p_{k,n} i!} I_2 + \sum_{i=0}^{\infty} \frac{p_{k,ni}}{p_{k,n} i!} P_{k,1}^n,$$

and the exponential representation of the nth element of the Pell matrix sequence is

$$e^{P_n} = \sum_{k=0}^{\infty} \frac{p_{nk}}{p_n k!} P_1^n - (-1)^n \sum_{k=0}^{\infty} \frac{p_{nk-n}}{p_n k!} I_2.$$

Proof. The eigenvalues of $P_1(s,t) = \begin{pmatrix} 2s & t \\ 1 & 0 \end{pmatrix}$ are $r_1(s,t) = s + \sqrt{s^2+t}$ and $r_2(s,t) = s - \sqrt{s^2+t}$. By Lemma 2.1, we know that $P_n(s,t) = P_1^n(s,t)$. Therefore, the eigenvalues of $P_n(s,t)$ are $r_1^n(s,t) = \left(s+\sqrt{s^2+t}\right)^n$ and $r_2^n(s,t) = \left(s-\sqrt{s^2+t}\right)^n$. By the equality $e^{P_n(s,t)} = a_1 P_n(s,t) + a_0 I_2$, we get

$$e^{r_1^n(s,t)} = a_1 r_1^n(s,t) + a_0, \; e^{r_2^n(s,t)} = a_1 r_2^n(s,t) + a_0.$$

By these equations, the values of a_0, a_1 are found. If we substitute the values of a_0, a_1, it is obtained that

$$e^{P_n(s,t)} = \left(\frac{r_1^n(s,t) e^{r_2^n(s,t)} - r_2^n(s,t) e^{r_1^n(s,t)}}{r_1^n(s,t) - r_2^n(s,t)} \right) I_2 + \left(\frac{e^{r_1^n(s,t)} - e^{r_2^n(s,t)}}{r_1^n(s,t) - r_2^n(s,t)} \right) P_n(s,t).$$

Applying the Maclaurin series expansion of $\exp x$ and (1.4), (1.5), we obtain the following:

$$\frac{e^{r_1^n(s,t)} - e^{r_2^n}(s,t)}{r_1^n(s,t) - r_2^n(s,t)} = \sum_{k=0}^{\infty} \left(\frac{r_1^{nk}(s,t) - r_2^{nk}(s,t)}{r_1^n(s,t) - r_2^n(s,t)} \right) \frac{1}{k!} = \sum_{k=0}^{\infty} \frac{p_{nk}(s,t)}{p_n(s,t) k!},$$

$$r_1^n(s,t) e^{r_2^n(s,t)} = r_1^n(s,t) \sum_{k=0}^{\infty} r_2^{nk}(s,t) \frac{1}{k!} = (-t)^n \sum_{k=0}^{\infty} r_2^{nk-n}(s,t) \frac{1}{k!},$$

and

$$r_2^n(s,t) e^{r_1^n(s,t)} = r_2^n(s,t) \sum_{k=0}^{\infty} r_1^{nk}(s,t) \frac{1}{k!} = (-t)^n \sum_{k=0}^{\infty} r_1^{nk-n}(s,t) \frac{1}{k!}.$$

If the results are combined

$$\begin{aligned} e^{P_n(s,t)} &= \left(\frac{r_1^n(s,t)e^{r_2^n(s,t)} - r_2^n(s,t)e^{r_1^n(s,t)}}{r_1^n(s,t) - r_2^n(s,t)}\right) I_2 + \left(\frac{e^{r_1^n(s,t)} - e^{r_2^n(s,t)}}{r_1^n(s,t) - r_2^n(s,t)}\right) P_n(s,t) \\ &= \left[(-t)^n \sum_{k=0}^{\infty} r_2^{nk-n}(s,t)\frac{1}{k!} - (-t)^n \sum_{k=0}^{\infty} r_1^{nk-n}(s,t)\frac{1}{k!}\right] I_2 / \left(r_1^n(s,t) - r_2^n(s,t)\right) \\ &\quad + \sum_{k=0}^{\infty} \frac{p_{nk}(s,t)}{p_n(s,t)k!} P_n(s,t) \\ &= -(-t)^n \sum_{k=0}^{\infty} \frac{p_{nk-n}(s,t)}{p_n(s,t)k!} I_2 + \sum_{k=0}^{\infty} \frac{p_{nk}(s,t)}{p_n(s,t)k!} P_1^n(s,t). \end{aligned}$$

If we choose $s = t = 1$, we can apply this result for classic Pell matrix sequence defined in (2.1). The eigenvalues of $P_1 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ matrix are $r_1 = 1 + \sqrt{2}$ and $r_2 = 1 - \sqrt{2}$. By $r_1 r_2 = -1$

$$\begin{aligned} e^{P_n} &= \left(\frac{r_1^n e^{r_2^n} - r_2^n e^{r_1^n}}{r_1^n - r_2^n}\right) I_2 + \left(\frac{e^{r_1^n} - e^{r_2^n}}{r_1^n - r_2^n}\right) P_n \\ &= \sum_{k=0}^{\infty} \frac{p_{nk}}{p_n k!} P_1^n - (-1)^n \sum_{k=0}^{\infty} \frac{p_{nk-n}}{p_n k!} I_2 \end{aligned}$$

where P_n is the *nth* element of the classic Pell matrix sequence.

If we choose $s = 1$, $t = k$, we can apply this result for k-Pell matrix sequence defined in (2.2). The eigenvalues of $P_{k,1} = \begin{pmatrix} 2 & 1 \\ k & 0 \end{pmatrix}$ matrix are $1+\sqrt{1+k}$ and $1-\sqrt{1+k}$. By Lemma 2.1, the eigenvalues of $P_{k,n}$ matrix as $r_{k,1}^n = \left(1 + \sqrt{1+k}\right)^n$ and $r_{k,2}^n = \left(1 - \sqrt{1+k}\right)^n$. By the Binet formula for $P_{k,n}$ and $r_{k,1} r_{k,2} = -k$

$$\begin{aligned} e^{P_{k,n}} &= \left(\frac{r_{k,1}^n e^{r_{k,2}^n} - r_{k,2}^n e^{r_{k,1}^n}}{r_{k,1}^n - r_{k,2}^n}\right) I_2 + \left(\frac{e^{r_{k,1}^n} - e^{r_{k,2}^n}}{r_{k,1}^n - r_{k,2}^n}\right) P_{k,n} \\ &= \left[(-k)^n \sum_{i=0}^{\infty} r_{k,2}^{ni-n} \frac{1}{i!} - (-k)^n \sum_{i=0}^{\infty} r_{k,1}^{ni-n} \frac{1}{i!}\right] I_2 / \left(r_{k,1}^n - r_{k,2}^n\right) + \sum_{i=0}^{\infty} \frac{p_{k,ni}}{p_{k,n} i!} P_{k,n} \\ &= -(-k)^n \sum_{i=0}^{\infty} \frac{p_{k,ni-n}}{p_{k,n} i!} I_2 + \sum_{i=0}^{\infty} \frac{p_{k,ni}}{p_{k,n} i!} P_{k,1}^n \end{aligned}$$

where $P_{k,n}$ is the *nth* element of the k-Pell matrix sequence.

The following theorem shows us a second way for expressing the exponential representation of the *nth* element of (s,t)-Pell matrix sequence. □

Theorem 3.2. *For any integer $n \geq 0$, the exponential representation of the nth element of (s,t)-Pell matrix sequence is in the following form:*

$$e^{P_n(s,t)} = U \begin{bmatrix} e^{r_1^n(s,t)} & 0 \\ 0 & e^{r_2^n(s,t)} \end{bmatrix} U^{-1}$$

where U is an invertible matrix and

$$\begin{bmatrix} e^{r_1^n(s,t)} & 0 \\ 0 & e^{r_2^n(s,t)} \end{bmatrix} = \sum_{i=0}^{\infty} t \frac{p_{ni-1}(s,t)}{i!} I_2 - \sum_{i=0}^{\infty} \frac{p_{ni}(s,t)}{i!} \begin{bmatrix} r_1(s,t) & 0 \\ 0 & r_2(s,t) \end{bmatrix}.$$

This result has more advantage for finding the exponential representations of the nth element of (s,t)-Pell matrix sequence. Because we only need the elements of the sequence $(p_n(s,t))$.

Proof. Because of the eigenvalues of $P_n(s,t)$ are $(r_1(s,t))^n$ and $(r_2(s,t))^n$, there is an invertible U matrix such that

$$P_n(s,t) = U \begin{bmatrix} (r_1(s,t))^n & 0 \\ 0 & (r_2(s,t))^n \end{bmatrix} U^{-1}.$$

Therefore, the exponential form is

$$e^{P_n(s,t)} = U \begin{bmatrix} e^{(r_1(s,t))^n} & 0 \\ 0 & e^{(r_2(s,t))^n} \end{bmatrix} U^{-1}.$$

Then, we obtain

$$\begin{aligned} \begin{bmatrix} e^{(r_1(s,t))^n} & 0 \\ 0 & e^{(r_2(s,t))^n} \end{bmatrix} &= \left(\frac{r_1(s,t) e^{(r_2(s,t))^n} - r_2(s,t) e^{(r_1(s,t))^n}}{r_1(s,t) - r_2(s,t)} \right) I_2 \\ &\quad + \left(\frac{e^{(r_2(s,t))^n} - e^{(r_1(s,t))^n}}{r_1(s,t) - r_2(s,t)} \right) \begin{bmatrix} r_1(s,t) & 0 \\ 0 & r_2(s,t) \end{bmatrix} \\ &= \left[\sum_{i=0}^{\infty} (-t) \left(\frac{r_2^{ni-1}(s,t) - r_1^{ni-1}(s,t)}{r_1(s,t) - r_2(s,t)} \right) \frac{1}{i!} I_2 \right] \\ &\quad + \sum_{i=0}^{\infty} \left(\frac{r_2^{ni}(s,t) - r_1^{ni}(s,t)}{r_1(s,t) - r_2(s,t)} \right) \frac{1}{i!} \begin{bmatrix} r_1(s,t) & 0 \\ 0 & r_2(s,t) \end{bmatrix} \\ &= \sum_{i=0}^{\infty} t \frac{p_{ni-1}(s,t)}{i!} I_2 - \sum_{i=0}^{\infty} \frac{p_{ni}(s,t)}{i!} \begin{bmatrix} r_1(s,t) & 0 \\ 0 & r_2(s,t) \end{bmatrix}. \end{aligned}$$

If we choose $s = t = 1$, we can apply this result for classic Pell matrix sequence as

$$e^{P_n} = U \begin{bmatrix} e^{r_1^n} & 0 \\ 0 & e^{r_2^n} \end{bmatrix} U^{-1}$$

where

$$\begin{aligned} \left[\begin{array}{cc} e^{r_1^n} & 0 \\ 0 & e^{r_2^n} \end{array}\right] &= \left(\frac{r_1 e^{r_2^n} - r_2 e^{r_1^n}}{r_1 - r_2}\right) I_2 + \left(\frac{e^{r_1^n} - e^{r_2^n}}{r_1 - r_2}\right) \left[\begin{array}{cc} r_1 & 0 \\ 0 & r_2 \end{array}\right] \\ &= \sum_{i=0}^{\infty} \left(\frac{r_1^{ni-1} - r_2^{ni-1}}{r_1 - r_2}\right) \frac{1}{i!} I_2 + \sum_{i=0}^{\infty} \left(\frac{r_1^{ni} - r_2^{ni}}{r_1 - r_2}\right) \frac{1}{i!} \left[\begin{array}{cc} r_1 & 0 \\ 0 & r_2 \end{array}\right] \\ &= \sum_{i=0}^{\infty} \frac{p_{ni-1}}{i!} I_2 + \sum_{i=0}^{\infty} \frac{p_{ni}}{i!} \left[\begin{array}{cc} r_1 & 0 \\ 0 & r_2 \end{array}\right]. \end{aligned}$$

If we choose $s = 1,\ \ t = k$, we can apply this result for k-Pell matrix sequence. The eigenvalues of matrix are $1 + \sqrt{1+k}$ and $1 - \sqrt{1+k}$. By Lemma 2.1, the eigenvalues of $P_{k,n}$ matrix as $r_{k,1}^n = \left(1 + \sqrt{1+k}\right)^n$ and $r_{k,2}^n = \left(1 - \sqrt{1+k}\right)^n$ .By the Binet formula for $P_{k,n}$ and $r_{k,1} r_{k,2} = -k$, we get

$$e^{P_{k,n}} = U \left[\begin{array}{cc} e^{r_{k,1}^n} & 0 \\ 0 & e^{r_{k,2}^n} \end{array}\right] U^{-1}$$

where

$$\begin{aligned} \left[\begin{array}{cc} e^{r_{k,1}^n} & 0 \\ 0 & e^{r_{k,2}^n} \end{array}\right] &= \left(\frac{r_{k,1} e^{r_{k,2}^n} - r_{k,2} e^{r_{k,1}^n}}{r_{k,1} - r_{k,2}}\right) I_2 + \left(\frac{e^{r_{k,1}^n} - e^{r_{k,2}^n}}{r_{k,1} - r_{k,2}}\right) \left[\begin{array}{cc} r_{k,1} & 0 \\ 0 & r_{k,2} \end{array}\right] \\ &= \sum_{i=0}^{\infty} k \left(\frac{r_{k,1}^{ni-1} - r_{k,2}^{ni-1}}{r_{k,1} - r_{k,2}}\right) \frac{1}{i!} I_2 + \sum_{i=0}^{\infty} \left(\frac{r_{k,1}^{ni} - r_{k,2}^{ni}}{r_{k,1} - r_{k,2}}\right) \frac{1}{i!} \left[\begin{array}{cc} r_{k,1} & 0 \\ 0 & r_{k,2} \end{array}\right] \\ &= k \left(\sum_{i=0}^{\infty} \frac{p_{k,ni-1}}{i!}\right) I_2 + \sum_{i=0}^{\infty} \frac{p_{k,ni}}{i!} \left[\begin{array}{cc} 1 + \sqrt{1+k} & 0 \\ 0 & 1 - \sqrt{1+k} \end{array}\right]. \end{aligned}$$

□

Theorem 3.3. *For $n \geq 0$, the exponential representation of the $(2n)$th element of Pell matrix sequence is given in the following form:*

$$e^{P_{2n}} = \left(p_1 + \sum_{k=1}^{\infty} \frac{p_{2nk-1}}{k!}\right) I_2 + \sum_{k=0}^{\infty} \frac{p_{2nk}}{k!} \left[\begin{array}{cc} r_1 & 0 \\ 0 & r_2 \end{array}\right].$$

Proof. By using Lemma 2.2, there is an invertible U matrix such that

$$P_{2n} = P_n P_n = U \left[\begin{array}{cc} r_1^{2n} & 0 \\ 0 & r_2^{2n} \end{array}\right] U^{-1}.$$

By using the properties of the exponential matrix, we have

$$e^{P_{2n}} = U \left[\begin{array}{cc} e^{r_1^{2n}} & 0 \\ 0 & e^{r_2^{2n}} \end{array}\right] U^{-1}.$$

We also obtain

$$\begin{bmatrix} e^{r_1^{2n}} & 0 \\ 0 & e^{r_2^{2n}} \end{bmatrix} = \left(\frac{r_1 e^{r_2^{2n}} - r_2 e^{r_1^{2n}}}{r_1 - r_2}\right) I_2 + \left(\frac{e^{r_2^{2n}} - e^{r_1^{2n}}}{r_1 - r_2}\right) \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix},$$

$$\frac{e^{r_1^{2n}} - e^{r_2^{2n}}}{r_1 - r_2} = \sum_{k=0}^{\infty} \left(\frac{r_1^{2nk} - r_2^{2nk}}{r_1 - r_2}\right) \frac{1}{k!} = \sum_{k=0}^{\infty} \frac{p_{2nk}}{k!},$$

and

$$\frac{r_1 e^{r_2^{2n}} - r_2 e^{r_1^{2n}}}{r_1 - r_2} = \sum_{i=0}^{\infty} \left(\frac{r_1^{2ni-1} - r_2^{2ni-1}}{r_1 - r_2}\right) \frac{1}{i!} = \sum_{i=0}^{\infty} \frac{p_{2ni-1}}{i!}.$$

By combining the results, the proof is completed. □

Exponential representation of $(2n)th$ k-Pell and (s,t)-Pell matrix sequences can be obtained by using the same procedure. We give the results as

$$e^{P_{k,2n}} = k\left(\sum_{i=1}^{\infty} \frac{p_{k,2ni-1}}{i!}\right) I_2 + \sum_{i=0}^{\infty} \frac{p_{k,2ni}}{i!} \begin{bmatrix} r_{k,1} & 0 \\ 0 & r_{k,2} \end{bmatrix},$$

$$e^{P_{2n}(s,t)} = t\left(\sum_{k=1}^{\infty} \frac{p_{2nk-1}(s,t)}{k!}\right) I_2 + \sum_{k=0}^{\infty} \frac{p_{2nk}(s,t)}{k!} \begin{bmatrix} r_1(s,t) & 0 \\ 0 & r_2(s,t) \end{bmatrix}.$$

Theorem 3.4. *For $n \geq 0$, the exponential representation of the $(2n)th$ element of (s,t)-Pell matrix sequence is computed in the following form:*

$$e^{P_{2n}(s,t)} = -t^{2n} \sum_{k=0}^{\infty} \frac{p_{2nk-2n}(s,t)}{p_{2n}(s,t)k!} I_2 + \sum_{k=0}^{\infty} \frac{p_{2nk}(s,t)}{p_{2n}(s,t)k!} P_1^{2n}(s,t)$$

Proof. The eigenvalues of $P_1(s,t)$ are $r_1(s,t) = s + \sqrt{s^2+t}$ and $r_2(s,t) = s - \sqrt{s^2+t}$. The eigenvalues of $P_{2n}(s,t)$ are $r_1^{2n}(s,t) = \left(s + \sqrt{s^2+t}\right)^{2n}$ and $r_2^{2n}(s,t) = \left(s - \sqrt{s^2+t}\right)^{2n}$. By the equality $e^{P_{2n}(s,t)} = a_1 P_{2n}(s,t) + a_0 I_2$, we get

$$e^{r_1^{2n}(s,t)} = a_1 r_1^{2n}(s,t) + a_0, e^{r_2^{2n}(s,t)} = a_1 r_2^{2n}(s,t) + a_0.$$

By these equations, the values of a_0, a_1 are found. If we substitute the values of a_0,

a_1, it is obtained that

$$\begin{aligned}
e^{P_{2n}(s,t)} &= \left(\frac{r_1^{2n}(s,t)e^{r_2^{2n}(s,t)} - r_2^{2n}(s,t)e^{r_1^{2n}(s,t)}}{r_1^{2n}(s,t) - r_2^{2n}(s,t)}\right) I_2 \\
&\quad + \left(\frac{e^{r_1^{2n}(s,t)} - e^{r_2^{2n}(s,t)}}{r_1^{2n}(s,t) - r_2^{2n}(s,t)}\right) P_{2n}(s,t) \\
&= \left[\sum_{k=0}^{\infty} \frac{1}{k!} \left(r_2^{2nk-2n}(s,t) - r_1^{nk-n}(s,t)\right)\right] \frac{I_2}{r_1^{2n}(s,t) - r_2^{2n}(s,t)} \\
&\quad + \sum_{k=0}^{\infty} \frac{p_{2nk}(s,t)}{p_{2n}(s,t)k!} P_{2n}(s,t) \\
&= -t^{2n} \sum_{k=0}^{\infty} \frac{p_{2nk-2n}(s,t)}{p_{2n}(s,t)k!} I_2 + \sum_{k=0}^{\infty} \frac{p_{2nk}(s,t)}{p_{2n}(s,t)k!} P_1^{2n}(s,t)
\end{aligned}$$

If we choose $s = t = 1$, we can apply this result for the classic Pell matrix sequence

$$e^{P_n} = \sum_{k=0}^{\infty} \frac{p_{2nk}}{p_{2n}k!} P_1^{2n} - \sum_{k=0}^{\infty} \frac{p_{2nk-2n}}{p_{2n}k!} I_2.$$

If we choose $s = 1,\ t = k$, we can apply this result for k-Pell matrix sequence

$$e^{P_{k,n}} = -k^{2n} \sum_{i=0}^{\infty} \frac{p_{k,2ni-2n}}{p_{k,2n}i!} I_2 + \sum_{i=0}^{\infty} \frac{p_{k,2ni}}{p_{k,2n}i!} P_{k,1}^{2n}.$$

□

4. Conclusion

The exponential representations of the Pell matrix sequence and its generalized matrix sequences are investigated in this study. The elements of the sequences are 2×2 matrices. This study can be extended to 3×3 matrices. The other special integer sequences can also be used for finding exponential representations of them.

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