J. Appl. Math. & Informatics Vol. **42**(2024), No. 5, pp. 1211 - 1225
https://doi.org/10.14317/jami.2024.1211

SINGULAR PERTURBATIONS AND SMALL DELAYS THROUGH LIOUVILLE'S GREEN TRANSFORMATION

DANY JOY, DINESH KUMAR S*

Abstract. In this paper, we introduce a numerical method for solving singularly perturbed delay differential equation using Liouville - Green transformation. As an initial step, we transformed the statement equation into a singular perturbation problem with boundary conditions and then we used Liouville - Green transformation to solve it. Almost second-order accuracy is achieved with the scheme derived. The algorithm's performance is assessed through the examination of multiple test scenarios that involve different perturbation settings and delay parameters. The results of the proposed method are compared with those of other numerical techniques already available. The numerical scheme is described together with error estimates and a convergence rate.

AMS Mathematics Subject Classification : 65L06, 65L11, 65L20, 65L50.
Key words and phrases : Singular perturbation problems, delay differential equations, Liouville-Green transformations, Thomas algorithm.

1. Introduction

In the scientific literature, singularly perturbed delay differential equations (SPDDEs) are those that have at least one positive or negative shift parameter and a highest order derivative that is multiplied by a small positive parameter. The numerical approaches to the treatment of these equations have attracted more attention in recent years. The mathematical modelling of the human pupil light reflex [17], vibrational issues in control theory [7], physiological kinetics [3], predator–prey model [1] etc are commonly encounters these kinds of problems. Ecology and epidemiology also play significant roles in the study and application of singularly perturbed delay differential equations. These fields often involve systems with processes occuring at vastly different time scales, which is where singular perturbation techniques become highly useful. The works

Received April 6, 2024. Revised May 29, 2024. Accepted June 11, 2024. *Corresponding author.

in [10], [9], [11], [8] can also be extended to give applications in SPDDEs as they deal with aforementioned disciplines. Fluid mechanics, quantum mechanics, optimal control, elasticity, chemical reactor theory, aerodynamics, geophysics, and many other areas of applied mathematics often deal with the area of singular perturbation. In a similar way, there has been a rise in interest in delay differential equations because they are used in so many different fields, such as medicine, life sciences, robotics, economics, physical sciences, and so on.

Lange and Miura [16] investigated a class of boundary-value problems and discussed an asymptotic method to approximatively solve this type of differential equation. In [2], Amiraliyev and Cimen carried out an exponentially fitted difference scheme on a uniform mesh. This is achieved by the use of the method of integral identities, which involves the utilisation of exponential basis functions and interpolating quadrature rules that are formulated with the weight and remainder term in integral form. Phaneendra and Lalu [18] applied Gaussian quadrature two point equation is applied to obtain a tridiagonal system and is solved with rate of convergence one. In [14], Kanth and Kumar developed a hybrid numerical technique consists of the tension spline technique in the boundary layer region and the midpoint approximation on piecewise uniform mesh in the outer region. Chakravarthy and Kumar [21] approached these problems via Numerov's method and the authors in [4] developed an initial value technique to solve these type of problems. In [19], a fitted parameter exponential spline method to solve SPDDEs of convection diffusion type. The authors in [26] presented Hermite polynomial approach to solve these equations by converting the main problem to a matrix equation. In [13] they considered a standard upwind finite difference scheme on a special type of mesh to tackle the delay argument. A numerical integration method by introducing an exponential integrating factor for solving singularly perturbed delay problems is studied by Challa and Reddy in [24]. Chakravarthy and Rao [5] developed modified Numerov method to solve the problem with delay and advance term. In [12], the authors cosider the singularly perturbed system of delay differential equations and a non polynomial spline technique is used to solve the problem. The authors in [15] proposed a fourth order finite difference scheme with fitting factors for the solution of SPDDEs with mixed shifts. In [22] they proposed a cubic spline in compression technique to solve the problem with integral boundary conditions. Ranjan and Prasad [20] studied an exponentially fitted three-term finite difference approach singularly perturbed delay differential equations with small shifts.

In this work, we take into account convection diffusion problems, which fall under the category of singularly perturbed delay differential equations. In Sect. 2, we define the problem and assumptions on the parameter. In Sect. 3 we employ Liouville-Green transformation to arrive at its numerical solution. Using Taylor series, an equivalent equation is developed to approximate the given problem and the Green transformation is applied to get a recurrence relation, which is then solved by Thomas Algorithm. The convergence analysis of the proposed method is also discussed in Sect. 4. In Sect. 5 different model examples are computed with varying delay and perturbation parameters, and their results are shown. Discussion and Conclusion follow in Sect. 6.

2. Statement of the problem

To illustrate the method, we use the following singularly perturbed delay differential equation of convection diffusion type:

$$\mu x''(z) + c(z)x'(z-\eta) + d(z)x(z) = g(z), 0 < z < 1, \tag{1}$$

with interval and boundary conditions,

$$x(z) = \psi(z), -\eta \le z \le 0 \quad \text{and} \quad x(1) = \gamma. \tag{2}$$

where μ $(0 < \mu << 1)$ is perturbation parameter and η $(0 < \eta < 1)$ is a small delay parameter. As $\eta < \mu$, for $c(z) \ge N > 0, (\mu - \eta c(z)) > 0, \forall z \in [0,1]$, then the boundary layer exist on the left side and for $c(z) \le \theta < 0$, the boundary layer exist on the right side. We assume $d(z) \le -\theta < 0$, where θ is a positive constant, $c(z), d(z), g(z)$ and $\psi(z)$ are bounded functions having continuous derivatives and γ is a fixed constant. When the shift parameter η is smaller than μ the use of Taylor's series expansion for the term containing shift argument is valid [25]. In this work, the case when $\eta < \mu$ is considered. Thus, applying the expansion of Taylor Series in the region around z, we obtain

$$x(z-\eta) = x(z) - \eta x'(z) + O(\eta^2). \tag{3}$$

Using (3) in (1), we develop the following problem containing small perturbation parameter ϵ.

$$\epsilon x''(z) + c(z)x'(z) + d(z)x(z) = g(z), \tag{4}$$

with,

$$x(0) = \psi_0, \; x(1) = \gamma, \tag{5}$$

where,

$$\epsilon \quad = \quad \mu - c(z)\eta.$$

3. Liouville - Green Transformation

3.1. Numerical Algorithm.

The subsequent procedure is suggested for acquiring the numerical solution of the problem:

Step 1: Introduce the uniform mesh by partitioning the domain $[0,1]$ into N mesh intervals.

Step 2: We make use of Taylor series expansion of first order derivatives.

Step 3: Liouville - Green Transformation is applied to the statement obtained in Step 2.

Step 4: We employ the scheme obtained in Step 3 and find the solution of the problem using Gauss elimination method.

3.2. The proposed numerical scheme.
Consider the equation,

$$\epsilon x''(z) + c(z)x'(z) + d(z)x(z) = g(z), z \in [0,1]. \tag{6}$$

The Liouville-Green Transformation [23] is given by,

$$\begin{aligned} u &= \alpha(z) = \tfrac{1}{\epsilon}\int c(z)dz, \\ \beta(z) &= \alpha'(z) = \tfrac{1}{\epsilon}c(z), \end{aligned} \tag{7}$$

$$w(u) = \beta(z).x(z). \tag{8}$$

Differentiate equation (8) with respect to z, we have,

$$\frac{dx}{dz} = \frac{\alpha'(z)}{\beta(z)}\frac{dw}{du} - \frac{\beta'(z)}{\beta^2(z)}w(u), \tag{9}$$

$$\frac{d^2x}{dz^2} = \frac{\alpha'^2(z)}{\beta(z)}\frac{d^2w}{du^2} + \left[\frac{\alpha''(z)}{\beta(z)} - \frac{2\beta'(z)\beta'(z)}{\beta(z)}\right]\frac{dw}{du} - \left[\frac{\beta''(z)}{\beta^2(z)} - 2\frac{\beta'^2(z)}{\beta^3(z)}\right]w. \tag{10}$$

Use the equations (9) and (10) in (6) to get,

$$\frac{d^2w}{du^2} + P(z)\frac{dw}{du} + Q(z)w(u) = R(z), \tag{11}$$

where,

$$\begin{aligned} P(z) &= 1 - \epsilon\frac{c'(z)}{c^2(z)}, \\ Q(z) &= 2\epsilon\frac{c'^2(z)}{c^4(z)} - \epsilon^2\frac{c''(z)}{c^3(z)} - \epsilon\frac{c'(z)}{c^2(z)} + \epsilon^2\frac{d(z)}{c^2(z)}, \\ R(z) &= \frac{g(z)}{c(z)}. \end{aligned}$$

Now, we use approximation for w_i' and w_i'' with the help of Taylor series of w_{i+1} and w_{i-1} upto $O(h^5)$,

$$w_i' \simeq \frac{w_{i+1} - w_{i-1}}{2h} - \frac{h^2}{6}w_i^{(3)} + T_1, \tag{12}$$

$$w_i'' \simeq \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} - \frac{h^2}{6}w_i^{(4)} + T_2, \tag{13}$$

where,

$$T_1 = -\frac{h^4}{120}w_i^{(5)} + O(h^5) \text{ and } T_2 = -\frac{h^4}{360}w_i^{(6)} + O(h^5).$$

Use equations (12) and (13) in (11), we get,

$$\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} - \frac{h^2}{6}w_i^{(4)} + T_2$$

$$+ P(z)\left(\frac{w_{i+1}-w_{i-1}}{2h} - \frac{h^2}{6}w_i^{(3)} + T_1\right) + Q(z)w_i = R(z). \quad (14)$$

From (11), we have

$$w_i'' \quad = \quad R_i - P_i w_i' - Q_i w_i. \quad (15)$$

We make use of (14) to get w_i''' and w_i^{iv}.

$$\begin{aligned} w_i''' &= R_i' - Q_i' w_i - (P_i' + Q_i)w_i' - P_i w_i'', \\ w_i^{(iv)} &= R_i'' - P_i R_i' + (P_i Q_i' - Q_i'')w_i + (P_i P_i' + P_i Q_i + 2Q_i' - P_i'')w_i' \\ &\quad +(P_i^2 - 2P_i' - Q_i)w_i''. \end{aligned}$$

Substituting the above equations, we have,

$$\begin{aligned} &\left[\frac{1}{h^2} - \frac{P_i}{2h} - \frac{h}{12}(2Q_i' + P_i'') + \frac{1}{6}(2P_i' + Q_i)\right] w_{i-1} \\ &+ \left[-\frac{2}{h^2} + \frac{2Q_i}{3} - \frac{h^2 Q_i''}{6} - \frac{2P_i'}{3}\right] w_i \\ &+ \left[\frac{1}{h^2} + \frac{P_i}{2h} - \frac{h}{12}(-2Q_i' - P_i'') - \frac{1}{6}(-2P_i' - Q_i)\right] w_{i+1} \\ &= R_i + \frac{h^2}{6}R_i'' - \frac{h^2}{6}P_i R_i' + T, \end{aligned} \quad (16)$$

where,

$$T = \frac{h^4}{36}(2Q_i' + P_i'')w_i^{(3)} + \frac{h^4}{36}(2P_i' + Q_i)w_i^{(4)} + \frac{h^4}{360}w_i^{(6)} + \frac{h^4}{120}P_i w_i^{(5)}.$$

Finally, we obtain the following recurrence relation by multiplying both sides of equation (16) by h^2,

$$E_i w_{i-1} + F_i w_i + G_i w_{i+1} \quad = \quad H_i, \quad for \;\; i = 1(1)N-1, \quad (17)$$

where,

$$\begin{aligned} E_i &= 1 - \frac{hP_i}{2} + \frac{h^3}{12}(-2Q_i' - P_i'') - \frac{h^2}{6}(-2P_i' - Q_i), \\ F_i &= -2 + \frac{2h^2 Q_i}{3} - \frac{h^4 Q_i''}{6} - \frac{2h^2 P_i'}{3}, \\ G_i &= 1 + \frac{hP_i}{2} - \frac{h^3}{12}(-2Q_i' - P_i'') - \frac{h^2}{6}(-2P_i' - Q_i), \\ H_i &= h^2 R_i + \frac{h^4}{6}R_i'' - \frac{h^4}{6}P_i R_i'. \end{aligned}$$

The system (17) is of tridiagonal form and it can be solved using Thomas Algorithm.

4. Stability and Convergence Analysis

For the modified problem, we develop certain results regarding the solutions and their derivatives. Let $L_{\epsilon,\eta}$ be the differential operator for the problem (4) and (5).

Lemma 4.1. *Let x be a smooth function defined on $\Gamma = (0,1)$ and it satisfies $x(0) \geq 0$ and $x(1) \geq 0$. Then $L_{\epsilon,\eta}x(z) \geq 0,\ z \in \Gamma$ implies that $x(z) \geq 0,\ \forall z \in \bar{\Gamma}$.*

Proof. Let us consider any aribtrary point in Γ, say z^* which satisfies $x(z^*) = \min_{z\in\bar{\Gamma}}\{x(z)\}$ and assume that $x(z^*) < 0$. It is clear that $z^* \notin \{0,1\}$, also $x'(z^*) = 0$ and $x''(z^*) \geq 0$. From (4), the operator,

$$L_{\epsilon,\eta}x(z^*) = \epsilon x''(z^*) + c(z^*)x'(z^*) + d(z^*)x(z^*) \leq 0,$$

which is a contradiction to our assumption $L_{\epsilon,\eta} \geq 0$. Then it follows that $x(z^*) \geq 0$. Here we choose z^* as arbitrary, we have $x(z) \geq 0,\ \forall x \in \bar{\Gamma}$. □

Lemma 4.2. *The solution $x(z)$ of the problem in (4) and (5) is bounded as*

$$\|x(z)\| \leq \theta^{-1}\|g\| + \max\{|\ \psi(0)\ |, |\ \gamma(1)\ |\}.$$

Proof. Let us consider the barrier function,

$$\tau^{\pm}(z) = \theta^{-1}\|g\| + \max\{|\ \psi(0)\ |, |\ \gamma(1)\ |\} \pm x(z).$$

Then we get $\tau^{\pm}(0) \geq 0$ and $\tau^{\pm}(1) \geq 0$, also we have,

$$\begin{aligned} L_{\epsilon,\eta}\tau^{\pm}(z) &= \epsilon(\tau^{\pm}(z))'' + c(z)(\tau^{\pm}(z))' + d(z)(\tau^{\pm}(z)) \\ &= d(z)\left(\theta^{-1}\|g\| + \max\{|\ \psi(0)\ |, |\ \gamma(1)\ |\}\right) \pm L_{\epsilon,\eta}x(z) \\ &= d(z)\left(\theta^{-1}\|g\| + \max\{|\ \psi(0)\ |, |\ \gamma(1)\ |\}\right) \pm g(z) \end{aligned}$$

Since $d(z) < 0$, we have $d(z)\theta^{-1} \leq -1$, thus we obtain,

$$L_{\epsilon,\eta}\tau^{\pm}(z) \leq (-\|g\| \pm g(z)) + d(z)\max\{|\ \psi(0)\ |, |\ \gamma(1)\ |\} \leq 0.$$

Since $g(z) \leq \|g\|$ for all $z \in \Gamma$. From the above lemma we get $\tau^{\pm}(z) \geq 0$ for all $z \in \Gamma$. Thus we obtain,

$$\|x(z)\| \leq \theta^{-1}\|g\| + \max\{|\ \psi(0)\ |, |\ \gamma(1)\ |\}.$$

□

Lemma 4.3. *The derivatives of the solution $x(z)$ of the problem in (4) and (5) satisfy,*

$$\|x^k\| \leq M(\mu - \eta N)^{-k},\ k = 1,2,3.$$

Proof. For the proof, the reader can refer [2]. □

Lemma 4.4. *Let U be the coefficient matrix associated with the system in (17) , then for all $\epsilon > 0$, the matrix U is irreducible and diagonally dominant.*

Proof. The equation (17) can be represented in matrix form as $UW = V$, where U is a tri-diagonal matrix with F_i as diagonal elements and E_i, G_i as co-diagonal elements. It is easily seen that $E_i \neq 0$ and $G_i \neq 0$, $\forall i = 1, 2, ..., N-1$. Hence U is irreducible and by our assumption $d(z) < 0$, we obtain $|E_i + G_i| < |F_i|$. Thus, the matrix U is diagonally dominant. Hence, the scheme in equation (17) is stable [6]. □

Theorem 4.5. *Let $x(z_i)$ denote the analytical solution for the problem described in* (4) *and* (5) *and the computational solution W^N for the discretized problem described in* (17)*. Then*

$$\sup_{0 \leq \mu \leq 1} | x(z_i) - W^N | \leq C_1 N^{-2},$$

holds for sufficiently large values of N.

Proof. We write (17) in matrix form as follows:

$$UW = V, \tag{18}$$

where $U = (u_{ij}), i = 1(1)N-1$, is a tridiagonal matrix of order $N-1$. Here,

$$\begin{aligned} u_{ii-1} &= 1 - \frac{hP_i}{2} + \frac{h^3}{12}(-2Q_i' - P_i'') - \frac{h^2}{6}(-2P_i' - Q_i), \\ u_{ii} &= -2 + \frac{2h^2P_i}{3} - \frac{h^4Q_i''}{6} - \frac{2h^2P_i'}{3}, \\ u_{ii+1} &= 1 + \frac{hP_i}{2} - \frac{h^3}{12}(-2Q_i' - P_i'') - \frac{h^2}{6}(-2P_i' - Q_i). \end{aligned}$$

$V = (v_i)$ is a column vector such that

$$\begin{aligned} v_1 &= -[\psi_0 E_1 - H_1], \\ v_i &= H_i, \quad i = 2(1)N-2, \\ v_{N-1} &= -[\gamma G_{N-1} - H_{N-1}]. \end{aligned}$$

with a local truncation error,

$$D_i(h) = \frac{h^4}{36}\bar{K} + O(h^5),$$

where,

$$\bar{K} = (2Q_i' + P_i'')w_i^{(3)} + (2P_i' + Q_i)w_i^{(4)} + \frac{w_i^{(6)}}{10} + \frac{3P_iw_i^{(5)}}{10}.$$

(18) can also be written in error form as:

$$U\overline{W} - D(h) = V. \tag{19}$$

Here $\overline{W}$ is the exact solution and $D(h)$ is the truncation error. From (18) and (19), we get $U(\overline{W} - W) = D(h)$, that is,

$$UE = D(h). \tag{20}$$

For the matrix U, the sum of elements of the i^{th} row be denoted by $\bar{S}_i$.

$$\bar{S}_1 = -1 + \frac{hP_i}{2} + h^2\left[\frac{2Q_i}{3} - \frac{2P_i'}{3} - \frac{1}{6}(2P_i' - Q_i)\right] + h^3\left[-\frac{1}{12}(-2Q_i' - P_i'')\right] + h^4\left(-\frac{Q_i''}{6}\right),$$

$$\bar{S}_i = h^2 Q_i + O(h^4), \qquad i = 2(1)N-2,$$

$$\bar{S}_{N-1} = -1 - \frac{hP_i}{2} + h^2\left[\frac{2Q_i}{3} - \frac{2P_i'}{3} - \frac{1}{6}(2P_i' - Q_i)\right] + h^3\left[\frac{1}{12}(-2Q_i' - P_i'')\right] + h^4\left(-\frac{Q_i''}{6}\right).$$

From theory of matrices, we get,

$$E = U^{-1}D(h). \tag{21}$$

Hence,

$$\|E\| \leq \|U^{-1}\|\|D(h)\|. \tag{22}$$

Let the (j,i)-th element of U^{-1} be $\overline{u}_{j,i}$, which are non-negative. Then,

$$\sum_{i=1}^{N-1} \overline{u}_{j,i}\bar{S}_i = 1, \qquad j = 1(1)N-1.$$

Hence,

$$\sum_{i=1}^{N-1} \overline{u}_{j,i} \leq \frac{1}{\min_{1\leq i\leq N-1} S_i} \leq \frac{1}{h^2 \mid Q_{i0} \mid}, \tag{23}$$

for some i_0 between 1 and $N-1$.
Equations (18), (21), (22) and (23) give,

$$e_i = \sum_{j=1}^{N-1} \overline{u}_{j,i} D_i(h) \qquad , i = 1(1)N-1,$$

which implies

$$\begin{aligned} e_i &\leq \left(\sum_{j=1}^{N-1} \overline{u}_{j,i}\right) \max_{1\leq i\leq N-1} \mid D_i(h) \mid \\ &\leq \frac{1}{h^2 \mid Q_{i0} \mid} \times \frac{h^4 \bar{K}}{36} = \frac{h^2 \bar{K}}{36 \mid Q_{i0} \mid}, \end{aligned}$$

where $\bar{K}$ does not depend on h. So $\|E\| = O(h^2)$ and hence our method is of second order convergent. □

5. Numerical Experiments

In this section, to validate the theoretical results, four numerical experiments are taken into consideration to illustrate the applicability of the proposed method having layer and oscillatory behaviour. We give tabulated solution of some problems with varying μ and η. Since the exact solutions of the problems are not known, the maximum absolute errors for the examples are determined using the following double mesh principle,

$$E^N = max_i \mid x_i^N - x_{2i}^{2N} \mid . \tag{24}$$

Table 1. The maximum absolute error of Example 1

η	N				
	100	200	300	400	500
0.03	1.4413e-08	3.6031e-09	1.6014e-09	9.0082e-10	5.7664e-10
0.05	2.7449e-08	6.8622e-09	3.0497e-09	1.7155e-09	1.0982e-09
0.09	4.9408e-08	1.2352e-08	5.4895e-09	3.0880e-09	1.9762e-09

Table 2. The maximum absolute error of Example 1

μ	N				
	100	200	300	400	500
10^{-3}	5.3595e-08	1.3399e-08	5.9551e-09	3.3500e-09	2.1435e-09
10^{-4}	5.3999e-08	1.3499e-08	5.9995e-09	3.3753e-09	2.1600e-09
10^{-5}	5.4039e-08	1.3510e-08	6.0040e-09	3.3779e-09	2.1616e-09
10^{-6}	5.4043e-08	1.3511e-08	6.0044e-09	3.3779e-09	2.1621e-09
10^{-7}	5.4044e-08	1.3511e-08	6.0047e-09	3.3781e-09	2.1613e-09
10^{-8}	5.4044e-08	1.3511e-08	6.0046e-09	3.3782e-09	2.1625e-09

Table 3. Rate of convergence ρ of Example 1

h	$\frac{h}{2}$	E_h	$\frac{h}{4}$	$E_{\frac{h}{2}}$	ρ
1/100	1/200	3.3620e-06	1/400	8.4048e-07	2.0000
1/200	1/400	8.4048e-07	1/800	2.1013e-07	1.9999
1/300	1/600	3.7354e-07	1/1200	9.3439e-08	1.9992

Example 1: Consider

$$\mu x''(z) + x'(z - \eta) - x(z) = 0$$

with

$$x(z) = 1, \ -\eta \le z \le 0, \ x(1) = 0$$

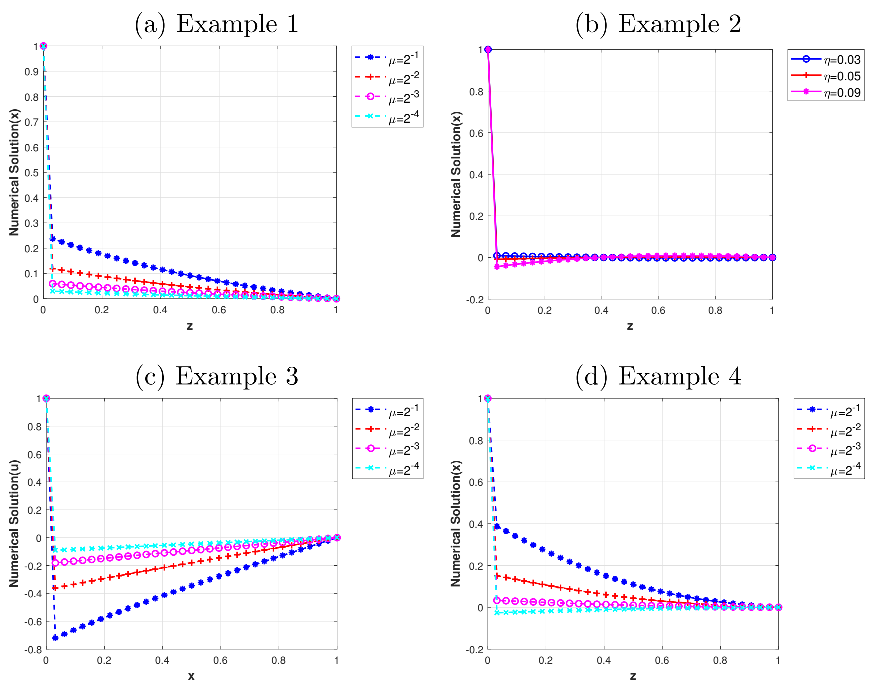


FIGURE 1. The numerical solution for different values of μ

TABLE 4. The maximum absolute error of Example 2

μ	N				
	100	200	300	400	500
0.03	3.9385e-08	9.8465e-09	4.3763e-09	2.4616e-09	1.5755e-09
0.05	3.6111e-08	9.0277e-09	4.0123e-09	2.2569e-09	1.4444e-09
0.09	1.4702e-07	3.6754e-08	1.6335e-08	9.1884e-09	5.8806e-09

Table 5. Rate of convergence ρ of Example 2

h	$\frac{h}{2}$	E_h	$\frac{h}{4}$	$E_{\frac{h}{2}}$	ρ
1/100	1/200	6.4406e-06	1/400	1.6101e-06	2.0001
1/200	1/400	1.6101e-06	1/800	4.0252e-07	2.0000
1/300	1/600	7.1561e-07	1/1200	1.7890e-07	2.0000

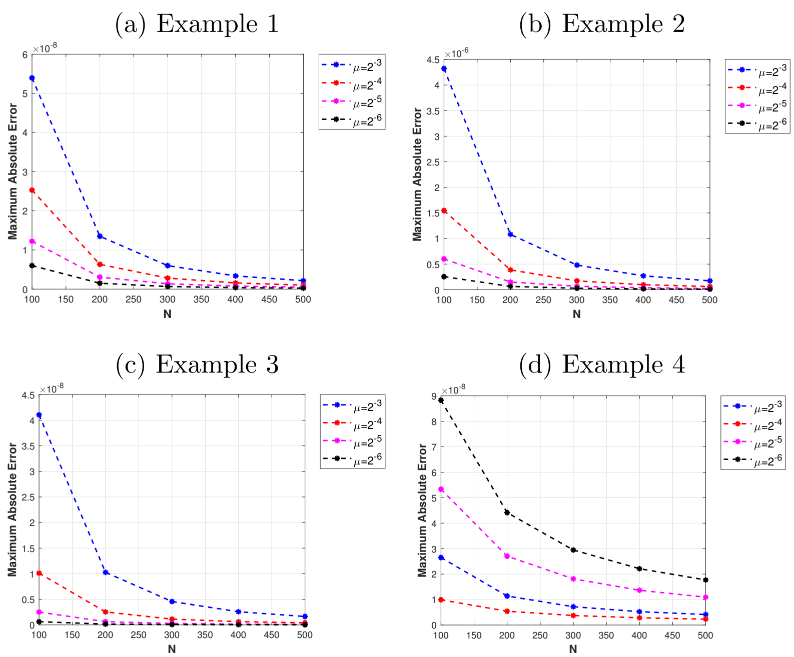
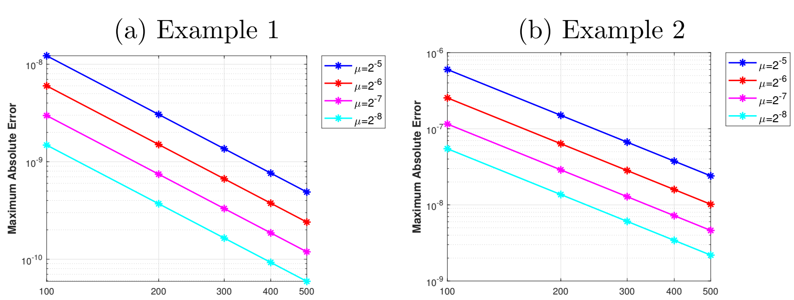


Figure 2. The MAE for different values of μ and $\eta = 0.5 * \mu$



(c) Example 3 (d) Example 4

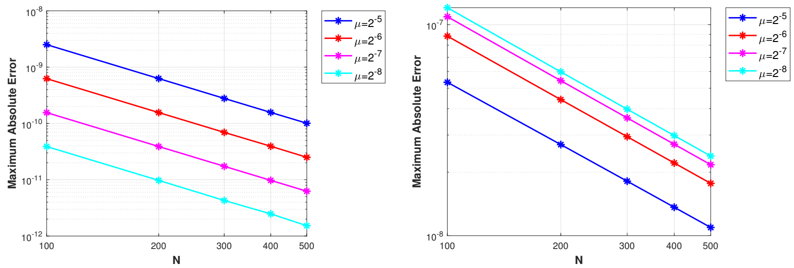


FIGURE 3. Loglog plot of maximum point-wise errors

Example 2: Consider

$$\mu x''(z) + 0.25x'(z-\eta) - x(z) = 1$$

with

$$x(z) = 1, \ -\eta \leq z \leq 0, \ x(1) = 0$$

TABLE 6. The maximum absolute error of Example 3

μ	N				
	100	200	300	400	500
0.03	1.8246e-09	4.5622e-10	2.0277e-10	1.1444e-10	7.2773e-11
0.05	4.1191e-09	1.0300e-09	4.5768e-10	2.5818e-10	1.6406e-10
0.09	1.1518e-08	2.8799e-09	1.2801e-09	7.1950e-10	4.5888e-10

Example 3: Consider

$$\mu x''(z) - x'(z-\eta) - x(z) = 1$$

with

$$x(z) = 1, \ -\eta \leq z \leq 0, \ x(1) = 0$$

Example 4: Consider

$$\mu x''(z) - (z^2+1)x'(z-\eta) - (z+1)x(z) = z^2$$

with

$$x(z) = 1, \ -\eta \leq z \leq 0, \ x(1) = 0$$

The maximum absolute errors for Examples were assessed by varying the values of delay parameter and the findings are presented in Tables 1, 4, 6 and 8 which is then compared with the results in [2] and [19]. The information presented in Figure 2 illustrate a noticeable trend, wherein the maximum absolute

Table 7. Rate of convergence ρ of Example 3

h	$\frac{h}{2}$	E_h	$\frac{h}{4}$	$E_{\frac{h}{2}}$	ρ
1/100	1/200	5.3293e-06	1/400	1.3323e-06	2.0000
1/200	1/400	1.3323e-06	1/800	3.3307e-07	2.0000
1/300	1/600	5.9213e-07	1/1200	1.4801e-07	2.0002

Table 8. The maximum absolute error of Example 4

μ	N				
	100	200	300	400	500
0.03	4.7000e-09	2.6886e-09	1.8892e-09	1.4560e-09	1.1844e-09
0.05	2.3072e-08	1.2055e-08	8.1760e-09	6.1874e-09	4.9775e-09
0.09	1.0306e-07	5.1339e-08	3.4221e-08	2.5668e-08	2.0538e-08

errors consistently decrease with an increase in the step size. The numerical solution of this SPDDE by setting delay parameter as $0.5 * \mu$ with 2^5 subintervals is presented in Figure 1. The loglog plot of the maximum pointwise error is also given in Figure 3 and the rate of convergence table is presented in Table 3, Table 5 and Table 7.

The Rate of Convergence (ρ)**.** We define $E_{h/2}$ in the same way as equation (24) as follows:

$$E_{h/2} = max_i \mid x_i^{h/2} - x_i^{h/4} \mid, \; for \; i = 1(1)2N-1.$$

The computational rate of convergence ρ is also obtained by using the double mesh principle and is defined as

$$\rho = \frac{\log(E_h) - \log\left(E_{h/2}\right)}{\log 2}$$

6. Discussion & Conclusion

The study introduces a numerical approximation for approaching SPDDEs with small delay. Four examples have been taken into consideration, which are not having exact solutions for various values of μ and η in order to show the effectiveness of this method. The results are summarized in terms of maximum absolute errors (Tables 1, 4, 6, 8) and it is discovered that the discussed transformation gives an improvement to the findings of [2] and [19]. Also, graphs have been used to show numerical solution of the problem to study the effect of η on the solution profile. In addition to this, as h decreases(N increases), the absolute error also decreases. By using truncation error, the rate of convergence of the proposed scheme is determined to be two. The approach we used in addressing singularly perturbed small delay differential equations can be extended to partial differential equation problems as well as the problems with delay and advanced parameters.

Conflicts of interest : The authors declare that they have no conflict of interest, relevant to the content of this article.

Data availability : No data was used for the research described in the article.

References

1. N. Anggriani, H.S. Panigoro, E. Rahmi, O.J. Peter, S.A. Jose, *A predator–prey model with additive Allee effect and intraspecific competition on predator involving Atangana–Baleanu–Caputo derivative*, Results Phys. **49** (2023), 106489.
2. G.M. Amiraliyev, E. Cimen, *Numerical method for a singularly perturbed convection–diffusion problem with delay*, Appl. Math. Comput. **216** (2010), 2351-2359.
3. C.T. Baker, G.A. Bocharov, F.A. Rihan, *A report on the use of delay differential equations in numerical modelling in the biosciences*, Manchester Centre for Computational Mathematics, 1999.
4. P.P. Chakravarthy, K. Kumar, *A novel method for singularly perturbed delay differential equations of reaction-diffusion type*, Differ. Equ. Dyn. Syst. **29** (2021), 723-34.
5. P.P. Chakravarthy, R.N. Rao, *A modified Numerov method for solving singularly perturbed differential–difference equations arising in science and engineering*, Results Phys. **2** (2012), 100-3.
6. G. File, G. Gadisa, T. Aga, Y.N. Reddy, *Numerical solution of singularly perturbed delay reaction-diffusion equations with layer or oscillatory behaviour.* Am. J. Numer. Anal. **5** (2017), 1-0.
7. V.Y. Glizer, *Asymptotic analysis and solution of a finite-horizon $H\infty$ control problem for singularly-perturbed linear systems with small state delay*, J. Optimiz. Ttheory. App. **117** (2003), 295-325.
8. S.A. Jose, R. Raja, J. Dianavinnarasi, D. Baleanu, A. Jirawattanapanit. *Mathematical modeling of chickenpox in Phuket: Efficacy of precautionary measures and bifurcation analysis*, Biomed. Signal Process. Control. **84** (2023), 104714.
9. S.A. Jose, R. Raja, Q. Zhu, J. Alzabut, M. Niezabitowski, V.E. Balas, *Impact of strong determination and awareness on substance addictions: A mathematical modeling approach*, Math. Method. Appl. Sci. **45** (2022), 4140-60.
10. S.A. Jose, R. Ramachandran, D. Baleanu, H.S. Panigoro, J. Alzabut, V.E. Balas, *Computational dynamics of a fractional order substance addictions transfer model with Atangana-Baleanu-Caputo derivative*, Math. Method. Appl. Sci. **46** (2023), 5060-85.
11. S.A. Jose, Z. Yaagoub, D. Joseph, R. Ramachandran, A. Jirawattanapanit, *Computational dynamics of a fractional order model of chickenpox spread in Phuket province*, Biomed. Signal Process. Control. **91** (2024), 105994.
12. D. Joy, S.D. Kumar, *Non-polynomial spline approach for solving system of singularly perturbed delay differential equations of large delay*, Math. Comput. Model. Dyn. Syst. **30** (2024), 179-201.
13. M.K. Kadalbajoo, K.K. Sharma, *A numerical method based on finite difference for boundary value problems for singularly perturbed delay differential equations*, Appl. Math. Comput. **197** (2008), 692-707.

14. A.R. Kanth, P.M. Kumar, *Computational results and analysis for a class of linear and nonlinear singularly perturbed convection delay problems on Shishkin mesh*, Hacet. J. Math. Stat. **49** (2020), 221-235.
15. D. Kumara Swamy, K. Phaneendra, Y.N. Reddy, *Accurate numerical method for singularly perturbed differential-difference equations with mixed shifts*, Khayyam J. Math. **4** (2018), 110-122.
16. C.G. Lange, R.M. Miura, *Singular perturbation analysis of boundary-value problems for differential-difference equations. VI. Small shifts with rapid oscillations*, SIAM J. Appl. Math. **54** (1994), 273-83.
17. A. Longtin, J.G. Milton, *Complex oscillations in the human pupil light reflex with mixed and delayed feedback*, Math. Biosci. **90** (1988), 183-199.
18. K. Phaneendra, M. Lalu, *Numerical solution of singularly perturbed delay differential equations using gaussion quadrature method*, J. Phys. Conf. Ser. **1344** (2019), 012013.
19. E.S. Prasad, R. Omkar, K. Phaneendra, *Fitted parameter exponential spline method for singularly perturbed delay differential equations with a large delay*, Comput. Math. Methods. **2022** (2022).
20. R. Ranjan, H.S. Prasad, *A novel approach for the numerical approximation to the solution of singularly perturbed differential-difference equations with small shifts*, J. Appl. Math. Comput. **65** (2021), 403-427.
21. R.N. Rao, P.P. Chakravarthy, *An initial value technique for singularly perturbed differential-difference equations with a small negative shift*, J. Appl. Math. Inform. **31** (2013), 131-45.
22. A.M. Regal, D. Kumar, *Singular Perturbations and Large Time Delays Through Accelerated Spline-Based Compression Technique*, Contemp. Math. (2024), 1072-92.
23. M. Sharma, A. Kaushik, C. Li, *Analytic approximation to delayed convection dominated systems through transforms*, J. Math. Chem. **52** (2014), 2459-2474.
24. C.L. Sirisha, Y.N. Reddy, *Numerical integration of singularly perturbed delay differential equations using exponential integrating factor*, Math. Commun. **22** (2017), 251-264.
25. H. Tian, *The exponential asymptotic stability of singularly perturbed delay differential equations with a bounded lag*, J. Math. Anal. Appl. **270** (2002), 143-9.
26. S. Yuzbasi, N. Savasaneril, *Hermite polynomial approach for solving singular perturbated delay differential equations*, J. Sci. Arts. **20** (2020), 845-854.

Dany Joy recieved M.Sc from Mahatma Gandhi University, Kerala, India and currently pursuing Ph.D. at Vellore Institute of Technology, Vellore, India. His research interests include Numerical Analysis, Singular Perturbation Problems and Delay Differential Equations.

Department of Mathematics, Vellore Institute of Technology, Vellore-632014, Tamil Nadu, India.
e-mail: `danyjoy4@gmail.com`

Dr Dinesh Kumar S received M.Sc. from Anna University Chennai, India and Ph.D. from Visvesvaraya National Institute of Technology, Nagpur, India. Presently he is working as Assistant Professor in the School of Advanced Sciences, Department of Mathematics at Vellore Institute of Technology, Vellore, India. His research interests include Numerical Analysis, Singular Perturbation Problems and Delay Differential Equations.

Department of Mathematics, Vellore Institute of Technology, Vellore-632014, Tamil Nadu, India.
e-mail: `mathdinesh005@gmail.com`