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A REMARK ON p-ADIC DEDEKIND SUMS ASSOCIATED WITH THE QUASI-PERIODIC EULER FUNCTIONS[†]

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Abstract. In this note, we investigate new types of p-adic Dedekind sums. They interpolate the Apostol-Dedekind sum associated with quasi-periodic Euler functions in different ways. We also obtain the integral representation and a reciprocity relation, respectively.

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1. History of the subject

To recall the definition for the classical Dedekind sums, we first introduce the following symbol

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & (x \neq \text{integer}) \\ 0 & (x \text{ integer}). \end{cases} \tag{1}$$

Here for $x \in \mathbb{R}$, $[x]$ denotes the greatest integer not exceeding x and $\{x\}$ denotes the fractional part of real number x, thus

$$\{x\} = x - [x]. \tag{2}$$

If h and k are coprime integers with $k > 0$, then the classical Dedekind sum $s(h,k)$ is defined by

$$s(h,k) = \sum_{\mu=0}^{k-1} \left(\left(\frac{h\mu}{k} \right) \right) \left(\left(\frac{\mu}{k} \right) \right). \tag{3}$$

In 1892, Dedekind [10] introduced this sum, which is connected with the transformation formula for the Dedekind η-function. And he also deduced the following

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well-known reciprocity formula

$$12hk\{s(h,k)+s(k,h)\}=h^2-3hk+k^2+1. \tag{4}$$

In the later years, several mathematicians generalized $s(h,k)$ and showed that their generalizations also satisfy the reciprocity formulas (see [1, 2, 3, 6, 7, 19, 17, 21, 24, 26, 29, 31, 32] and the references therein). Among these, Apostol's generalized Dedekind sum is given by

$$s_m(h,k)=\sum_{\mu=0}^{k-1}\overline{B}_m\left(\frac{h\mu}{k}\right)\overline{B}_1\left(\frac{\mu}{k}\right), \tag{5}$$

where $\overline{B}_m(x)$ is the m-th Bernoulli function defined by

$$\overline{B}_m(x)=B_m(\{x\}) \quad \text{for } m>1, \qquad \overline{B}_1(x)=((x)). \tag{6}$$

Here $B_m(x)$ denotes the m-th Bernoulli polynomial which are defined as the coefficient of $\frac{t^m}{m!}$ in the generating function

$$\frac{te^{xt}}{e^t-1}=\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!}, \tag{7}$$

where $|t|<2\pi$. More than 70 years ago, Apostol [1, Theorem 1] proved a reciprocity formula for (5).

As an companion of the above periodic Bernoulli function $\overline{B}_m(x)$, the m-th quasi-periodic Euler function $\overline{E}_m(x), m\geq 0$, is defined by (see [8, 19])

$$\overline{E}_m(x)=E_m(x) \quad \text{for } 0\leq x<1, \quad \overline{E}_m(x+1)=-\overline{E}_m(x). \tag{8}$$

Here $E_m(x)$ denotes the m-th Euler polynomials which is defined by the generating function

$$\frac{2e^{xt}}{e^t+1}=\sum_{m=0}^{\infty}E_m(x)\frac{t^m}{m!}, \tag{9}$$

where $|t|<\pi$. See [3, 12, 11, 18, 19, 27] for an account of further properties of the Euler polynomials and numbers. For $x\in\mathbb{R}$ and $r\in\mathbb{Z}$, we have

$$\overline{E}_m(x)=(-1)^{[x]}E_m(\{x\}), \quad \overline{E}_m(x+r)=(-1)^r\overline{E}_m(x). \tag{10}$$

For further properties of the quasi-periodic Euler functions, we refer to [3, 8, 19].

The Apostol-Dedekind sum $T_m(h,k)$ associated with quasi-periodic Euler functions is defined by (see [11, p. 56, (1.9)] and [19, p. 269, (1.10)])

$$T_m(h,k)=\sum_{\mu=0}^{k-1}(-1)^{\mu}\overline{E}_m\left(\frac{h\mu}{k}\right)\overline{E}_1\left(\frac{\mu}{k}\right), \tag{11}$$

which is an analogue of the generalized Dedekind sums (5) for quasi-periodic Euler functions, where h and k are positive integers. Notice that, as indicated by Carlitz in [8, p.661, 2nd paragraph], the Bernoulli function are periodic, but the Euler functions are just quasi-periodic (also comparing with Eqs. (6) and

(10) above), thus the signs $(-1)^{\mu}$ in the definition of the Apostol-Dedekind sum $T_m(h,k)$ with quasi-periodic Euler functions (11) are necessary.

The Apostol-Dedekind sum $T_m(h,k)$ associated with quasi-periodic Euler functions has been investigated by Hu, Kim and Kim in [11], and they also obtained its reciprocity formula by applying the Euler-Boole summation formula (see [5, p. 684, Lemma 2]). T. Kim [15] and Simesek [30] considered another type of Dedekind sum associated with Euler numbers and polynomials, which is named the Dedekind type DC-sum. In particular, Simsek [30] has established many interesting properties for this sum, including its analytic properties and trigonometric representations of this sum, its relations between Clausen functions, Polylogarithm function, Hurwitz zeta function, generalized Lambert series (G-series), Hardy-Berndt sums. Furthermore, he also gave many applications related to these sums and functions.

In this note, we will consider the p-adic analogue of Dedekind sums. This topic has been investigated by several authors before. In 1985 and 1988, Rosen and Snyder [26] and Snyder [31] gave a p-adic interpolation of Apostol's Dedekind sum. They commenced this sums by interpolating certain Bernoulli functions. Then Kudo [21] discussed the p-adic Dedekind sums in a different way from Rosen and Synder. The p-adic Dedekind sums associated with Euler numbers and polynomials have been studied by T. Kim [16] and Simsek [29] respectively.

The previous definitions of p-adic Dedekind sums are all based on variations of the p-adic Hurwitz zeta functions $H_p(s;a,k)$ given by Washington in [34], and are all restricted to $s \in \mathbb{Z}_p$. Recently, by applying Cohen [9]'s and Tangedal and Young [33]'s definitions of p-adic Hurwitz zeta functions for $s \in \mathbb{C}_p$, Hu and Kim [12] extended the definition domain of the p-adic Dedekind sums from $s \in \mathbb{Z}_p$ to $s \in \mathbb{C}_p$. They considered p-adic analogue of Carlitz's higher order Dedekind sums associated with both periodic Bernoulli and quasi-periodic Euler functions, and obtained the reciprocity relations for the special values of these p-adic Dedekind sums.

This note will introduce three new types of p-adic Dedekind sums associated with the quasi-periodic Euler functions (see Definitions 3.1, 3.4 and 3.7), they interpolate the above Apostol-Dedekind sum $T_m(h,k)$ associated with quasi-periodic Euler functions in different ways (see Theorems 3.3, 3.6 and 3.8). We prove the integral representation and a reciprocity relation, respectively (see Proposition 3.2 and Theorem 3.8). Our main tool is the fermionic p-adic integral which will be recalled in the next section.

2. The fermionic p-adic integral

In this section, we shall recall the definition of the fermionic p-adic integral for a p-adic function f and the associated p-adic Hurwitz-Washington functions.

Let p be an odd prime number. We denote by $\mathbb{Z}_p, \mathbb{Q}_p$, and $\mathbb{C}_p$ the ring of p-adic integers, the field of p-adic numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{Z}_p^{\times} = \mathbb{Z}_p \backslash p\mathbb{Z}_p$ and let $|\cdot|_p$ be the p-adic valuation normalized so that $|p|_p = p^{-1}$. For each $a \in \mathbb{Z}_p$ with $p \nmid a$, a can be uniquely written in the form

$$\langle a\rangle = \omega^{-1}(a)a, \tag{12}$$

where ω is the Teichmüller character. If $x \in \mathbb{Z}_p$, let

$$\binom{x}{j} = \frac{1}{j!}x(x-1)\cdots(x-j+1), \quad j>0; \quad \binom{x}{0} = 1. \tag{13}$$

Of cause $\binom{x}{j}$ simply reduces to a binomial coefficient if x is a nonnegative integer.

We say that $f : \mathbb{Z}_p \to \mathbb{C}_p$ is uniformly differentiable function (or, equivalently, strictly differentiable function) at a point $t \in \mathbb{Z}_p$, and we write $f \in UD(\mathbb{Z}_p)$, if the difference quotients $\Phi_f : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{C}_p$ such that

$$\Phi_f(x,y) = \frac{f(x)-f(y)}{x-y} \tag{14}$$

have a limit $f'(t)$ as $(x,y) \to (t,t), x \neq y$ (see [25, p. 218]).

We also recall the μ_{-1}-measure

$$\mu_{-1}(t+p^N\mathbb{Z}_p) = (-1)^t \tag{15}$$

for $t \in \mathbb{Z}_p$, which was independently found by Katz [13, p. 486] (in Katz's notation, the $\mu^{(2)}$-measure), Shiratani and Yamamoto [28], Osipov [23], Lang [22] (in Lang's notation, the $E_{1,2}$-measure), T. Kim [14] from very different viewpoints. Obviously, in contrast with the Haar distribution, the μ_{-1}-measure is bounded under the p-adic valuation, so it can be applied to integrate the continuous functions on $\mathbb{Z}_p$ (see [20, p. 39, Theorem 6]). The p-adic integral $I_{-1}(f)$ on $\mathbb{Z}_p$, in the fermionic sense, is defined by

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(t)d\mu_{-1}(t) = \lim_{N\to\infty}\sum_{t=0}^{p^N-1} f(t)(-1)^t, \tag{16}$$

where $f \in UD(\mathbb{Z}_p)$. It has also been defined independently by Shiratani and Yamamoto [28] in order to interpolate the Euler numbers p-adically. Osipov [23] gave a new proof of the existence of the Kubota-Leopoldt p-adic zeta function by using the integral representation

$$I_\varepsilon(f) = \int_{\mathbb{Z}_p} f(t)d\mu_\varepsilon(t) = \lim_{N\to\infty}\sum_{t=0}^{p^{lN}-1} f(t)\varepsilon^t, \tag{17}$$

where $f \in UD(\mathbb{Z}_p), \varepsilon^k = 1, \varepsilon \neq 1, (k,p) = 1$, and $k \mid (p^l - 1)$. Note that when $k = 2$, $I_\varepsilon(f)$ is the fermionic p-adic integral $I_{-1}(f)$ on $\mathbb{Z}_p$.

The p-adic Hurwitz-Washington function $H_p(s;a,N)$ is defined by (see [26, p. 24])

$$H_p(s;a,N) = \omega^{-1}(a)\langle a\rangle^s \sum_{j=0}^{\infty}\binom{s}{j}\left(\frac{N}{a}\right)^j E_j(0) \tag{18}$$

for $a \in \mathbb{Z}_p^\times$, $s \in \mathbb{Z}_p$, $\binom{s}{j} = \frac{s(s-1)\cdots(s-j+1)}{j!}$, and $p \mid N$. Since

$$\left|\binom{s}{j}\right|_p \leq 1, \quad \left|\frac{N}{a}\right|_p < 1,$$

and

$$|E_j(0)|_p \leq 1$$

for all $j \geq 0$ (by $2^j E_j(0) \in \mathbb{Z}$), the functions $H_p(s; a, N)$ converge and define a p-adic continuous function for s on $\mathbb{Z}_p$ (see [26, p. 24] and [34, Theorem 5.9]).

As in [26, Section 1], let $s = m$, where m is a positive integer, for $a \in \mathbb{Z}_p^\times$ and $p \mid N$, we have

$$\begin{aligned} H_p(m; a, N) &= \omega^{-1}(a)\langle a\rangle^m \sum_{j=0}^{m} \binom{m}{j} \left(\frac{N}{a}\right)^j E_j(0) \\ &= \omega^{-1}(a) \left(\frac{\langle a\rangle}{a}\right)^m N^m \sum_{j=0}^{m} \binom{m}{j} \left(\frac{a}{N}\right)^{m-j} E_j(0) \\ &= \omega^{-m-1}(a) N^m E_m\left(\frac{a}{N}\right). \end{aligned} \tag{19}$$

In particular, if $m + 1 \equiv 0 \pmod{p-1}$, then

$$H_p(m; a, N) = N^m E_m\left(\frac{a}{N}\right), \tag{20}$$

where a and N are positive integers such that $(p, a) = 1$ and $p \mid N$. Therefore, $H_p(s; a, N)$ is a p-adic continuous extension of $N^m E_m\left(\frac{a}{N}\right)$.

From (19), we have the following integral representation.

Lemma 2.1. *For $p \mid N$ and $p \nmid a$, we have*

$$H_p(s; a, N) = \omega^{-1}(a)\langle a\rangle^s \int_{\mathbb{Z}_p} \left(1 + \frac{Nt}{a}\right)^s d\mu_{-1}(t)$$

and

$$E_m\left(\frac{a}{N}\right) = \int_{\mathbb{Z}_p} \left(\frac{a}{N} + t\right)^m d\mu_{-1}(t),$$

where m is a positive integer and $N \neq 0$.

Proof. Let $s \in \mathbb{Z}_p$. Then (see [25, p. 266] and [31, p. 295])

$$\left(1 + \frac{Nt}{a}\right)^s = \sum_{j=0}^{\infty} \binom{s}{j} \left(\frac{N}{a}\right)^j t^j,$$

since $t \in \mathbb{Z}_p$ and $\frac{N}{a} \in p\mathbb{Z}_p$. From (16), we deduce easily that (see [14])

$$\int_{\mathbb{Z}_p} t^j d\mu_{-1}(t) = E_j(0),$$

where j is a nonnegative integer. Combining the above identity with (19), we get the desired result. $\square$

3. p-adic interpolations of Apostol-Dedekind type sums

Set

$$t_m(h,k) := \sum_{\mu=1}^{k-1} (-1)^{\mu+\left[\frac{h\mu}{k}\right]} \frac{\mu}{k} \overline{E}_m\left(\frac{h\mu}{k}\right). \tag{21}$$

The p-adic function $H_p(s;a,N)$ allows us to interpolate $k^m t_m(h,k)$, where $p \mid k$ and $(h\mu, p) = 1$ for each $\mu = 1, 2, \ldots, k-1$:

$$k^m t_m(h,k) = \sum_{\mu=1}^{k-1} (-1)^{\mu} \frac{\mu}{k} H_p(m; (h\mu)_k, k), \tag{22}$$

where $m+1 \equiv 0 \pmod{p-1}$ (see (20)).

Here $(\alpha)_k$ denotes the integer $x \in [0,k)$ such that $\alpha \equiv x \pmod{k}$.

Inspired by (11) and an argument similar with (22), in the following, for $p \mid k$ but $p \nmid h$, we define a new type of p-adic Dedekind sum which is associated with the quasi-periodic Euler functions.

Definition 3.1. Let h and k be integers such that $k > 0, p \mid k$ but $p \nmid h$. Then

$$T_p(s;h,k) = \sum_{\substack{\mu=0 \\ p\nmid\mu}}^{k-1} (-1)^{\mu+\left[\frac{h\mu}{k}\right]} \overline{E}_1\left(\frac{\mu}{k}\right) H_p(s; (h\mu)_k, k)$$

for all $s \in \mathbb{Z}_p$, where $(\alpha)_k$ denotes the integer $x \in [0,k)$ such that $\alpha \equiv x \pmod{k}$.

Using Lemma 2.1 and Definition 3.1, we have its integral representation.

Proposition 3.2 (Integral representation). *Let h, k be integers such that $k > 0, p \mid k$ but $p \nmid h$. Then we have*

$$T_p(s;h,k) = \sum_{\substack{\mu=0 \\ p\nmid\mu}}^{k-1} (-1)^{\mu+\left[\frac{h\mu}{k}\right]} \overline{E}_1\left(\frac{\mu}{k}\right) \frac{\langle (h\mu)_k \rangle^s}{\omega(h\mu)} \int_{\mathbb{Z}_p} \left(1 + \frac{kt}{(h\mu)_k}\right)^s d\mu_{-1}(t)$$

for all $s \in \mathbb{Z}_p$.

Theorem 3.3 (Interpolation). *For any integers $m, h,$ and k such that $m \geq 0, k > 0, p \mid k$ but $p \nmid h$, we have*

$$T_p(m;h,k) = k^m \sum_{\substack{\mu=0 \\ p\nmid\mu}}^{k-1} (-1)^{\mu} \overline{E}_1\left(\frac{\mu}{k}\right) \omega^{-m-1}(h\mu) \overline{E}_m\left(\frac{h\mu}{k}\right).$$

Moreover, if $m+1 \equiv 0 \pmod{p-1}$, then

$$T_p(m;h,k) = k^m T_m(h,k) - p^m \left(\frac{k}{p}\right)^m T_m\left(h, \frac{k}{p}\right).$$

Proof. The proof goes along the same line as in [31, p. 295, Proposition 2]. Note that by (8) and (10), we have

$$\begin{aligned}\overline{E}_m\left(\frac{h\mu}{k}\right) &= \overline{E}_m\left(\frac{(h\mu)_k}{k}+\left[\frac{h\mu}{k}\right]\right)\\ &= (-1)^{\left[\frac{h\mu}{k}\right]}\overline{E}_m\left(\frac{(h\mu)_k}{k}\right)\\ &= (-1)^{\left[\frac{h\mu}{k}\right]}E_m\left(\frac{(h\mu)_k}{k}\right),\end{aligned}\tag{23}$$

since

$$\frac{h\mu}{k}=\frac{(h\mu)_k}{k}+\left[\frac{h\mu}{k}\right] \quad \text{with} \quad 0\le \frac{(h\mu)_k}{k}<1.$$

The first formula follows from (19), (23) and Definition 3.1 by observing that $\omega^{-m-1}((h\mu)_k)=\omega^{-m-1}(h\mu)$ for $p \mid k$ and ω^{-m} has period dividing by p.

To see the second formula, if $m+1\equiv 0 \pmod{p-1}$, then from the first part and (11), we get

$$\begin{aligned}T_p(m;h,k) &= k^m\sum_{\mu=0}^{k-1}(-1)^{\mu}\overline{E}_1\left(\frac{\mu}{k}\right)\overline{E}_m\left(\frac{h\mu}{k}\right)\\ &\quad - k^m\sum_{\substack{\mu=0\\ p\mid\mu}}^{k-1}(-1)^{\mu}\overline{E}_1\left(\frac{\mu}{k}\right)\overline{E}_m\left(\frac{h\mu}{k}\right)\\ &= k^m T_m(h,k) - k^m\sum_{\mu=0}^{\frac{k}{p}-1}(-1)^{p\mu}\overline{E}_1\left(\frac{\mu}{\frac{k}{p}}\right)\overline{E}_m\left(\frac{h\mu}{\frac{k}{p}}\right)\\ &= k^m T_m(h,k) - k^m T_m\left(h,\frac{k}{p}\right),\end{aligned}\tag{24}$$

since p is an odd prime. This completes the proof. $\square$

In order to extent the Definition 3.1 to the case of $p\nmid hk$, as the above situation, we now consider the problem on interpolating the term $N^mE_m\left(\frac{a}{N}\right)$ p-adically, where a and N are positive integers such that $p \nmid aN$. We try to reduce this problem to the above case (where $p \mid N$) (see (20)). Notice that, by a Raabe type theorem (see [3, p. 355, (1.2.13)]), we have (see [4, p. 335, (2.8)] and [8, p. 661, (1.3)])

$$\begin{aligned}N^mE_m\left(\frac{a}{N}\right) &= N^mp^m\sum_{j=0}^{p-1}(-1)^jE_m\left(\frac{\frac{a}{N}+j}{p}\right)\\ &= \sum_{j=0}^{p-1}(-1)^j(pN)^mE_m\left(\frac{a+jN}{pN}\right)\end{aligned}\tag{25}$$

for $m \in \mathbb{N}_0$. If $(p, N) = 1$ and a is an arbitrary integer, then the set

$$\{0N + a, 1N + a, 2N + a, \ldots, (p-1)N + a\}$$

form a complete residue system modulo p. So, among $j = 0, 1, 2, \ldots, p-1$, we have just one term satisfying $a + jN \equiv 0 \pmod p$, where $p \nmid aN$ and $(a, N) = 1$. So due to the definition (18), each term in the last sum of (25) can be interpolated p-adically except for one term which satisfying $a + jN \equiv 0 \pmod p$, which is $j = (N^{-1}(p-a))_p$. Thus the exceptional term will be written as

$$(-1)^{(N^{-1}(p-a))_p}(pN)^m E_m\left(\frac{a + (N^{-1}(p-a))_p N}{pN}\right),$$

and by subtracting it from (25), we may possible to interpolate

$$N^m E_m\left(\frac{a}{N}\right) - (-1)^{(N^{-1}(p-a))_p}(pN)^m E_m\left(\frac{a + (N^{-1}(p-a))_p N}{pN}\right) \tag{26}$$

p-adically. Therefore if we define

$$H_p(s; a, N) = \sum_{\substack{j=0 \\ a+jN \not\equiv 0 \ (\mathrm{mod}\ p)}}^{p-1} (-1)^{j+\left[\frac{a+jN}{pN}\right]} H_p(s; (a+jN)_{pN}, pN), \tag{27}$$

then by (20), the definition of the quasi-periodic Euler function $\overline{E}_m(x)$ (8) and (25), we see that

$$\begin{aligned}
H_p(m; a, N) &= \sum_{\substack{j=0 \\ a+jN \not\equiv 0 \ (\mathrm{mod}\ p)}}^{p-1} (-1)^{j+\left[\frac{a+jN}{pN}\right]} H_p(m; (a+jN)_{pN}, pN) \\
&= \sum_{\substack{j=0 \\ a+jN \not\equiv 0 \ (\mathrm{mod}\ p)}}^{p-1} (-1)^{j+\left[\frac{a+jN}{pN}\right]} (pN)^m E_m\left(\frac{(a+jN)_{pN}}{pN}\right) \\
&= \sum_{\substack{j=0 \\ a+jN \not\equiv 0 \ (\mathrm{mod}\ p)}}^{p-1} (-1)^{j+\left[\frac{a+jN}{pN}\right]} (pN)^m \overline{E}_m\left(\frac{(a+jN)_{pN}}{pN}\right) \\
&= \sum_{\substack{j=0 \\ a+jN \not\equiv 0 \ (\mathrm{mod}\ p)}}^{p-1} (-1)^{j} (pN)^m \overline{E}_m\left(\frac{a+jN}{pN}\right) \\
&= \sum_{\substack{j=0 \\ a+jN \not\equiv 0 \ (\mathrm{mod}\ p)}}^{p-1} (-1)^{j} (pN)^m E_m\left(\frac{a+jN}{pN}\right) \\
&= N^m E_m\left(\frac{a}{N}\right) - (-1)^{(N^{-1}(p-a))_p}(pN)^m E_m\left(\frac{a + (N^{-1}(p-a))_p N}{pN}\right)
\end{aligned} \tag{28}$$

for any positive integer m such that $m+1 \equiv 0 \pmod{p-1}$. Therefore we get

$$H_p(m;a,N) = N^m E_m\left(\frac{a}{N}\right) - (-1)^{(N^{-1}(p-a))_p}(pN)^m \\ \times E_m\left(\frac{a+(N^{-1}(p-a))_p N}{pN}\right).$$

Inspired by (22) and (27), for $p \nmid hk$, we define another type of p-adic Dedekind sums $T_p(s;h,k)$ associated with the quasi-periodic Euler functions.

Definition 3.4. Let h,k be integers such that $k>0$ and $p \nmid hk$. Then

$$T_p(s;h,k) = \sum_{\mu=0}^{k-1}(-1)^{\mu}\overline{E}_1\left(\frac{\mu}{k}\right)\sum_{\substack{j=0 \\ p\nmid(h\mu+kj)}}^{p-1}(-1)^{j+\left[\frac{h\mu+kj}{pk}\right]} \\ \times H_p(s,(h\mu+kj)_{pk},pk)$$

for all $s \in \mathbb{Z}_p$.

Lemma 3.5 ([11, p. 58, Lemma 2.3]). *For odd positive integers a and b with $(a,b)=1$, we have*

$$\sum_{j=0}^{b-1}(-1)^j\overline{E}_m\left(\frac{x+aj}{b}\right) = b^{-m}\overline{E}_m(x)$$

for $m \geq 0$ and arbitrary real numbers x.

Theorem 3.6 (Interpolation). *For any integers m,h, and k such that $m \geq 0, k>0$ and $p \nmid hk$, we have*

$$T_p(m;h,k) = \sum_{\mu=0}^{k-1}(-1)^{\mu}\overline{E}_1\left(\frac{\mu}{k}\right)\sum_{\substack{j=0 \\ p\nmid(h\mu+kj)}}^{p-1}(-1)^j\frac{(pk)^m}{\omega^{m+1}(h\mu+kj)} \\ \times \overline{E}_m\left(\frac{h\mu+kj}{pk}\right).$$

Moreover, if $m+1 \equiv 0 \pmod{p-1}$, then

$$T_p(m;h,k) = k^m(T_m(h,k) - p^m T_m^*((p^{-1}h)_k,k)),$$

where $(p^{-1}h)_k$ denotes the integer $x \in [0,k)$ such that $px \equiv h \pmod{k}$ and

$$T_m^*((p^{-1}h)_k,k) := \sum_{\mu=0}^{k-1}(-1)^{\mu+(k^{-1}(p-h\mu))_p}\overline{E}_1\left(\frac{\mu}{k}\right)\overline{E}_m\left(\frac{(p^{-1}h)_k\mu}{k}\right).$$

Proof. The first formula follows from (23) and Definition 3.4 as in the proof of Theorem 3.3.

To see the second formula, if $m+1 \equiv 0 \pmod{p-1}$, then from Lemma 3.5 we get

$$\begin{aligned}
T_p(m;h,k) &= \sum_{\mu=0}^{k-1}(-1)^{\mu}\overline{E}_1\left(\frac{\mu}{k}\right)k^m p^m \sum_{j=0}^{p-1}(-1)^j \overline{E}_m\left(\frac{h\mu+kj}{pk}\right) \\
&\quad - \sum_{\mu=0}^{k-1}(-1)^{\mu}\overline{E}_1\left(\frac{\mu}{k}\right)k^m p^m \sum_{\substack{j=0 \\ p\mid(h\mu+kj)}}^{p-1}(-1)^j \overline{E}_m\left(\frac{h\mu+kj}{pk}\right) \\
&= k^m \sum_{\mu=0}^{k-1}(-1)^{\mu}\overline{E}_1\left(\frac{\mu}{k}\right)\overline{E}_m\left(\frac{h\mu}{k}\right) \qquad (29) \\
&\quad - (kp)^m \sum_{\mu=0}^{k-1}(-1)^{\mu+(k^{-1}(p-h\mu))_p}\overline{E}_1\left(\frac{\mu}{k}\right)\overline{E}_m\left(\frac{(p^{-1}h)_k\mu}{k}\right) \\
&= k^m T_m(h,k) - (pk)^m T_m^*((p^{-1}h)_k,k) \\
&= k^m (T_m(h,k) - p^m T_m^*((p^{-1}h)_k,k)),
\end{aligned}$$

because

$$h\mu + kj \equiv 0 \pmod{p}$$

and

$$h\mu + kj \equiv h\mu \pmod{k}$$

implies that

$$h\mu + kj \equiv p(p^{-1}h)_k\mu \pmod{pk}.$$

This completes the proof. $\square$

Finally, motived by the definition (11) and Lemma 2.1, we define the following type of p-adic Dedekind sums based on the double fermionic p-adic integral.

Definition 3.7. Let h and k be positive integers such that $(h,k)=1$ and $p \nmid hk$. For any $s \in \mathbb{Z}_p$, we put

$$\begin{aligned}
\Pi_p(s;h,k) = &\sum_{\substack{i,j=0 \\ i\neq j}}^{p-1}(-1)^{j+\left[\frac{hj}{p}\right]+i+\left[\frac{ki}{p}\right]}\langle k(hj)_p - h(ki)_p\rangle^{s+1} \\
&\times \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\left(1+\frac{p(kt_1-ht_2)}{k(hj)_p-h(ki)_p}\right)^{s+1} d\mu_{-1}(t_1)d\mu_{-1}(t_2).
\end{aligned}$$

In what follows, we investigate it properties. Let m be a positive integer such that $m+1 \equiv 0 \pmod{p-1}$. From Definition 3.7, we obtain

$$\begin{aligned}\Pi_p(m;h,k) &= \sum_{\substack{i,j=0\\ i\neq j}}^{p-1} (-1)^{j+\left[\frac{hj}{p}\right]+i+\left[\frac{ki}{p}\right]} \\ &\quad \times \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} (k(hj)_p - h(ki)_p + p(kt_1 - ht_2))^{m+1}\, d\mu_{-1}(t_1)d\mu_{-1}(t_2) \\ &= \sum_{\substack{i,j=0\\ i\neq j}}^{p-1} (-1)^{j+\left[\frac{hj}{p}\right]+i+\left[\frac{ki}{p}\right]} \sum_{l=0}^{m+1} (-1)^l \binom{m+1}{l} k^{m+1-l} h^l \\ &\quad \times p^{m+1-l} \int_{\mathbb{Z}_p} \left(\frac{(hj)_p}{p} + t_1\right)^{m+1-l} d\mu_{-1}(t_1) \\ &\quad \times p^l \int_{\mathbb{Z}_p} \left(\frac{(ki)_p}{p} + t_2\right)^{l} d\mu_{-1}(t_2).\end{aligned} \tag{30}$$

Thus by (10), (30) and Lemma 3.5, we find that

$$\begin{aligned}\Pi_p(m;h,k) &= \sum_{l=0}^{m+1} (-1)^l \binom{m+1}{l} k^{m+1-l} h^l \\ &\quad \times p^{m+1} \sum_{\substack{i,j=0\\ i\neq j}}^{p-1} (-1)^{j+\left[\frac{hj}{p}\right]+i+\left[\frac{ki}{p}\right]} E_{m+1-l}\left(\frac{(hj)_p}{p}\right) E_l\left(\frac{(ki)_p}{p}\right) \\ &= \sum_{l=0}^{m+1} (-1)^l \binom{m+1}{l} k^{m+1-l} h^l \\ &\quad \times \left(p^{m+1} \sum_{i,j=0}^{p-1} (-1)^{j+i} \overline{E}_{m+1-l}\left(\frac{hj}{p}\right) \overline{E}_l\left(\frac{ki}{p}\right)\right. \\ &\qquad \left. - p^{m+1} \sum_{i=0}^{p-1} (-1)^{i+i} \overline{E}_{m+1-l}\left(\frac{hi}{p}\right) \overline{E}_l\left(\frac{ki}{p}\right)\right).\end{aligned} \tag{31}$$

Therefore, by (25) we have

$$\begin{aligned}\Pi_p(m;h,k) &= \sum_{l=0}^{m+1} (-1)^l \binom{m+1}{l} k^{m+1-l} h^l E_{m+1-l}(0) E_l(0) \\ &\quad - p^{m+1}(kT^* - h)^{m+1} \binom{h\ k}{p} \\ &= (kE(0) - hE(0))^{m+1} - p^{m+1}(kT^* - h)^{m+1} \binom{h\ k}{p},\end{aligned} \tag{32}$$

where

$$(kE(0)-hE(0))^{m+1}=\sum_{l=0}^{m+1}(-1)^l\binom{m+1}{l}k^{m+1-l}h^lE_{m+1-l}(0)E_l(0)$$

and

$$(kT^*-h)^{m+1}\binom{h\ k}{p}:=\sum_{l=0}^{m+1}(-1)^l\binom{m+1}{l}k^{m+1-l}h^lT^*_{m+1-l,l}\binom{h\ k}{p}$$

with

$$T^*_{m,n}\binom{h\ k}{p}:=\sum_{i=0}^{p-1}\overline{E}_m\left(\frac{hi}{p}\right)\overline{E}_n\left(\frac{ki}{p}\right)$$

(cf. [17, p. 450, (10)]). Now we consider the integrand in (30). It is easily seen that

$$\begin{aligned}(k(hj)_p-h(ki)_p+p(kt_1-ht_2))^{m+1}\\ =(-1)^{m+1}(h(ki)_p-k(hj)_p+p(ht_2-kt_1)^{m+1}.\end{aligned}\tag{33}$$

Then by the first identity of (30) and notice that p is an odd prime by our assumption, for $m+1\equiv 0 \pmod{p-1}$, we have

$$\Pi_p(m;h,k)=\Pi_p(m;k,h).$$

In conclusion, we have obtained the following results.

Theorem 3.8 (Reciprocity relation and interpolation)**.** *Let $\Pi_p(m;h,k)$, etc., be defined as above and let m,h,k be odd positive integers such that $(h,k)=1, p\nmid hk$ and $m+1\equiv 0 \pmod{p-1}$. Then we have*

$$\Pi_p(m;h,k)=\Pi_p(m;k,h)$$

and

$$\Pi_p(m;h,k)=(kE(0)-hE(0))^{m+1}-p^{m+1}(kT^*-h)^{m+1}\binom{h\ k}{p}.$$

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References

1. T.M. Apostol, *Generalized Dedekind sums and transformation formulae of certain Lambert series*, Duke Mathematical J. **17** (1950), 147-157.
2. T.M. Apostol, *Theorems on generalized Dedekind sums*, Pacific J. of Mathematics **2** (1952), 1-9.
3. A. Bayad, *Arithmetical properties of elliptic Bernoulli and Euler numbers*, Int. J. Algebra **4** (2010), 353-372.
4. B.C. Berndt, *Analytic Eisenstein series, theta-functions, and series relations in the spirit of Ramanujan*, J. Reine Angew. Math. **303/304** (1978), 332-365.
5. J.M. Borwein, P.B. Borwein and K. Dilcher, *Pi, Euler numbers, and asymptotic expansions*, Amer. Math. Monthly **96** (1989), 681-687.
6. L. Carlitz, *Generalized Dedekind sums*, Math. Z. **85** (1964), 83-90.
7. L. Carlitz, *The reciprocity theorem for Dedekind sums*, Pacific J. of Mathematics **3** (1953), 523-527.
8. L. Carlitz, *Multiplication formulas for products of Bernoulli and Euler polynomials*, Pacific J. Math. **9** (1959), 661-666.
9. H. Cohen, *Number Theory Vol. II: Analytic and Modern Tools*, Graduate Texts in Mathematics, 240, Springer, New York, 2007.
10. R. Dedekind, *Erläuterungen zu zwei Fragmenten von Riemann,*, in "B. Riemann's Gesammelte Mathematische Werke" (H. Weber, Ed.), 2nd ed., pp. 466-472, Berlin, 1892. [Reprinted by Dover, New York, 1953.] [See also: "R. Dedekind's Gesammelte Mathematische Werke" (R. Fricke, E. Noether, and Ö. Ore, Eds.), Band I. F. Vieweg, Braunschweig, 1930, pp. 159-173.]
11. S. Hu, D. Kim and M.-S. Kim, *On reciprocity formula of Apostol-Dedekind sum with quasi-periodic Euler functions*, J. Number Theory **162** (2016), 54-67.
12. S. Hu and M.-S. Kim, *The p-adic analytic Dedekind sums*, J. Number Theory **171** (2017), 112-127.
13. N.M. Katz, *p-adic L-functions via moduli of elliptic curves, Algebraic geometry*, Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974), pp. 479-506, Amer. Math. Soc., Providence, R. I., 1975.
14. T. Kim, *On the analogs of Euler numbers and polynomials associated with p-adic q-integral on $\mathbb{Z}_p$ at $q=-1$*, J. Math. Anal. Appl. **331** (2007), 779-792.
15. T. Kim, *Note on Dedekind type DC sums*, Adv. Stud. Contemp. Math. **18** (2009), 249-260. arXiv:0812.2553v1[math.NT]
16. T. Kim, *Note on q-Dedekind-type sums related to q-Euler polynomials*, Glasg. Math. J. **54** (2012), 121-125.
17. M.-S. Kim, *A representation of Dedekind sums with quasi-periodicity Euler functions*, J. Appl. Math. Inform. **35** (2017), 449-457.
18. M.-S. Kim and S. Hu, *On p-adic Hurwitz-type Euler zeta functions*, J. Number Theory **132** (2012), 2977-3015.
19. M.-S. Kim and J.-W. Son, *On generalized Dedekind sums involving quasi-periodic Euler functions*, J. Number Theory **144** (2014), 267-280.
20. N. Koblitz, *p-adic Numbers, p-adic Analysis and Zeta-Functions*, 2nd ed., Springer-Verlag, New York, 1984.
21. A. Kudo, *On p-adic Dedekind sums*, Nagoya Math. J. **144** (1996) 155-170.
22. S. Lang, *Cyclotomic Fields I and II*, Combined 2nd ed., Springer-Verlag, New York, 1990.
23. Ju.V. Osipov, *p-adic zeta functions*, Uspekhi Mat. Nauk **34** (1979), 209-210.
24. H. Rademacher, *Some remarks on certain generalized Dedekind sums*, Acta Arith. **9** (1964), 97-105.
25. A.M. Robert, *A course in p-adic analysis*, Graduate Texts in Mathematics, 198, Springer-Verlag, New York, 2000.

26. K.H. Rosen and W.M. Snyder, *p-adic Dedekind sums*, J. Reine Angew. Math. **361** (1985), 23-26.
27. C.S. Ryoo *A note on the reflection symmetries of the Genocchi polynomials*, J. Appl. Math. Inform. **27** (2009), 1397-1404.
28. K. Shiratani and S. Yamamoto, *On a p-adic interpolation function for the Euler numbers and its derivatives*, Mem. Fac. Sci. Kyushu Univ. Ser. A **39** (1985), 113-125.
29. Y. Simsek, *q-Dedekind type sums related to q-zeta function and basic L-series*, J. Math. Anal. Appl. **318** (2006), 333-351.
30. Y. Simsek, *Special functions related to Dedekind-type DC-sums and their applications*, Russ. J. Math. Phys. **17** (2010), 495-508.
31. C. Snyder, *p-adic interpolation of Dedekind sums*, Bull. Austral. Math. Soc. **37** (1988), 293-301.
32. L. Takács, *On generalized Dedekind sums*, J. Number Theory **11** (1979), 264-272.
33. B.A. Tangedal and P.T. Young, *On p-adic multiple zeta and log gamma functions*, J. Number Theory **131** (2011), 1240-1257.
34. L.C. Washington, *Introduction to cyclotomic fields*, Second edition, Graduate Texts in Mathematics, 83, Springer-Verlag, New York, 1997.

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