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A STUDY ON DEGENERATE (p, q, h)-BERNOULLI POLYNOMIALS AND NUMBERS†

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ABSTRACT. This paper introduces a more generalized form of the degenerated q-Bernoulli polynomial, termed (p,q)-Bernoulli polynomial, and presents their properties. Various properties including symmetry were investigated, yet properties of symmetry were not identified. However, in the process, another property was discovered, and the purpose is to introduce this newly found property.

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1. Introduction

The well-known traditional Bernoulli polynomials, which have been known for a long time, are defined by the following generating function.

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!} \quad \text{(cf. [1-13])}. \tag{1.1}$$

When $x=0$, the value of the nth Bernoulli number is denoted by $B_n(0)=B_n$.

We first introduce the basic concepts and sevaral exponential functions.

Let $n, q \in \mathbb{R}$ and $q \neq 1$. Jackson defined the q-number as below:

$$[n]_q=\frac{1-q^n}{1-q}$$

and $\lim_{q\to 1}[n]_q=n$. Also we introduce the (p,q) numbers as below:

Let $n, p, q \in \mathbb{R}$, p and q are not 1, and $p \neq q$,

$$[n]_{p,q}=\frac{p^n-q^n}{p-q}$$

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and $\lim_{p\to 1, q\to 1}[n]_{p,q} = n$.

For $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$ and $[0]_q = 1$, q-binomial is defined by

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[n-r]_q![r]_q!}$$

and $\lim_{q\to 1} \begin{bmatrix} n \\ r \end{bmatrix}_q = \binom{n}{r}$.

In this paper, we use the (p,q)-binomial to be extended as belows:

For $[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q} \cdots [2]_{p,q}[1]_{p,q}$ and $[0]_{p,q} = 1$, (p,q)-binomial is defined by

$$\begin{bmatrix} n \\ r \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-r]_{p,q}![r]_{p,q}!}$$

and $\lim_{q\to 1} \begin{bmatrix} n \\ r \end{bmatrix}_{1,q} = \binom{n}{r}$.

Many mathematicians have studied various exponential functions. Let's examine how these diverse exponential functions have evolved.

e^t is well known classical exponential function. Also, since e^t is an analytic function on complex number field, e^t is capable of series expansion as belows:

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

$e_q(t)$ is a q-exponential function defiened as

$$e_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!}.$$

For real $q > 1$, the function $e_q(t)$ is an entire function of t. For $0 < q < 1$, $e_q(t)$ is regular in the disk $|z| < 1/(1-q)$. Also, the inverse of $e_q(t)$ is $e_{q^{-1}}(-t)$, i.e., $e_q(t)e_{q^{-1}}(-t) = 1$.

$e_{p,q}(t)$ is a (p,q)-exponential function defiened as

$$e_{p,q}(t) = \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}} t^n}{[n]_{p,q}!} \quad \text{and} \quad E_{p,q}(t) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} t^n}{[n]_{p,q}!}.$$

Here, $[n]_{p,q} = \frac{p^n - q^n}{p-q}$.

$(1+\lambda t)^{\frac{1}{\lambda}}$ is a degenerated exponetial function and $\lim_{\lambda\to 0}(1+\lambda t)^{\frac{1}{\lambda}} = e^t$.

More recently, Burcu Silindir and Ahmet Yantir [14] have generalized exponential function with Definition 1.1 and 1.2 in the following way.

Definition 1.1. For $0 < q < 1$, one define the generalized quantum binomial, (q, h)-analogue of $(x - x_0)^n$, as the polynomial

$$(x-x_0)^n_{q,h} = \begin{cases} 1 & if \quad n = 0, \\ \prod_{i=1}^{n} (x - (q^{i-1}x_0 + [i-1]_q h)) & if \quad n > 0, \end{cases}$$

where $x_0 \in \mathbb{R}$ (see [8],[10]).

When $q \to 1$ and $h \to 0$, the generalized quantum binomial approximates the ordinary binomial as belows:

$$\lim_{(q,h)\to(1,0)} (x - x_0)^n_{(q,h)} = (x - x_0)^n.$$

In this paper, $(x - 0)^n_{(q,h)}$ is denoted as $(x)^n_{q,h}$ for convenience.
That is, $(x - 0)^n_{(q,h)} = (x)^n_{q,h} = x(x - [1]_q h)(x - [2]_q h) \cdots (x - [n-1]_q h)$.

Definition 1.2. For $0 < q < 1$, one define the degenerated q-exponential function, denote by

$$e_{q,h}(x : t) = \sum_{n=0}^{\infty} \frac{(x)^n_{q,h}}{[n]_q!} t^n,$$

where $(0)^0_{q,h} = 1$ and $e_{q,h}(0, t) = 1$.

Using Definition 1.1 and 1.2, we have already defined the q-Bernoulli polynomials and numbers as follows, and we intend to extend these. (see [8]).

Definition 1.3. For $0 < q < 1$ and $t \in \mathbb{R}$, we define degenerate q-Bernoulli polynomials and numbers by the following generating functions

$$\frac{t}{e_{q,h}(1 : t) - 1} e_{q,h}(x : t) = \sum_{n=0}^{\infty} \beta_{n,q}(x : h) \frac{t^n}{[n]_q!}$$

and

$$\frac{t}{e_{q,h}(1 : t) - 1} = \sum_{n=0}^{\infty} \beta_{n,q}(h) \frac{t^n}{[n]_q!}.$$

When $q \to 1$ and $h = 0$ it is equal to the classical Bernoulli polynomial.

In this paper, the concept is extended more generally to define (p, q)-binomials and (p, q)-exponential function as follows.

Definition 1.4. For $0 < p \leq 1$ and $0 < q < 1$, we define (p,q)- bionomial, $(x)_{p,q,h}$ as below.

$$\begin{aligned}(x)^n_{p,q,h} &= \prod_{i=1}^{n}(p^{i-1}x - [i-1]_{p,q}h)\\ &= x(px - [1]_{p,q}h)(p^2x - [2]_{p,q}h)\cdots(p^{n-1}x - [n-1]_{p,q}h).\end{aligned}$$

From Definition 1.3, we get

$$(x)^n_{p,q,h} = (x)^n_{\frac{q}{p},\frac{h}{p}}, \quad \lim_{h\to 0}(x)^n_{p,q,h} = p^{\binom{n}{2}}x^n \quad \text{and} \quad \lim_{p\to 1}(x)^n_{p,q,h} = (x)^n_{q,h}.$$

Definition 1.5. For $0 < p \leq 1$ and $0 < q < 1$, one define the degenerated (p,q,h)-exponential function, denote by

$$e_{p,q,h}(x:t) = \sum_{n=0}^{\infty}\frac{(x)^n_{p,q,h}}{[n]_{p,q}!}t^n = \sum_{n=0}^{\infty}p^{\binom{n}{2}}(x)_{\frac{q}{p},\frac{h}{p}}\frac{t^n}{[n]_{p,q}},$$

where $(0)^0_{p,q,h} = 1$ and $e_{p,q,h}(0,t) = 1$.

Here are some important properties of the (p,q,h)-exponential to be used in this paper.

$$\begin{aligned}e_{p,q,h}(x:abt) &= \sum_{n=0}^{\infty}(x)^n_{p,q,h}\frac{a^nb^nt^n}{[n]_{p,q}!}\\
=&\sum_{n=0}^{n}x(px-h)(p^2x-[2]_{p,q}h)\cdots(p^{i-1}x-[i-1]_{p,q}h)\\
&\qquad\cdots(p^{n-1}x-[n-1]_{p,q}h)\frac{a^nb^nt^n}{[n]_{p,q}!}\\
=&\sum_{n=0}^{n}ax(pax-ah)(p^2ax-[2]_{p,q}ah)\cdots(p^{i-1}ax-[i-1]_{p,q}ah)\\
&\qquad\cdots(p^{n-1}ax-[n-1]_{p,q}ah)\frac{b^nt^n}{[n]_{p,q}!}\\
=&e_{p,q,ah}(ax:bt).\end{aligned}$$

Theorem 1.6. *Let n be a positive integers, $0 < p \leq 1, 0 < q < 1$ and $k \in \mathbb{Z}$. We have*

$$e_{p,q,h}(x:abt) = e_{p,q,ah}(ax:bt) = e_{p,q,abh}(abx:t).$$

2. A degenerate (p, q, h)-Bernoulli polynomial and it's properties

Definition 2.1. For $0 < p \leq 1$, $0 < q < 1$ and $t \in \mathbb{R}$, we define a degenerate (p, q, h)-Bernoulli polynomials and numbers by the following generating functions

$$\frac{t}{e_{p,q,h}(1:t)-1} e_{p,q,h}(x:t) = \sum_{n=0}^{\infty} \beta_{n,p,q}(x:h) \frac{t^n}{[n]_{p,q}!}$$

and

$$\frac{t}{e_{p,q,h}(1:t)-1} = \sum_{n=0}^{\infty} \beta_{n,p,q}(h) \frac{t^n}{[n]_{p,q}!}.$$

When p, $q \to 1$ and $h = 0$ it is equal to the classical Bernoulli polynomial.

By the Definition 2.1,

$$\begin{aligned} \frac{t}{e_{p,q,h}(1:t)-1} e_{p,q,h}(x:t) &= \sum_{n=0}^{\infty} \beta_{n,p,q}(h) \frac{t^n}{[n]_{p,q}!} \times \sum_{n=0}^{\infty} (x)_{p,q,h}^n \frac{t^n}{[n]_{p,q}!} \\ &= \sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \beta_{n-k,p,q}(h)(x)_{p,q,h}^k \frac{t^n}{[n]_{p,q}!}. \end{aligned} \tag{2.1}$$

By the Definition 2.1, equation (2.1) and comparing their coefficients, we can derive the following theorem.

Theorem 2.2. *Let n be a positive integer, $0 < p \leq 1$, $0 < q < 1$ and $k \in \mathbb{Z}$. We have*

$$\beta_{n,p,q}(x:h) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \beta_{n-k,p,q}(h)(x)_{p,q,h}^k.$$

By the Definition 2.1

$$\begin{aligned} \frac{t^2}{e_{p,q,h}(1:t)-1} e_{p,q,h}(x:t) =& t \sum_{n=0}^{\infty} \beta_{n,p,q}(x:h) \frac{t^n}{[n]_{p,q}!} \\ =& \sum_{n=0}^{\infty} \beta_{n,p,q}(x:h) \frac{t^{n+1}}{[n]_{p,q}!} \\ =& \sum_{n=1}^{\infty} [n]_{p,q} \beta_{n-1,p,q}(x:h) \frac{t^n}{[n]_{p,q}}. \end{aligned} \tag{2.2}$$

Furthermore, by slightly modifying the generating function introduced above, the following result is obtained.

$$
\begin{aligned}
&\frac{t^2}{e_{p,q,h}(1:t)-1}e_{p,q,h}(x:t)\\
=&\frac{t}{e_{p,q,h}(1:t)-1}e_{p,q,h}(x:t)\left(\frac{t}{e_{p,q,h}(1:t)-1}e_{p,q,h}(1:t)-\frac{t}{e_{p,q,h}(1:t)-1}\right)\\
=&\frac{t}{e_{p,q,h}(1:t)-1}e_{p,q,h}(x:t)\frac{t}{e_{p,q,h}(1:t)-1}e_{p,q,h}(1:t)\\
&\qquad-\frac{t}{e_{p,q,h}(1:t)-1}e_{p,q,h}(x:t)\frac{t}{e_{p,q,h}(1:t)-1}\\
&\qquad=\sum_{n=0}^{\infty}\beta_{n,p,q}(x:h)\frac{t^n}{[n]_{p,q}!}\cdot\sum_{n=0}^{\infty}\beta_{n,p,q}(x:h)\frac{t^n}{[n]_{p,q}!}\\
&\qquad\qquad-\sum_{n=0}^{\infty}\beta_{n,p,q}(x:h)\frac{t^n}{[n]_{p,q}!}\cdot\sum_{n=0}^{\infty}\beta_{n,p,q}(h)\frac{t^n}{[n]_{p,q}!}\\
=&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\beta_{n-k,p,q}(x:h)\beta_{k,p,q}(x:h)\right)\frac{t^n}{[n]_{p,q}!}\\
&\qquad-\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\beta_{n-k,p,q}(x:h)\beta_{k,p,q}(h)\right)\frac{t^n}{[n]_{p,q}!}\\
=&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\beta_{n-k,p,q}(x:h)\beta_{k,p,q}(x:h)\right.\\
&\qquad\left.-\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\beta_{n-k,p,q}(x:h)\beta_{k,p,q}(h)\right)\frac{t^n}{[n]_{p,q}!}\\
=&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\beta_{n-k,p,q}(x:h)\left(\beta_{k,p,q}(x:h)-\beta_{k,p,q}(h)\right)\right)\frac{t^n}{[n]_{p,q}!}. \qquad (2.3)
\end{aligned}
$$

Equation (2.2) and equation (2.3) are expressions obtained using the same generating function. Comparing their coefficients, we can derive the following theorem.

Theorem 2.3. *Let n be a positive integer, $0 < p \leq 1$, $0 < q < 1$ and $k \in \mathbb{Z}$. We have*

$$\beta_{n-1,p,q}(x:h) = \frac{1}{[n]_{p,q}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\beta_{n-k,p,q}(x:h)\left(\beta_{k,p,q}(x:h)-\beta_{k,p,q}(h)\right).$$

By Definition 2.1,

$$\begin{aligned}
&\sum_{n=0}^{\infty}\beta_{n,p,q}(x:h)\frac{t^n}{[n]_{p,q}!}=\frac{t}{e_{p,q,h}(1:t)-1}e_{p,q,h}(x:t)\\
=&\frac{1}{1-t}\frac{t}{e_{p,q,h}(1:t)-1}e_{p,q,h}(x:t)(1-t)\\
=&\frac{1}{1-t}\frac{t}{e_{p,q,h}(1:t)-1}e_{p,q,h}(x:t)-\frac{1}{1-t}\frac{t^2}{e_{p,q,h}(1:t)-1}e_{p,q,h}(x:t)\\
=&\sum_{n=0}^{\infty}t^n\cdot\sum_{n=0}^{\infty}\beta_{n,p,q}(x:h)\frac{t^n}{[n]_{p,q}!}-\sum_{n=0}^{\infty}t^{n+1}\cdot\sum_{n=0}^{\infty}\beta_{n,p,q}(x:h)\frac{t^n}{[n]_{p,q}!}\\
=&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\beta_{n-k,p,q}(x:h)\frac{t^n}{[n-k]_{p,q}!}\right)-\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\beta_{n-k,p,q}(x:h)\frac{t^{n+1}}{[n-k]_{p,q}!}\right)\\
=&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{[n]_{p,q}!}{[n-k]_{p,q}!}\beta_{n-k,p,q}(x:h)\frac{t^n}{[n]_{p,q}!}\right)\\
&\quad-\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{[n]_{p,q}!}{[n-k]_{p,q}!}\beta_{n-k,p,q}(x:h)\frac{t^{n+1}}{[n]_{p,q}!}\right)\\
=&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{[n]_{p,q}!}{[n-k]_{p,q}!}\beta_{n-k,p,q}(x:h)-\sum_{k=0}^{n-1}\frac{[n]_{p,q}!}{[n-1-k]_{p,q}!}\beta_{n-1-k,p,q}(x:h)\right)\frac{t^n}{[n]_{p,q}!}.
\end{aligned}$$

From the given equation, we can derive the following theorem.

Theorem 2.4. *Let n be a positive integer, $0<p\le 1, 0<q<1$ and $k\in\mathbb{Z}$. We have*

$$\begin{aligned}
&\beta_{n.p,q}(x:h)\\
=&\sum_{k=0}^{n}\frac{[n]_{p,q}!}{[n-k]_{p,q}!}\beta_{n-k,p,q}(x:h)-\sum_{k=0}^{n-1}\frac{[n]_{p,q}!}{[n-1-k]_{p,q}!}\beta_{n-1-k,p,q}(x:h)\\
=&\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}[k]_{p,q}!\beta_{n-k,p,q}(x:h)-\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}_{p,q}[n]_{p,q}[k]_{p,q}!\beta_{n-1-k,p,q}(x:h).
\end{aligned}$$

The following property to be discussed is one discovered by the author while investigating symmetry. Although previously sought-after symmetric properties couldn't be found, a similar result resembling symmetry was obtained, hence it is introduced.

$$(1)\quad \frac{abt^2 e_{p,q,\frac{h}{a}}(x:at)e_{p,q,\frac{h}{b}}(x:bt)}{(e_{p,q,\frac{h}{a}}(1:at)-1)(e_{p,q,\frac{h}{b}}(1:bt)-1)}$$
$$=\frac{at}{e_{p,q,\frac{h}{a}}(1:at)-1}e_{p,q,\frac{h}{a}}(x:at)\times\frac{bt}{e_{p,q,\frac{h}{b}}(1:bt)-1}e_{p,q,\frac{h}{b}}(x:bt)$$
$$=\sum_{n=0}^{\infty}\beta_{n,p,q}\left(x:\frac{h}{a}\right)\frac{(at)^n}{[n]_{p,q}}\times\sum_{n=0}^{\infty}\beta_{n,p,q}\left(x:\frac{h}{b}\right)\frac{(bt)^n}{[n]_{p,q}}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a^{n-k}b^k\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\beta_{n-k,p,q}\left(x:\frac{h}{a}\right)\beta_{k,p,q}\left(x:\frac{h}{b}\right)\right)\frac{t^n}{[n]_{p,q}!}.$$

$$(2)\quad \frac{abt^2 e_{p,q,\frac{h}{a}}(x:at)e_{p,q,\frac{h}{b}}(x:bt)}{(e_{p,q,\frac{h}{a}}(1:at)-1)(e_{p,q,\frac{h}{b}}(1:bt)-1)}$$
$$=\frac{bt}{e_{p,q,\frac{h}{a}}(1:bt)-1}e_{p,q,\frac{h}{a}}(x:at)\times\frac{at}{e_{p,q,\frac{h}{b}}(1:at)-1}e_{p,q,\frac{h}{b}}(x:bt)$$
$$=\frac{bt}{e_{p,q,\frac{h}{a}}(1:bt)-1}e_{p,q,\frac{h}{b}}(\frac{a}{b}x:bt)\times\frac{at}{e_{p,q,\frac{h}{b}}(1:at)-1}e_{p,q,\frac{h}{a}}(\frac{b}{a}x:at)$$
$$=\sum_{n=0}^{\infty}\beta_{n,p,q}\left(\frac{b}{a}x:\frac{h}{a}\right)\frac{(bt)^n}{[n]_{p,q}!}\times\sum_{n=0}^{\infty}\beta_{n,p,q}\left(\frac{a}{b}x:\frac{h}{b}\right)\frac{(at)^n}{[n]_{p,q}!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a^{k}b^{n-k}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\beta_{n-k,p,q}\left(\frac{b}{a}x:\frac{h}{a}\right)\beta_{k,p,q}\left(\frac{a}{b}x:\frac{h}{b}\right)\right)\frac{t^n}{[n]_{p,q}!}.$$

Comparing the coefficient on both sides, we get following:

$$\sum_{k=0}^{n}a^{n-k}b^k\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\beta_{n-k,p,q}\left(x:\frac{h}{a}\right)\beta_{k,p,q}\left(x:\frac{h}{b}\right)$$
$$=\sum_{k=0}^{n}a^{k}b^{n-k}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\beta_{n-k,p,q}\left(\frac{b}{a}x:\frac{h}{a}\right)\beta_{k,p,q}\left(\frac{a}{b}x:\frac{h}{b}\right).$$

From this, we can obtain the following theorem.

Theorem 2.5. *Let n be a nonnegative integer, $a,b,h\in\mathbb{R}$ and $0<p\le 1$, $0<q<1$. We have*

$$\sum_{k=0}^{n}a^{n-k}b^k\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\beta_{n-k,p,q}\left(x:\frac{h}{a}\right)\beta_{k,p,q}\left(x:\frac{h}{b}\right)$$
$$=\sum_{k=0}^{n}a^{k}b^{n-k}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\beta_{n-k,p,q}\left(\frac{b}{a}x:\frac{h}{a}\right)\beta_{k,p,q}\left(\frac{a}{b}x:\frac{h}{b}\right).$$

When we substitute abx for x in the above theorem, we obtain the following result.

$$\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q} a^{n-k}b^{k}\beta_{n-k,p,q}(abx:\frac{h}{a})\beta_{k,p,q}(abx:\frac{h}{b})$$
$$=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q} a^{n-k}b^{k}\beta_{n-k,p,q}(b^2x:\frac{h}{a})\beta_{k,p,q}(a^2x:\frac{h}{b}).$$

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