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CYCLES IN T-GRAPHS

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Abstract. We are concerned with the T-Graphs, denoted byG_T, which are graphs defined based on the Topological structure of the given set. Precisely, for a given topology T on a set X, a T-Graph $G = (V, E)$ is an undirected graph. In this graph, the vertex set V is $P(X)$ and the edge set E consists of all unordered pairs of nodes u and v in V , denoted by (u, v). The pair of nodes satisfies the condition that $u \in T$ and $u^c \cap v \in T$, where u^c is the complement of the subset u of X. The primary objective of this paper is threefold: (i) Find that the number of triangles (ii) we intend to establish that the number of rectangles and (iii) the number of Hamiltonian cycles and Eulerian closed trials in a graph G_T.

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1. Introduction

Graph theory is a branch of mathematical modelling within mathematics that aims to determine whether real-life problems can be solved. Topology, on the other hand, is an import field of mathematics used to assess whether two given objects are equivalent, particularly when one can be continuously deformed, similar to concept of a rubber sheet. Graph theory and topology have closely intertwined histories, and these two areas share numerous common problems and techniques. For instance, the famous Konigsberg problem was a classic example of geometria situs until the development of topological ideas in the 19th century. Following the development of topologic al concepts, the Konigsberg problem transitioned into an example of analysis situs. The connection between graph theory and topology has given rise to a new field known as topological graph theory. In topological graph theory, the goal is to represent a graph on a plane or sphere without edges crossing, except at the vertices where they meet.

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There are two approaches to the study of combined work of topology and graph theory. One approach is from graph theory to topology, and the other is from topology to graph theory [1]. In the work by Dongseok Kim et.al [**?**], a one-to-one correspondence is established between the topologies on a non-empty set and their corresponding underlying digraphs. In various fields, such as image analysis in [15, 16], the study of molecular structures in [13], the exploration of important geometries of finite sets in [2], and cycles, including the possibility of cycles, the number of cycles, and their applications in network topology in [8, 9, 11, 12], finite topology plays crucial role. In [13, 14, 17], the counting of number of cycles in a graphs with specific length is not easy way and also some time not possible.

In this article, we use the following notations: T denotes the arbitrary topology on any given non-empty set X. That is T is a collection of subsets of X which is closed with respect to arbitrary union and finite intersection. E_T denotes the edge set of the T- Graph and $T-$ Graph is denoted by G_T . $P(X)$ be the vertex set of every T- Graph.

The pioneering work by Nagaratnamaiah et al. [3, 4, 5, 6], has paved the way for our study, focusing on the enumeration of various type of cycles in our T-Graphs. In this article, we present key propositions for counting the enumeration of triangles and the maximum number of triangles in a T-Graph based on counting principles. The article is organized as follows: Section 1: Introduction, which introduces the concepts of finite T-Graphs. Section 2: Counting of triangles in T-Graphs . Section 3: Counting of rectangles in T-Graphs. Section 4: Discussion of cycles other than triangles and rectangles in T-Graphs. Conclusion is in final section.

2. Range of Number of Triangles in T-Graphs

This section marks the initial part of our of our contributed work, with the central theme focusing on counting the number of triangles in our defined T-Graphs on a non-empty finite set, X . The main proposition in the section are as follows:

The first proposition establishes that the number of triangles in a T − Graphs on a non- empty set X falls within the range of zero to $2^3\binom{|P(X)|}{3}$ (including bounds).

The second proposition demonstrates that a triangle with nodes u, v and w exists in T − Graph if at least two of these nodes among u, v and w are within topology T , and their complements intersection with the remaining node is also within T.

Proposition 2.1. *The Range value of number of triangles in a T-graph G_T lies between zero to including bounds.*

Proof. As the fact of Topology, Trivial topology is the smallest among all topologies defined on a given set X and Discrete topology is the largest among all topologies defined on X.

Case 1: If T is a trivial topology on X, then we observed that every node v in $P(X)$ other than the vertex X has an edge with X but there are no edge between two nodes u and v , where u and v are both different from X. Also we observed that the nodes $\emptyset$ and X have two edges between them. Thus there is no cycle with three vertices in T-graph G_T in this case. $2^3\binom{|P(X)|}{3}$.

Case 2: If T is a discrete topology on X, then we observed that every pair of nodes u and v have two edges between them. According to the permutation counting principal, the number of triangles among any three distinct nodes u, v, w in this graph is equal to $2.2.2 = 2^3$. Hence the number of triangles in this case is $2^3\binom{|P(X)|}{3}$.

Case 3: If T is neither trivial nor discrete topology on X, then we observed that there exist at least one pair of subsets u and v of X as the nodes of T- graph G_T other than $\emptyset$ and X such that u is in topology T and v is not in topology T . Therefore, the nodes $\emptyset$ and u have two edges, the nodes $\emptyset$ and X have two edges, the nodes u and X have at least one edge and at most two, and the nodes v and $\emptyset$ have no any edge between them. Therefore, by using permutation counting principal, the number of triangles with vertices$\emptyset$,X and u is $2.2.1 = 2^2$ or $2.2.2 = 2^3$. In the similar manner the number of triangles with vertices $\emptyset$,X,v is $2.1.0 = 0$. Hence the number of triangles in this case is less than $2^3\binom{|P(X)|}{3}$. □

Proposition 2.2. *In T-graph G_T, there is at least two triangles among three distinct nodes u, v and w if u and v lie in topology T such that $u^c \cap w$ and $v^c \cap w$ both are in topology T and T is closed with respect to complement.*

Proof. Since the nodes u and v are in topology T, and $u^c \cap w$ and $v^c \cap w$ in T, then $(u, w) \in E_T$ and $(v, w) \in E_T$. u^c and v^c in T because T is closed with respect to complement. Thus $(u, v) \in E_T$ and $(v, u) \in E_T$. That is the nodes u and v have two edges, the nodes u and w have one edge and the nodes v and w have only one edge. Thus the number of triangles with nodes u, v and w is two. □

Remark 2.1. In any T-Graph, three nodes u, v and w have atmost eight triangles. By our definition of T-Graph every pair of nodes have at most two edges, so the number of triangles among three distinct nodes $u, v$and w is eight.

3. Range of Number of Rectangles in T-Graphs

This section count the number of rectangles among four nodes of T-Graphs as well as total number of rectangles in a graph with the variation of topology on the giving non-empty finite set.

Proposition 3.1. *The number of rectangles in a graph* (G, E_T) *lies between zero to* $2^4\binom{|P(X)|}{4}$ *(including bounds).*

Proof. **Case 1:** T be a trivial topology. In this case there are no loops of length more than two. Therefore, the number of rectangles whose length four is zero.
Case 2: T is a discrete topology. Every pair of nodes in T- Graph have two edges. Thus the number of rectangles whose length four among any four nodes of the graph is $2.2.2.2 = 2^4$ and hence the total number of rectangles in T- Graph is $2^4\binom{|P(X)|}{4}$.
Case 3: T is neither trivial nor discrete topology. If $|T| = 3$, In this case there are no loops of length more than three. Then there is no rectangle of size four. If $|T| > 3$, Therefore the topology T has atleast four nodes. Since T is neither trivial nor discrete topology and $|T| > 3$, so there exist atleast two node u and v other than $\emptyset$ and X which belongs to topology T and there exist atleast one node w which does not belongs to the topology T. Thus the nodes $\emptyset$ and u have two edges $\emptyset$ and v have two edges and the two nodes u and X have atleast one edge and the nodes v and X have atleast one edge. It follows that the number of rectangles among these four nodes of the graph is $2.2.1.1 = 2^2$. The total number of rectangles is $2^2\binom{|P(X)|}{4}$. ☐

Proposition 3.2. *In graph* (G, E_T)*, the nodes* u, v, w *and* y *are forms a rectangle if atleast two of them is a member of* T *such that their complement intersection with other two are also members of* T *and* T *is closed with respect to complement.*

Proof. Suppose u and v in topology T and w and y not in T .
Since $u, v, u^{\complement} \cap w, u^{\complement} \cap y, v^{\complement} \cap w$ and $v^{\complement} \cap y \in T$, and so $(u, w) \in E_T, (u, y) \in E_T, (v, w) \in E_T$ *and* $(v, y) \in E_T$. Thus there is a rectangle among the nodes u, v, w and y ☐

Remark 3.1. In graph (G, E_T), the nodes u, v, w and y have at most sixteen rectangles.

Proof. Case 1: u, v, w and $y \notin T$. The maximum number of edges between any two nodes in T-Graphs are two. Thus the maximum number of rectangles among four nodes is sixteen. ☐

4. Range of Number of Cycles in T-Graphs

This section shows that if the Hamiltonian and/or Eulerian cycles are exist in T − Graph then count the number of that cycles.

Proposition 4.1. *Let* T *be a discrete topology in a graph* (G, E_T) *then Hamiltonian and the number of Hamiltonian cycles in the graph* (G, E_T) *is*

$$2^{2^{|X|}}[\frac{(2^{|X|}(2^{|X|}+1))}{2}]$$

.

Proof. Since T be a discrete topology in a graph ,so the vertex set of the graph (G, E_T) is T. There is a cycle passing through each node of the graph because every pair of nodes have two edges. Hence the graph (G, E_T) Hamiltonian . Next count the number of Hamilton cycles in T-Graph is $2^{2^{|X|}}$. If the cardinality of X is n , then the size of the graph is $2^|X|$. Suppose index this vertex set $T = \{v_1, v_2, ..., v_{2^{|X|}}\}$. Consider any node v_1 (say). There are two edges with v_1 to other nodes of the graph. Next consider that node is v_2 (say). Proceed like this ,we get $2^{2^{|X|}}$ number of cycles start with the node v_1. First node is chosen in $2^{|X|}$ ways and next node is $2^{|X|} - 1$ ways and so on and final node is chosen in one way ,therefore the number of ways to select each node in these $2^{2^{|X|}}$ number of cycles is

$$\sum_{n=1}^{2^{|X|}} n = [\frac{(2^{|X|}(2^{|X|}+1))}{2}]$$

Thus the total number of Hamiltonian cycles in the graph (G, E_T) is

$$2^{2^{|X|}}[\frac{(2^{|X|}(2^{|X|}+1))}{2}]$$

□

Proposition 4.2. *The number of Hamiltonian cycles in a graph* (G, E_T) *is atmost* $2^{2^{|X|}}[\frac{(2^{|X|})(2^{|X|}+1)}{2}]$.

Proof. If T be a discrete topology . By proposition 4.1, there are

$$2^{2^{|X|}}[\frac{(2^{|X|})(2^{|X|}+1)}{2}]$$

in Hamiltonian. If T be a non discrete topology . So there exist a node u such that $u \notin T$, then $\emptyset^{\complement} \cap u = u \notin T$. Therefore there is no any edge between the nodes $\emptyset$ and u and hence there is no Hamiltonian cycles with $\emptyset$ is a predecessor or succession of the node u. It follows that the number of Hamiltonian cycles in the graph (G, E_T) is less than

$$2^{2^{|X|}}[\frac{(2^{|X|})(2^{|X|}+1)}{2}]$$

□

Proposition 4.3. *Let* T *be a discrete topology in a graph* (G, E_T) *then Eulerian and the number of Eulerian cycles in the graph* (G, E_T) *is*

$$2^{2^{|X|}}[\frac{(2^{|X|}(2^{|X|}+1)}{2}]$$

.

Proof. Since T be a discrete topology in a graph, so the vertex set of the graph (G, E_T) is T. Because every pair of nodes have two edges, so start from any node and move to any another node through any one of edge between them and proceeds like this up to reach the staring node and then return in the same order with another edge which is different from previously traversed one. Thus there is a Eulerian cycle passing through each edge of the graph. Hence the graph (G, E_T) then Eulerain. Now count the number of Eulerian cycles in graph (G, E_T). Suppose the cardinality of the set X is n, the order of the graph is $2^{|X|}$ and size of the graph is $2^{2^{|X|}}$. That is order of the vertex set is $2^{|X|}$ and the order of the edge set is $2^{2^{|X|}}$. Suppose index this edge set $E_T = \{v_1, v_2, \dots, v_(2^{2^{|X|}}\}$. Consider any edge incident on the node v_1 (say). There are two edges with v_1 to other nodes of the graph. Next consider another incident node of the edges incident from is v_2 (say). Proceed like this , up to the start node through another edge which are not taken in previously exactly in reverse order then we get $2^{2^{|X|}}$ number of Eulerian cycles start with the node v_1 First node is chosen in $2^{|X|}$ ways, next node $2^{|X|} - 1$ and so on and final node is chosen in one way,therefore the number of ways to select each node in these $2^{2^{|X|}}$ number of cycles is

$$\sum_{n=1}^{2^{|X|}} n = [\frac{(2^{|X|}(2^{|X|}+1))}{2}]$$

. □

Proposition 4.4. *The number of Eulerian cycles in a graph (G, E_T) is atmost* $2^{2^{|X|}}[\frac{(2^{|X|}(2^{|X|}+1))}{2}]$.

Proof. If T be a discrete topology , then there are $2^{2^{|X|}}[\frac{(2^{|X|}(2^{|X|}+1))}{2}]$ in Eulerian. If T be a non discrete topology, so there exist a node u such that $u \notin T$, then $\emptyset^{\mathsf{C}} \cap u = u \notin T$. There is no edge between the nodes $\emptyset$ and u. There is no Eulerain cycles with $\emptyset$ is a predecessor or succession of the node u. The number of Eulerain cycles in the graph (G, E_T) is less than $2^{2^{|X|}}[\frac{(2^{|X|}(2^{|X|}+1))}{2}]$. □

5. Conclusion

In this paper we have worked on counting of various type of cycles of length more than two on a new family of graphs , namely T-Graphs. We have proved that the range of triangles lies between zero to $2^3|P(X)|_{C_3}$ and attains its maximum value when the topology is discrete topology. Extended the work to prove the range of rectangles lie between $2^4|P(X)|_{C_4}$ and finally counted the range of number of Hamiltonian cycles is between zero to $2^{2^{|X|}}[\frac{(2^{|X|}(2^{|X|}+1))}{2}]$ and the range of number of Eulerian cycles lies between zero to $2^{2^{|X|}}[\frac{(2^{|X|}(2^{|X|}+1))}{2}]$.

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