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NONCONFORMING MIXED DISCRETIZATION FOR SECOND-ORDER ELLIPTIC PROBLEMS IN $\mathbb{R}^{3\dagger}$

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ABSTRACT. The main aim of this paper is to suggest modified patch conditions for nonconforming mixed finite element(NMFE) method and introduce a new family of the NMFE space of higher order on parallelepiped grids. Also, we provide a framework for error estimates.

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1. Introduction

The mixed finite element method has been investigated in many fields[2, 4, 5, 6]. However, when a common mixed approach was used in semiconductor application problems or problems with large jumps in the diffusion coefficient, a major drawback was revealed due to poor local approximation of the flux[7, 8, 10]. In this case, it is necessary to enhance the discrete space for the flux. Hence a nonconforming technique is applied. So far, the nonconforming mehtods have been widely studied for Lagrangian finite elements, but has not received much attention for mixed finite elements[3, 9]. In a recent work, Jo and Kim[12] improved Hiptmair's result[11] for nonconforming mixed finite element(NMFE) method and provided a family of the NMFE space which satisfies the some patch conditions. In this paper, we suggest modified patch conditions and construct another NMFE spaces of higher order ($k \geq 1$) which has fewer degrees of freedom than the existing ones.

The organization of this paper is as follows: In Section 2, we present the model problem. Next, we provide modified patch conditions for nonconforming mixed discretization. In Section 4, we introduce new nonconforming mixed finite element spaces on parallelepiped grids. Finally, an error estimates of the discretization is given.

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2. Model problem

We are concerned with the second-order elliptic boundary value problem which have large jumps in the diffusion coefficient

$$\begin{cases} -\mathrm{div}(\kappa\nabla p) &= f, \quad \text{in } \Omega, \\ p &= 0, \quad \text{on } \Gamma, \end{cases} \tag{1}$$

where Ω is a simply connected bounded Lipschitz polyhedral domain with connected boundary Γ, f is a given funtion of the space $L^2(\Omega)$ and $\kappa \in L^\infty(\Omega)$ is assumed to be uniformly positive definite and bounded. Introducing the auxiliary variable $\mathbf{u} = \kappa\nabla p$, the problem (1) may be written as the system

$$\begin{cases} \mathbf{u} - \kappa\nabla p &= 0, \quad \text{in } \Omega, \\ \mathrm{div}\,\mathbf{u} + f &= 0, \quad \text{in } \Omega, \\ p &= 0, \quad \text{on } \Gamma. \end{cases} \tag{2}$$

Then the mixed formulation of (2) is to find $(\mathbf{u}, p) \in H(\mathrm{div}\,, \Omega) \times L^2(\Omega)$ such that

$$\int_\Omega \kappa^{-1}\mathbf{u}\cdot\mathbf{v}d\mathbf{x} + \int_\Omega p\mathrm{div}\,\mathbf{v}d\mathbf{x} = 0, \quad \forall \mathbf{v} \in H(\mathrm{div}\,, \Omega), \tag{3}$$

$$\int_\Omega \mathrm{div}\,\mathbf{u}qd\mathbf{x} + \int_\Omega fqd\mathbf{x} = 0, \quad \forall q \in L^2(\Omega), \tag{4}$$

where $H(\mathrm{div}\,, \Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 : \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\}$. For convenience of the presentation, we let

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \kappa^{-1}\mathbf{u}\cdot\mathbf{v}d\mathbf{x},$$

$$b(\mathbf{v}, p) = \int_\Omega p\ \mathrm{div}\,\mathbf{v}d\mathbf{x}.$$

Then the equations (3) and (4) can be expressed simply as follows: find $(\mathbf{u}, p) \in H(\mathrm{div}\,, \Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= 0, \quad \forall \mathbf{v} \in H(\mathrm{div}\,, \Omega), \\ b(\mathbf{u}, q) + (f, q) &= 0, \quad \forall q \in L^2(\Omega), \end{aligned} \tag{5}$$

where $(\cdot,\cdot)$ indicates the inner product in $L^2(\Omega)$. If the well-known inf-sup condition is satisfied, then the problem (5) has a unique solution[1].

3. Nonconforming mixed discretization

Let $\Omega = [0, 1]^3$ and $\mathcal{T}_h$ be a regular triangulation of Ω into parallelepiped K. And let $Q_{\ell,m}(K)$ or $Q_{\ell,m,n}(K)$ be the space of polynomials of degree less or equal to ℓ, m, n respectively, in each variable. The nonconforming discretization of equations (3) and (4) is to find $(\mathbf{u}_h, p_h) \in \mathbf{X}_h \times W_h$ such that

$$\int_\Omega \kappa^{-1}\mathbf{u}_h\cdot\mathbf{v}_h d\mathbf{x} + \int_\Omega p_h \mathrm{div}\,\mathbf{v}_h d\mathbf{x} = 0, \quad \forall \mathbf{v}_h \in \mathbf{X}_h, \tag{6}$$

$$\int_\Omega \operatorname{div} \mathbf{u}_h q_h d\mathbf{x} + \int_\Omega f q_h d\mathbf{x} \quad = \quad 0, \quad \forall q_h \in W_h, \tag{7}$$

where the two finite-dimensional spaces $\mathbf{X}_h$ and W_h have to satisfy the following three conditions:

C1. $\mathbf{X}_h = \mathbf{V}_h + H(\operatorname{div}, \Omega)$, where $\mathbf{V}_h = \{\mathbf{v} \in (L^2(\Omega))^3 \mid \mathbf{v}|_K \in H(\operatorname{div}, K),\ \forall K \in \mathcal{T}_h\}$ and $\mathbf{V}_h \nsubseteq H(\operatorname{div}, \Omega)$. Thst is, $\mathbf{V}_h$ space is locally in $H(\operatorname{div}, K)$ but not in $H(\operatorname{div}, \Omega)$.

C2. (Modified patch conditions) For all horizontal inter-element boundaries $f_H = \partial K_i \cap \partial K_j (i \neq j)$ and $\forall \mathbf{v}_h \in \mathbf{V}_h$, we have

$$\int_{f_H} q \left(\mathbf{v}_h|_{K_i} \cdot \mathbf{n}_i + \mathbf{v}_h|_{K_j} \cdot \mathbf{n}_j \right) dA = 0,\ \forall q \in Q_{k,k}(f_H),$$

where $\mathbf{n}_i$ denotes the unit outward normal to ∂K_i. For all vertical inter-element boundaries $f_V = \partial K_i \cap \partial K_j$,

$$\int_{f_V} q \left(\mathbf{v}_h|_{K_i} \cdot \mathbf{n}_i + \mathbf{v}_h|_{K_j} \cdot \mathbf{n}_j \right) dA = 0,$$
$$\forall q \in Q_{k,k}(f_V) \backslash \{x^k z^k\} \text{ or } Q_{k,k}(f_V) \backslash \{y^k z^k\}.$$

A lack of continuity of normal components across inter-element boundaries gives rise to a nonconforming appoximation[11].

C3. $\operatorname{div} \mathbf{V}_h = W_h$ and $W_h \subset L^2(\Omega)$.

Since $\mathbf{V}_h \nsubseteq H(\operatorname{div}, \Omega)$, we cannot guarantee that the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ make sense for functions of $\mathbf{V}_h$. So we define extensions to the larger space $\mathbf{X}_h$ as follows:

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = \sum_{K_i} \int_{K_i} \kappa^{-1} \mathbf{u}_h \cdot \mathbf{v}_h d\mathbf{x},\ \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{X}_h,$$
$$b_h(\mathbf{v}_h, p_h) = \sum_{K_i} \int_{K_i} p_h \operatorname{div} \mathbf{v}_h d\mathbf{x},\ \forall p_h \in W_h,\ \forall \mathbf{v}_h \in \mathbf{X}_h.$$

The $\mathbf{X}_h$ space have the following norm which is an extension of $\|\cdot\|_{H(\operatorname{div},\Omega)}$:

$$\|\mathbf{u}_h\|^2_{\mathbf{X}_h} = \sum_{K_i} \|\mathbf{u}_h\|^2_{H(\operatorname{div}, K_i)},\ \forall \mathbf{u}_h \in \mathbf{X}_h.$$

And the W_h space has L^2-norm. Then the mixed finite element problem corresponding to (5) is to find $(\mathbf{u}_h, p_h) \in \mathbf{X}_h \times W_h$ such that

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) &= 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b_h(\mathbf{u}_h, q_h) + (f, q_h) &= 0, \quad \forall q_h \in W_h. \end{aligned} \tag{8}$$

The purpose of this article is to construct a new nonconforming mixed finite element space $\mathbf{V}_h$ satisfying above three conditions for problem (8).

4. New nonconforming mixed finite element spaces

In this section, we introduce a new family of nonconforming mixed finite element space of higher order $k \geq 1$ on parallelepiped grids. First, we denote by $Q^1_{\ell,m,n}(K)$ and $Q^2_{\ell,m,n}(K)$ the set of all polynomials of $Q_{\ell,m,n}(K)$ except those having the form $x^i y^m z^n$ and $x^\ell y^j z^n$ for $i = 0, \dots \ell-1 (\ell \geq 1)$ and $j = 0, \dots m-1 (m \geq 1)$, respectively. Also, $Q^1_{0,m,n}(K) = Q_{0,m,n}(K)$ and $Q^2_{\ell,0,n}(K) = Q_{\ell,0,n}(K)$. To define the vector variable space $\mathbf{V}_h(K)$, we consider the following vectors for $i = 0, \cdots, k-1$:

$$\begin{aligned}
&\mathbf{a}_{11} = (x^{k+1}y^i z^k, 0, 0), \quad \mathbf{a}_{12} = (0, 0, x^k y^i z^{k+1}), \quad \mathbf{b}_1 = (x^{k+1}y^i z^k, 0, -x^k y^i z^{k+1}),\\
&\mathbf{a}_{21} = (0, x^i y^{k+1} z^k, 0), \quad \mathbf{a}_{22} = (0, 0, x^i y^k z^{k+1}), \quad \mathbf{b}_2 = (0, x^i y^{k+1} z^k, -x^i y^k z^{k+1}),\\
&\mathbf{c}_1 = (x^{k+1}y^k z^k, 0, 0), \quad \mathbf{c}_2 = (0, x^k y^{k+1} z^k, 0), \quad \mathbf{c}_3 = (0, 0, x^k y^k z^{k+1}),\\
&\mathbf{d} = (x^{k+1}y^k z^k, x^k y^{k+1} z^k, -2x^k y^k z^{k+1}).
\end{aligned}$$

Definition 4.1. For $i = 0, \cdots, k-1$, we let $\mathbf{V}^*_h(K)$ as follows:

$$\mathbf{V}^*_h(K) = \begin{pmatrix} Q^1_{k+1,k,k} \\ Q^2_{k,k+1,k} \\ Q_{k,k,k+1} \end{pmatrix} \oplus \begin{pmatrix} y^{k+1}z^i \\ x^{k+1}z^i \\ 0 \end{pmatrix}$$

Let $\mathbf{V}_h(K)$ be the subspace of $\mathbf{V}^*_h(K)$, where the elements $\mathbf{a}_{\ell m}$ are replaced by the elements $\mathbf{b}_\ell$ $(\ell, m = 1, 2)$ and the three elements $\mathbf{c}_n$ are replaced by the single element $\mathbf{d}$ $(n = 1, 2, 3)$.

With abuse of notation, we may write

$$\mathbf{a}_{11},\ \mathbf{a}_{12} \Rightarrow \mathbf{b}_1, \quad \mathbf{a}_{21},\ \mathbf{a}_{22} \Rightarrow \mathbf{b}_2, \quad \mathbf{c}_1,\ \mathbf{c}_2,\ \mathbf{c}_3 \Rightarrow \mathbf{d}.$$

Then the dimension of $\mathbf{V}_h(K)$ is $2\{(k+1)^2(k+2) - (k+1)\} + (k+1)^2(k+2) + 2k - (2k+2) = 3k^3 + 12k^2 + 13k + 2$. We give an example for $k = 1$: $\mathbf{u} = (u_1, u_2, u_3) \in \mathbf{V}_h(K)$, where

$$\begin{aligned}
u_1 &= P_1(x,y,z) + a_1 xy + a_2 zx + a_3 x^2 + a_4 x^2 y + a_5 y^2 + \alpha x^2 z + \gamma x^2 yz,\\
u_2 &= P_1(x,y,z) + b_1 xy + b_2 yz + b_3 y^2 + b_4 xy^2 + b_5 x^2 + \beta y^2 z + \gamma xy^2 z,\\
u_3 &= Q_1(x,y,z) + c_1 z^2 - \alpha xz^2 - \beta yz^2 - 2\gamma xyz^2,
\end{aligned}$$

where $P_\ell(K)$ be the space of polynomials of total degree ℓ.

To define the degrees of freedom, we define $\boldsymbol{\phi}_{\ell m},\ \boldsymbol{\psi}_\ell,\ \boldsymbol{\xi}_n$ and $\boldsymbol{\zeta}$ for $\ell, m = 1, 2,\ n = 1, 2, 3$ similarly to $\mathbf{a}_{\ell m},\ \mathbf{b}_\ell,\ \mathbf{c}_n$ and $\mathbf{d}$ except that the highest exponent $k+1$ is replaced by $k-1$, respectively. Also, we need an auxiliary space : $\boldsymbol{\Psi}_h(K)$ be the subspace of

$$\begin{pmatrix} Q^1_{k-1,k,k} \\ Q^2_{k,k-1,k} \\ Q_{k,k,k-1} \end{pmatrix}$$

where the elements $\boldsymbol{\phi}_{\ell m}$ are replaced by the elements $\boldsymbol{\psi}_\ell$ $(\ell, m = 1, 2)$ and the three elements $\boldsymbol{\xi}_n$ are replaced by the single element $\boldsymbol{\zeta}$ $(n = 1, 2, 3)$.

For any $\mathbf{u}_h = (u_1, u_2, u_3) \in \mathbf{V}_h(K)$, we consider the following degrees of fredom:

$$\int_{f_H} \mathbf{u}_h \cdot \mathbf{n}\, q\, dA,\ \forall q \in Q_{k,k}(f_H),\ \text{for each horizontal faces } f_H, \tag{9}$$

$$\int_{f_V} \mathbf{u}_h \cdot \mathbf{n}\, q\, dA,\ \forall q \in Q^*_{k,k}(f_V),\ \text{for each vertical faces } f_V, \tag{10}$$

$$\int_K \mathbf{u}_h \cdot \mathbf{q}\, d\mathbf{x},\ \forall \mathbf{q} \in (y^{k+1}z^i,\ x^{k+1}z^i,\ 0)^T,\ i = 0, \cdots, k-1, \tag{11}$$

$$\int_K \mathbf{u}_h \cdot \mathbf{q}\ d\mathbf{x},\ \ \forall \mathbf{q} \in \mathbf{\Psi}_h(K), \tag{12}$$

where $\mathbf{n}$ is a unit outward normal vector and we define $Q^*_{k,k}(f_V) = Q^*_{k,k}(y,z) = Q_{k,k}(y,z)\backslash\{y^k z^k\}$ for each vertical face f_V in $yz-$plane etc. Then the number of conditions is $2(k+1)^2 + 4\{(k+1)^2 - 1\} + 2k + 2\{k(k+1)^2 - (k-1)\} + k(k+1)^2 - (2k+2)$, which is also the dimension of $\mathbf{V}_h(K)$.

Theorem 4.2. *A vector function $\mathbf{u}_h = (u_1, u_2, u_3) \in \mathbf{V}_h(K)$ is uniquely determined by the degrees of freedom (9)-(12).*

Proof. Since the number of conditions equals the dimension of $\mathbf{V}_h(K)$, it suffices to show that if all the conditions are zero, then $\mathbf{u}_h = 0$. From (11) , we can rewrite $\mathbf{u}_h$ as follows:

$$u_1 = r_1 + r_2 x^k + v_1 x^{k+1},$$
$$u_2 = s_1 + s_2 y^k + v_2 y^{k+1},$$
$$u_3 = t_1 + t_2 z^k + v_3 z^{k+1},$$

where $r_i, s_i \in Q^*_{k,k}(f_V),\ t_i \in Q_{k,k}(f_H)$ for $i = 1,\ 2$ and $\mathbf{v}_h = (v_1, v_2, v_3) \in \mathbf{\Psi}_h(K)$. Then, the degrees of freedom (12) implies

$$u_1 = r_1^* + r_2^* x^k,$$
$$u_2 = s_1^* + s_2^* y^k,$$
$$u_3 = t_1^* + t_2^* x^k,$$

where $r_i^*, s_i^* \in Q^*_{k,k}(f_V),\ t_i^* \in Q_{k,k}(f_H)$ for $i = 1,\ 2$. From (9) and (10), we prove that $\mathbf{u}_h = 0$. □

Remark 4.1. This element has $k+2$ fewer degrees of freedom than Jo-Kim element on parallelepiped[12]. When $k = 1$, the new element has three normal component degrees of freedom per vertical face, four normal componet dof per horizontal face, and ten interior dof. On the other hand, the Jo-Kim element has four normal component dof per vertical face, three dof per horizontal face, and eleven interior dof.

Definition 4.3. For the scalar variable, we define

$$W_h(K) = Q_{k,k,k}(K)\backslash\{x^k y^i z^k,\ x^i y^k z^k,\ x^k y^k z^k\},$$

for $i = 0, \cdots, k-1$.

By simple computation, we know that the dimension of $W_h(K)$ is $(k+1)^3-2k-1 = k^3+3k^2+k$. Also, we know that $\operatorname{div} \mathbf{V}_h(K) = W_h(K)$ and $W_h \subset L^2(\Omega)$.

For error estimates, we define an interpolation operator $\mathbf{\Pi}_h : \mathbf{H}^{k+1}(K) \to \mathbf{V}_h(K)$ by

$$\int_{f_H} (\mathbf{u} - \mathbf{\Pi}_h \mathbf{u}) \cdot \mathbf{n}\, q\, dA,\ \forall q \in Q_{k,k}(f_H), \tag{13}$$

$$\int_{f_V} (\mathbf{u} - \mathbf{\Pi}_h \mathbf{u}) \cdot \mathbf{n}\, q\, dA,\ \forall q \in Q^*_{k,k}(f_V), \tag{14}$$

$$\int_{K} (\mathbf{u} - \mathbf{\Pi}_h \mathbf{u}) \cdot \mathbf{q}\, d\mathbf{x},\ \forall \mathbf{q} \in (y^{k+1}z^i,\ x^{k+1}z^i,\ 0)^T,\ i = 0, \cdots, k-1, \tag{15}$$

$$\int_{K} (\mathbf{u} - \mathbf{\Pi}_h \mathbf{u}) \cdot \mathbf{q}\ d\mathbf{x},\ \ \forall \mathbf{q} \in \mathbf{\Psi}_h(K). \tag{16}$$

Then we have the following lemma.

Lemma 4.4. *If $\mathbf{\Pi}_h \mathbf{u}$ is the interpolation of $\mathbf{u}$, then we have*

$$\int_K \operatorname{div}(\mathbf{u} - \mathbf{\Pi}_h \mathbf{u}) q\, d\mathbf{x} = 0,\ \forall \mathbf{u} \in \mathbf{V}_h(K),\ q \in W_h(K).$$

Proof. Let $q \in W_h(K)$. Then, $q|_{f_H} \in Q_{k,k}(f_H)$ for each horizontal faces f_H and $q|_{f_V} \in Q^*_{k,k}(f_V)$ for each vertical faces f_V. Also, we see that

$$\partial_x q \in Q^1_{k-1,k,k} \backslash \{x^{k-1}y^i z^k\}, \tag{17}$$

$$\partial_y q \in Q^2_{k,k-1,k} \backslash \{x^i y^{k-1} z^k\}, \tag{18}$$

$$\partial_z q \in Q_{k,k,k-1} \backslash \{x^i y^k z^{k-1},\ x^k y^i z^{k-1}\}, \tag{19}$$

for $i = 0, \cdots, k$. Hence $\nabla q \in \mathbf{\Psi}_h(K)$. From definition of $\mathbf{\Pi}_h$, we have

$$\begin{aligned}
\int_K (\operatorname{div} \mathbf{\Pi}_h \mathbf{u}) q\, d\mathbf{x} &= \int_{\partial K} \mathbf{\Pi}_h \mathbf{u} \cdot \mathbf{n} q\, dA - \int_K \mathbf{\Pi}_h \mathbf{u} \cdot \nabla q\, d\mathbf{x} \\
&= \left\{ \int_{f_H} \mathbf{\Pi}_h \mathbf{u} \cdot \mathbf{n} q\, dA + \int_{f_V} \mathbf{\Pi}_h \mathbf{u} \cdot \mathbf{n} q\, dA \right\} - \int_K \mathbf{\Pi}_h \mathbf{u} \cdot \nabla q\, d\mathbf{x} \\
&= \left\{ \int_{f_H} \mathbf{u} \cdot \mathbf{n} q\, dA + \int_{f_V} \mathbf{u} \cdot \mathbf{n} q\, dA \right\} - \int_K \mathbf{u} \cdot \nabla q\, d\mathbf{x} \\
&= \int_{\partial K} \mathbf{u} \cdot \mathbf{n} q\, dA - \int_K \mathbf{u} \cdot \nabla q\, d\mathbf{x} \\
&= \int_K \operatorname{div} \mathbf{u} q\, d\mathbf{x}.
\end{aligned}$$

□

5. Error estimates

To obtain error estimates, we first define $B : \mathbf{X}_h \to W'$ by $B(\mathbf{v}, q) = b(\mathbf{v}, q),\ \forall \mathbf{v} \in X_h,\ \forall q \in W$. And also, we define $B_h : \mathbf{V}_h \to W_h'$ by $B_h(\mathbf{v}_h, q_h) =$

$b_h(\mathbf{v}_h, q_h)$, $\forall \mathbf{v}_h \in V_h$, $\forall q_h \in W_h$. To confirm the existence of a unique discrete solution, the famous Babuška-Brezzi conditions must be checked. For this, we let

$$\begin{aligned} N(B) &= \{\mathbf{v} \in \mathbf{X}_h \mid \operatorname{div} \mathbf{v}|_K = 0,\ \forall K \in \mathcal{T}_h\}, \\ N(B_h) &= \{\mathbf{v}_h \in \mathbf{V}_h \mid b_h(\mathbf{v}_h, q_h) = 0,\ \forall q_h \in W_h\}, \\ N(B_h^*) &= \{p_h \in W_h \mid b_h(\mathbf{v}_h, p_h) = 0,\ \forall \mathbf{v}_h \in \mathbf{V}_h\}. \end{aligned}$$

Since $\operatorname{div} \mathbf{V}_h = W_h$ and $N(B_h) \subset N(B)$, we have the coercivity of $a_h(\cdot, \cdot)$ on $N(B_h)$

$$\sup_{\mathbf{v}_h \in\ N(B_h)} \frac{a_h(\mathbf{v}_h, \mathbf{w}_h)}{\|\mathbf{v}_h\|_{\mathbf{X}_h}} \geq \alpha \|\mathbf{w}_h\|_{\mathbf{X}_h},\ \forall \mathbf{w}_h \in N(B_h), \tag{20}$$

for some positive constant α independent of h. Also, we have

$$\sup_{\mathbf{v}_h \in\ \mathbf{X}_h} \frac{b_h(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{\mathbf{X}_h}} \geq \beta \|q_h\|_{W_h/N(B_h^*)},\ \forall q_h \in W_h, \tag{21}$$

by Lemma 4.4. From standard technique, problem (8) has a unique solution [1].

Theorem 5.1. *Let* $(\mathbf{u}, p) \in \mathbf{V} \times W$ *be the exact solution of the saddle point problem* (5) *and* $(\mathbf{u}_h, p_h) \in \mathbf{X}_h \times W_h$ *be the discrete solution of* (8). *If the inf-sup conditions (20) and (21) are satisfied, then the following a priori error estimates hold:*

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{X}_h} + \|p - p_h\|_{W_h} \\ &\leq C \left(\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{V}_h} + \inf_{q_h \in W_h} \|p - q_h\|_{W_h} + \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{|\ a_h(\mathbf{u}, \mathbf{v}_h) + b_h(\mathbf{v}_h, p)\ |}{\|\mathbf{v}_h\|_{\mathbf{X}_h}} \right). \end{aligned}$$

Proof. Pick an arbitrary $\mathbf{v}_h \in \mathbf{V}_h$ and let $\mathbf{w}_h \in \mathbf{V}_h$ be such that

$$b_h(\mathbf{w}_h, q_h) = b_h(\mathbf{u}_h - \mathbf{v}_h, q_h),\ \forall q_h \in W_h. \tag{22}$$

For $\mathbf{x}_h = \mathbf{v}_h + \mathbf{w}_h$, this implies

$$b_h(\mathbf{x}_h, q_h) = b_h(\mathbf{u}_h, q_h) = -(f, q_h),\ \forall q_h \in W_h,$$

since $\mathbf{u}_h$ satisfies the second equation of (8). Then $\mathbf{u}_h - \mathbf{x}_h \in N(B_h)$. From (20), we conclude that

$$\begin{aligned} &\alpha \|\mathbf{u}_h - \mathbf{x}_h\|_{\mathbf{X}_h} \\ &\leq \sup_{\mathbf{v}_h \in N(B_h)} \frac{|\ a_h(\mathbf{v}_h, \mathbf{u}_h - \mathbf{x}_h)\ |}{\|\mathbf{v}_h\|_{\mathbf{X}_h}} \\ &\leq \sup_{\mathbf{v}_h \in N(B_h)} \frac{|\ a_h(\mathbf{v}_h, \mathbf{u}_h - \mathbf{u}) + a_h(\mathbf{v}_h, \mathbf{u} - \mathbf{x}_h)\ |}{\|\mathbf{v}_h\|_{\mathbf{X}_h}} \\ &\leq \sup_{\mathbf{v}_h \in N(B_h)} \frac{|\ a_h(\mathbf{v}_h, \mathbf{u} - \mathbf{x}_h) - b_h(\mathbf{v}_h, q_h - p) - \{a_h(\mathbf{v}_h, \mathbf{u}) + b_h(\mathbf{v}_h, p)\}\ |}{\|\mathbf{v}_h\|_{\mathbf{X}_h}} \end{aligned}$$

for any $q_h \in W_h$. Then, we get

$$\alpha\|\mathbf{u}_h-\mathbf{x}_h\|_{\mathbf{X}_h} \leq c_1\|\mathbf{u}-\mathbf{x}_h\|_{\mathbf{X}_h}+c_2\|p-q_h\|_{W_h}+\sup_{\mathbf{v}_h\in N(B_h)}\frac{|\ a_h(\mathbf{u},\mathbf{v}_h)+b_h(\mathbf{v}_h,p)\ |}{\|\mathbf{v}_h\|_{\mathbf{X}_h}},$$

since both bilinear forms are bounded. If the inverse triangle inequality is applied to the left hand side of the above inequality, then we obtain

$$\begin{aligned}&\alpha\left(\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{X}_h}-\|\mathbf{u}-\mathbf{x}_h\|_{\mathbf{X}_h}\right)\\ &\leq c_1\|\mathbf{u}-\mathbf{x}_h\|_{\mathbf{X}_h}+c_2\|p-q_h\|_{W_h}+\sup_{\mathbf{v}_h\in N(B_h)}\frac{|\ a_h(\mathbf{u},\mathbf{v}_h)+b_h(\mathbf{v}_h,p)\ |}{\|\mathbf{v}_h\|_{X_h}}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{X}_h}\\ &\leq\left(1+\frac{c_1}{\alpha}\right)\|\mathbf{u}-\mathbf{x}_h\|_{\mathbf{X}_h}+\frac{c_2}{\alpha}\|p-q_h\|_{W_h}+\frac{1}{\alpha}\sup_{\mathbf{v}_h\in N(B_h)}\frac{|\ a_h(\mathbf{u},\mathbf{v}_h)+b_h(\mathbf{v}_h,p)\ |}{\|\mathbf{v}_h\|_{\mathbf{X}_h}}.\end{aligned}$$

Also, from (21) and (22), we have

$$\|\mathbf{u}-\mathbf{x}_h\|_{\mathbf{X}_h}\leq\|\mathbf{u}-\mathbf{v}_h\|_{\mathbf{X}_h}+\|\mathbf{w}_h\|_{\mathbf{X}_h}\leq\left(1+\frac{c_2}{\beta}\right)\|\mathbf{u}-\mathbf{v}_h\|_{\mathbf{X}_h}.$$

Hence

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{X}_h} \quad\leq\quad &\left(1+\frac{c_1}{\alpha}\right)\left(1+\frac{c_2}{\beta}\right)\|\mathbf{u}-\mathbf{v}_h\|_{\mathbf{X}_h}+\frac{c_2}{\alpha}\|p-q_h\|_{W_h}+\\ &\frac{1}{\alpha}\sup_{\mathbf{v}_h\in N(B_h)}\frac{|\ a_h(\mathbf{u},\mathbf{v}_h)+b_h(\mathbf{v}_h,p)\ |}{\|\mathbf{v}_h\|_{\mathbf{X}_h}}.\end{aligned} \tag{23}$$

From the first equation of (8), we know that

$$a_h(\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h)+b_h(\mathbf{v}_h,p-p_h)=a_h(\mathbf{u},\mathbf{v}_h)+b_h(\mathbf{v}_h,p).$$

Then we have

$$\begin{aligned}\|p-p_h\|_{W_h/N(B_h^*)}&\leq\frac{1}{\beta}\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{|\ b_h(\mathbf{v}_h,p-p_h)\ |}{\|\mathbf{v}_h\|_{\mathbf{X}_h}}\\ &\leq\frac{1}{\beta}\left(c_3\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{X}_h}+\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{|\ a_h(\mathbf{u},\mathbf{v}_h)+b_h(\mathbf{v}_h,p)\ |}{\|\mathbf{v}_h\|_{\mathbf{X}_h}}\right),\end{aligned} \tag{24}$$

by (21). Combining the previous inequalities (23) and (24) into a single line, we can obtain the desired result. □

To show convergence of our method, we have to control the following approximation error and the consistency error:

$$\inf_{\mathbf{v}_h\in\mathbf{V}_h}\|\mathbf{u}-\mathbf{v}_h\|_{\mathbf{V}_h}+\inf_{q_h\in W_h}\|p-q_h\|_{W_h}, \tag{25}$$

$$\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{|\ a_h(\mathbf{u},\mathbf{v}_h)+b_h(\mathbf{v}_h,p)\ |}{\|\mathbf{v}_h\|_{\mathbf{X}_h}}. \tag{26}$$

For the approximation error, we can easily obtain optimal order $k+1$, since $\mathbf{V}_h$ and W_h has polynomials of degree k. That is,

$$\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{V}_h} + \inf_{q_h \in W_h} \|p - q_h\|_{W_h} \leq Ch^{k+1} \|p\|_{H^{k+2}(\Omega)}. \tag{27}$$

Unfortunately, the consistency error is smaller than that of the approximation error. That is,

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{| \, a_h(\mathbf{u}, \mathbf{v}_h) + b_h(\mathbf{v}_h, p) \, |}{\|\mathbf{v}_h\|_{\mathbf{X}_h}} \leq Ch^k \|p\|_{H^{k+2}(\Omega)}. \tag{28}$$

Because of this loss, we have only suboptimal convergence. For proof, see [12].

Theorem 5.2. *Suppose the following regularity holds for* $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$, $p \in H^{k+2}(\Omega)$:

$$\|\mathbf{u}\|_{k+1} + \|p\|_{k+2} \leq C\|f\|_{H^k}(\Omega).$$

Then there exists a constant C *independent of* h *such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{X}_h} + \|p - p_h\|_{W_h} \leq Ch^k \|p\|_{H^{k+2}}(\Omega) \leq Ch^k \|f\|_{H^k}(\Omega). \tag{29}$$

Proof. This follows directly from Theorem 5.1, (27) and (28). □

6. Conclusion

We studied nonconforming approach based mixed finite element methods to solve second order elliptic problems on parallelepiped grids. For this method, we suggested modified patch conditions and constructed a new family of nonconforming mixed finite element space of higher order $k \geq 1$ which satisfies these conditions. This space has smaller number of degrees of freedom than the existing ones, however, one can still obtain the same convergence order theoretically. In future research, numerical experiments using this element space will be presented to support the theory.

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