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ON SIMPLICITY OF UNDIRECTED GRAPHS AND CORRESPONDING ADJACENT TOPOLOGICAL SPACES

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Abstract. In the theory of undirected simple graphs, we introduce the concept of adjacent topological spaces and analyze the minimization of open sets within these spaces. We define the continuity property of mappings between two adjacent topological spaces associated with undirected simple graphs and explore the connections between homeomorphic adjacent topological spaces and isomorphic graphs. Additionally, we investigate the compactness of adjacent topological spaces and examine the relationship between the connectedness of these spaces and the graphs.

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1. Introduction

In discrete mathematics, the graph theory is considered as topic for giving solutions of some problems. Many researchers studied some relations between graph theory and theory of topological spaces as constructing topological spaces on the set of vertices or the set of edges of simple graphs or directed graphs. This constructing is taken from the notion of a graph model and the digital image. In 2013, Amiri [6] introduced the subbasis family $S_G = \{A_x : x \in V\}$ such that A_x is the set of all adjacent vertices of x to construct a topology on the set of vertices of simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. In 2018, [1], Abdu and Kiliciman introduced topological spaces associated with simple graphs, called the incidence topological space $(\mathcal{V}, T_{IG})$, of a simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ without isolated vertex has a subbasis S_{IG} which is given by $S_{IG} = \{ends(\varepsilon) : \varepsilon \in \mathcal{E}(\mathcal{G})\}$. In [2], they used the directed graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ to introduce two constructions of topologies on the set $\mathcal{E}$, called compatible edge topology and incompatible edge topology. In 2022, [4], Othman et. al. used directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ to introduce the

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notion of pathless directed topological space on the set of vertices $\mathcal{V}$. In 2023, [5] and [4], Othman, Ayache and Saif introduced the L_2-directed topological spaces for directed graphs on the set of vertices $\mathcal{V}$.

This paper consists four sections. In Section 2, we give the notion of adjacent topological space for any undirected graph. We study minimalizing of open sets in these spaces with simple graphs. In Section 3, We introduce the notion of continuity property for adjacent topological spaces for two simple graphs and give the relations between the homeomorphically of adjacent topological spaces and the isomorphically for graphs. In Section 4, we study the compactness property for adjacent topological spaces and explain the relations between connectedness of adjacent topological spaces and graphs.

A graph $\mathcal{G}$ consists of a non-empty set $\mathcal{V}(\mathcal{G})$ of vertices, and a set $\mathcal{E}(\mathcal{G})$ of edges and we write $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$. A graph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ is called a finite graph if both sets $\mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{G})$ are finites. If the edge $\varepsilon \in \mathcal{E}(\mathcal{G})$ joins the two vertices x and y in $\mathcal{V}(\mathcal{G})$ then we write $J_\varepsilon = \{x, y\}$ and also we say that x and y are incidents with ε. A two different edges ε_1 and ε_2 in $\mathcal{E}(\mathcal{G})$ are called adjacent edges if $J_{\varepsilon_1} \cap J_{\varepsilon_2} \neq \emptyset$. An edge $\varepsilon \in \mathcal{E}(\mathcal{G})$ is called an isolated edge if there is no edge $\varepsilon' \in \mathcal{E}(\mathcal{G})$ such that $J_\varepsilon \cap J_{\varepsilon'} \neq \emptyset$, that is, there is no edge in $\mathcal{E}(\mathcal{G})$ adjacent with ε. A vertex $x \in \mathcal{V}(\mathcal{G})$ is called an isolated vertex if there is no edge in $\mathcal{E}(\mathcal{G})$ incident with it. For the edge $\varepsilon \in \mathcal{E}(\mathcal{G})$ we mean by $A(\varepsilon)$ the adjacent set of ε given by

$$\mathcal{A}(\varepsilon) = \{\varepsilon' \in \mathcal{E}(\mathcal{G}) : J_\varepsilon \cap J_{\varepsilon'} \neq \emptyset\}$$

and by $\partial(\varepsilon)$ we mean the degree of ε, that is, the number of adjacent edges with ε. Note that $\partial(\varepsilon) = |\mathcal{A}(\varepsilon)| - 1$, where $|\mathcal{A}(\varepsilon)|$ is the number of elements of $\mathcal{A}(\varepsilon)$. For the vertex $x \in \mathcal{V}(\mathcal{G})$ we mean by $I(x)$ the incident set of x given by

$$I(x) = \{y \in \mathcal{V}(\mathcal{G}) : J_\varepsilon = \{x, y\} \text{ for some } \varepsilon \in \mathcal{E}(\mathcal{G})\}$$

and by $\partial(x)$ we mean the degree of x, that is, the number of elements of $I(x)$. A graph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ is called connected graph if we can travel a long the edges from any vertix into any other vertix. In a graph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$, the loop is edge that starts and ends at the same vertex and the multiple edges are edges between two vertices. A simple graph is a graph without loops and multiple edges. A graph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ is called locally finite if $\partial(x)$ is finite number for all $x \in \mathcal{V}(\mathcal{G})$. All graphs in this papers assumed locally finites.

2. Main results

3. The adjacent topological spaces

Definition 3.1. *Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be any graph. Define the family of subsets of $\mathcal{E}(\mathcal{G})$ as follows:*

$$\mathcal{AG} = \{\mathcal{A}(\varepsilon) : \varepsilon \in \mathcal{E}(\mathcal{G})\}.$$

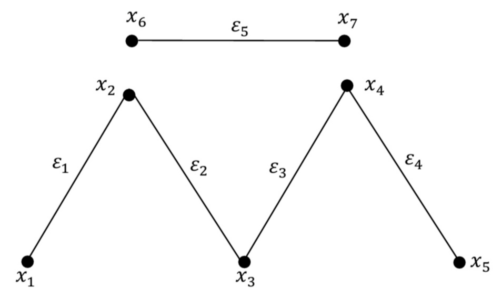

FIGURE 1

The adjacent topological space of a graph $\mathcal{G}$ is a pair $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$, where $T_{\mathcal{AG}}$ is a topology on $\mathcal{E}(\mathcal{G})$ induced by a subbasis $\mathcal{AG}$.

Example 3.2. *The graph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ in Figure(1) is given by*

$$\mathcal{V}(\mathcal{G}) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$$

and

$$\mathcal{E}(\mathcal{G}) = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\}.$$

The subbasis $\mathcal{AG}$ is given by

$$\mathcal{AG} = \{\mathcal{A}(\varepsilon_1), \mathcal{A}(\varepsilon_2), \mathcal{A}(\varepsilon_3), \mathcal{A}(\varepsilon_4), \mathcal{A}(\varepsilon_5)\}$$

where

$$\mathcal{A}(\varepsilon_1) = \{\varepsilon_1, \varepsilon_2\}, \ \ \mathcal{A}(\varepsilon_2) = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}, \ \ \mathcal{A}(\varepsilon_3) = \{\varepsilon_4, \varepsilon_2, \varepsilon_3\}$$

$$\mathcal{A}(\varepsilon_4) = \{\varepsilon_4, \varepsilon_3\}, \ \ \textit{and} \ \mathcal{A}(\varepsilon_5) = \{\varepsilon_5\}.$$

Then the adjacent topology space of a graph $\mathcal{G}$ is given by

$$T_{\mathcal{AG}} = \{\emptyset, \mathcal{E}(\mathcal{G}), \{\varepsilon_2\}, \{\varepsilon_3\}, \{\varepsilon_5\}, \{\varepsilon_1, \varepsilon_2\}, \{\varepsilon_2, \varepsilon_3\}, \{\varepsilon_2, \varepsilon_5\}, \{\varepsilon_3, \varepsilon_4\}, \{\varepsilon_3, \varepsilon_5\},$$
$$\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}, \{\varepsilon_2, \varepsilon_3, \varepsilon_4\}, \{\varepsilon_1, \varepsilon_2, \varepsilon_5\}, \{\varepsilon_3, \varepsilon_4, \varepsilon_5\}, \{\varepsilon_2, \varepsilon_3, \varepsilon_5\}, \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\},$$
$$\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_5\}, \{\varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\}\}.$$

Example 3.3. *The graph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ in Figure(2-A) is given by*

$$\mathcal{V}(\mathcal{G}) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$$

and

$$\mathcal{E}(\mathcal{G}) = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}.$$

The subbasis $\mathcal{AG}$ is given by

$$\mathcal{AG} = \{\mathcal{A}(\varepsilon_1), \mathcal{A}(\varepsilon_2), \mathcal{A}(\varepsilon_3)\}$$

where

$$\mathcal{A}(\varepsilon_1) = \{\varepsilon_1\}, \ \mathcal{A}(\varepsilon_2) = \{\varepsilon_2\}, \ \textit{and} \ \mathcal{A}(\varepsilon_3) = \{\varepsilon_3\}.$$

Then the adjacent topology space of a graph $\mathcal{G}$ *is given by* $T_{\mathcal{AG}} = P(\mathcal{E}(\mathcal{G}))$, *where* $P(\mathcal{E}(\mathcal{G}))$ *is the power of* $\mathcal{E}(\mathcal{G})$, *that is, the family of all subsets of* $\mathcal{E}(\mathcal{G})$. *This topology is called the discrete topology on* $\mathcal{E}(\mathcal{G})$.

The graph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ *in Figure(2-B) is given by*

$$\mathcal{V}(\mathcal{G}) = \{x_1, x_2, x_3\}$$

and

$$\mathcal{E}(\mathcal{G}) = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}.$$

The subbasis $\mathcal{AG}$ *is given by*

$$\mathcal{AG} = \{\mathcal{A}(\varepsilon_1), \mathcal{A}(\varepsilon_2), \mathcal{A}(\varepsilon_3)\}$$

where

$$\mathcal{A}(\varepsilon_1) = \mathcal{A}(\varepsilon_2) = \mathcal{A}(\varepsilon_3) = \mathcal{E}(\mathcal{G}).$$

Then the adjacent topology space of a graph $\mathcal{G}$ *is given by* $T_{\mathcal{AG}} = \{\emptyset, \mathcal{E}(\mathcal{G})\}$. *This topology is called the indiscrete topology on* $\mathcal{E}(\mathcal{G})$.

The graph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ *in Figure(2-C) is given by*

$$\mathcal{V}(\mathcal{G}) = \{x_1, x_2, x_3, x_4\}$$

and

$$\mathcal{E}(\mathcal{G}) = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\}.$$

The subbasis $\mathcal{AG}$ *is given by*

$$\mathcal{AG} = \{\mathcal{A}(\varepsilon_1), \mathcal{A}(\varepsilon_2), \mathcal{A}(\varepsilon_3), \mathcal{A}(\varepsilon_4), \mathcal{A}(\varepsilon_5), \mathcal{A}(\varepsilon_6)\}$$

where

$$\mathcal{A}(\varepsilon_1) = \mathcal{A}(\varepsilon_2) = \{\varepsilon_1, \varepsilon_2, \varepsilon_5, \varepsilon_6\},$$
$$\mathcal{A}(\varepsilon_3) = \mathcal{A}(\varepsilon_4) = \{\varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\},$$
$$\mathcal{A}(\varepsilon_5) = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\},$$

and

$$\mathcal{A}(\varepsilon_6) = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_6\}.$$

Then the adjacent topology space of a graph $\mathcal{G}$ *is given by*

$$T_{\mathcal{AG}} = \{\emptyset, \mathcal{E}(\mathcal{G}), \{\varepsilon_5\}, \{\varepsilon_6\}, \{\varepsilon_5, \varepsilon_6\}, \{\varepsilon_1, \varepsilon_2\}, \{\varepsilon_3, \varepsilon_4\}, \{\varepsilon_1, \varepsilon_2, \varepsilon_5\}, \{\varepsilon_1, \varepsilon_2, \varepsilon_6\},$$
$$\{\varepsilon_3, \varepsilon_4, \varepsilon_5\}, \{\varepsilon_3, \varepsilon_4, \varepsilon_6\}, \{\varepsilon_1, \varepsilon_2, \varepsilon_5, \varepsilon_6\}, \{\varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\}, \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\},$$
$$\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\}, \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_6\}\}.$$

A graph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ is called 2-simple graph if it is simple graph with $|\mathcal{V}(\mathcal{G})| > 3$ and $0 < \partial(x) < 3$ for all $x \in \mathcal{V}(\mathcal{G})$. It is clear that in 2-simple graph, $\partial(\varepsilon) < 3$ for all $\varepsilon \in \mathcal{E}(\mathcal{G})$.

Theorem 3.4. Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be any simple graph and $\varepsilon \in \mathcal{E}(\mathcal{G})$. Then:

(1) If ε is an isolated edge then $\{\varepsilon\}$ is an open set in $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$.
(2) If $\mathcal{G}$ is 2-simple and $\partial(\varepsilon) = 2$ then $\{\varepsilon\}$ is an open set in $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$.

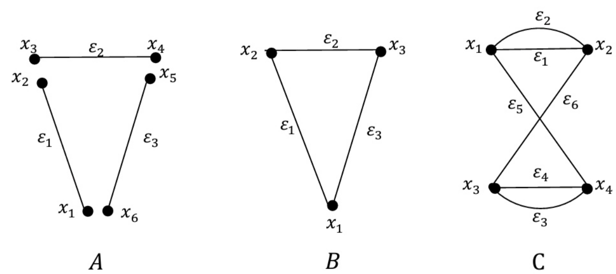

FIGURE 2

Proof. (1) Let ε be any edge in $\mathcal{E}(\mathcal{G})$. If ε is an isolated edge then only $J_\varepsilon \cap J_\varepsilon \neq \emptyset$, that is, $\mathcal{A}(\varepsilon) = \{\varepsilon\}$. Hence by definition of $\mathcal{AG}$, $\{\varepsilon\} \in \mathcal{AG} \subset T_{\mathcal{AG}}$, that is, $\{\varepsilon\}$ is an open set in $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$.
(2) Since $\mathcal{G}$ is 2-simple and $\partial(\varepsilon) = 2$ then there are at least two $\varepsilon', \varepsilon'' \in \mathcal{E}(\mathcal{G})$ such that $\varepsilon', \varepsilon'' \in \mathcal{A}(\varepsilon)$. Then we have

$$\mathcal{A}(\varepsilon') \cap \mathcal{A}(\varepsilon'') \cap \mathcal{A}(\varepsilon) = \{\varepsilon\}.$$

Hence by definition of $\mathcal{AG}$, $\{\varepsilon\} \in \mathcal{AG} \subset T_{\mathcal{AG}}$, that is, $\{\varepsilon\}$ is an open set in $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$. □

Corollary 3.5. Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be any 2-simple graph. If $\partial(\varepsilon) = 2$ for all $\varepsilon \in \mathcal{E}(\mathcal{G})$ then the adjacent topological space $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ is a discrete.

Proof. It is clear from theorem above that if $\partial(\varepsilon) = 2$ for all $\varepsilon \in \mathcal{E}(\mathcal{G})$ then $\{\varepsilon\}$ is an open set in $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ for all $\varepsilon \in \mathcal{E}(\mathcal{G})$. That is, $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ is a discrete. □

Remark 3.1. *The coresspounding adjacent topological space will be a discrete:*

(1) *The cycle graph with n vertices, denoted by C_n and $n > 3$, is a simple graph such that the number of vertices in C_n equals the number of edges and every vertex has degree 2.*
(2) *A complete graph with n vertices, denoted by K_n and $n > 3$, is a simple graph in which every pair of distinct vertices is connected by a unique edge.*
(3) *A complete bipartite graph, denoted by $K_{n,m}$ and $n, m > 1$, is a graph whose vertices can be partitioned into two subsets V_1 with n vertices and V_2 with m vertices such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph.*

Corollary 3.6. Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be any simple graph. If ε is an isolated edge for all $\varepsilon \in \mathcal{E}(\mathcal{G})$ then the adjacent topological space $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ is a discrete.

Proof. It is clear from Theorem(3.4) that if ε is an isolated edge for all $\varepsilon \in \mathcal{E}(\mathcal{G})$ then $\{\varepsilon\}$ is an open set in $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ for all $\varepsilon \in \mathcal{E}(\mathcal{G})$. That is, $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ is a discrete. □

Alexandroff space [8], is a topological space such that arbitrary intersection of open sets is an open set.

Theorem 3.7. Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be any simple graph. The adjacent topological space $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ is Alexandroff space (e.g., arbitrary intersection of open sets is an open set).

Proof. Let $\{\mathcal{A}(\varepsilon) : \varepsilon \in E \subseteq \mathcal{E}(\mathcal{G})\}$ be the collection of elements of $\mathcal{AG}$. We will prove that $\cap_{\varepsilon \in E}\mathcal{A}(\varepsilon)$ is open set. If $\varepsilon' \in \cap_{\varepsilon \in E}\mathcal{A}(\varepsilon)$ then $\varepsilon' \in \mathcal{A}(\varepsilon)$ for all $\varepsilon \in E$. Hence $\varepsilon \in \mathcal{A}(\varepsilon')$ for all $\varepsilon \in E$. That is, $E \subseteq \mathcal{A}(\varepsilon')$. Since $\mathcal{G}$ is locally finite then E is finite. Hence $\cap_{\varepsilon \in E}\mathcal{A}(\varepsilon)$ is open set. □

Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be any simple graph and $\varepsilon \in \mathcal{E}(\mathcal{G})$. Since the adjacent topological space $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ is Alexandroff space, $MO(\varepsilon)$ denotes the smallest open set containing ε which is the intersection of all open sets containing ε.

Remark 3.2. *It is clear that from Theorem(3.4) if the edge ε is an isolated or $\partial(\varepsilon) = 2$ in a 2-simple graph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ then $MO(\varepsilon) = \{\varepsilon\}$.*

Theorem 3.8. Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be any simple graph and $\varepsilon \in \mathcal{E}(\mathcal{G})$. Then

$$MO(\varepsilon) = \cap_{\varepsilon' \in \mathcal{A}(\varepsilon)}\mathcal{A}(\varepsilon').$$

Proof. It is clear from defifintion of $\mathcal{AG}$, $\mathcal{A}(\varepsilon)$ is an open set in $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$. So by Theorem(3.7), $\cap_{\varepsilon' \in \mathcal{A}(\varepsilon)}\mathcal{A}(\varepsilon')$ is open set. It is clear that if $\varepsilon' \in \mathcal{A}(\varepsilon)$ then $\varepsilon \in \mathcal{A}(\varepsilon')$. Hence $\cap_{\varepsilon' \in \mathcal{A}(\varepsilon)}\mathcal{A}(\varepsilon')$ is open set containing ε. By the definition of $MO(\varepsilon)$ as is the smallest open set containing ε then

$$MO(\varepsilon) \subseteq \cap_{\varepsilon' \in \mathcal{A}(\varepsilon)}\mathcal{A}(\varepsilon').$$

On the other hand, since $MO(\varepsilon)$ is the intersection of all open sets containing ε, then let

$$MO(\varepsilon) = \cap_{\varepsilon' \in B}\mathcal{A}(\varepsilon')$$

for some subset B of $\mathcal{E}(\mathcal{G})$. Then $\varepsilon \in \mathcal{A}(\varepsilon')$ for all $\varepsilon' \in B$. This implies $\varepsilon' \in \mathcal{A}(\varepsilon)$ for all $\varepsilon' \in B$. That is, $B \subseteq \mathcal{A}(\varepsilon)$. Hence

$$\cap_{\varepsilon' \in \mathcal{A}(\varepsilon)}\mathcal{A}(\varepsilon') \subseteq \cap_{\varepsilon' \in B}\mathcal{A}(\varepsilon') = MO(\varepsilon).$$

Therefore

$$MO(\varepsilon) = \cap_{\varepsilon' \in \mathcal{A}(\varepsilon)}\mathcal{A}(\varepsilon').$$

□

Remark 3.3. *If the edge ε with $\partial(\varepsilon) = 1$ in a simple graph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$, then $\mathcal{A}(\varepsilon) = \{\varepsilon, \varepsilon'\}$ for some $\varepsilon' \in \mathcal{E}(\mathcal{G})$. Then by Theorem(3.8),*

$$MO(\varepsilon) = \mathcal{A}(\varepsilon) \cap \mathcal{A}(\varepsilon') = \{\varepsilon, \varepsilon'\} = \mathcal{A}(\varepsilon).$$

Theorem 3.9. Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be any simple graph and $\varepsilon, \varepsilon' \in \mathcal{E}(\mathcal{G})$. Then $\varepsilon \in MO(\varepsilon')$ if and only if $\mathcal{A}(\varepsilon') \subseteq \mathcal{A}(\varepsilon)$.

Proof. Let $\varepsilon \in MO(\varepsilon')$. By Theorem(3.8),

$$MO(\varepsilon') = \cap_{\varepsilon'' \in \mathcal{A}(\varepsilon')} \mathcal{A}(\varepsilon'').$$

Then $\varepsilon \in \cap_{\varepsilon'' \in \mathcal{A}(\varepsilon')} \mathcal{A}(\varepsilon'')$. That is,

$$\varepsilon \in \mathcal{A}(\varepsilon'') \text{ for all } \varepsilon'' \in \mathcal{A}(\varepsilon').$$

This implies

$$\varepsilon'' \in \mathcal{A}(\varepsilon) \text{ for all } \varepsilon'' \in \mathcal{A}(\varepsilon').$$

Hence $\mathcal{A}(\varepsilon') \subseteq \mathcal{A}(\varepsilon)$.
Conversely, let $\mathcal{A}(\varepsilon') \subseteq \mathcal{A}(\varepsilon)$. Hence

$$\varepsilon \in \cap_{\varepsilon'' \in \mathcal{A}(\varepsilon)} \mathcal{A}(\varepsilon'') \subseteq \cap_{\varepsilon'' \in \mathcal{A}(\varepsilon')} \mathcal{A}(\varepsilon'') = MO(\varepsilon').$$

That is, $\varepsilon \in MO(\varepsilon')$. □

Corollary 3.10. Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be any simple graph and $\varepsilon, \varepsilon' \in \mathcal{E}(\mathcal{G})$. Then $MO(\varepsilon) = MO(\varepsilon')$ if and only if $\mathcal{A}(\varepsilon') = \mathcal{A}(\varepsilon)$.

Proof. Let$MO(\varepsilon) = MO(\varepsilon')$. Then $\varepsilon \in MO(\varepsilon')$ and $\varepsilon' \in MO(\varepsilon)$. Then by Theorem(3.9), $\mathcal{A}(\varepsilon') \subseteq \mathcal{A}(\varepsilon)$ and $\mathcal{A}(\varepsilon) \subseteq \mathcal{A}(\varepsilon')$. That is, $\mathcal{A}(\varepsilon') = \mathcal{A}(\varepsilon)$.
Conversely, let $\mathcal{A}(\varepsilon') = \mathcal{A}(\varepsilon)$. Then $\mathcal{A}(\varepsilon') \subseteq \mathcal{A}(\varepsilon)$ and $\mathcal{A}(\varepsilon) \subseteq \mathcal{A}(\varepsilon')$. Since $\mathcal{A}(\varepsilon') \subseteq \mathcal{A}(\varepsilon)$ then by Theorem(3.9), $\varepsilon \in MO(\varepsilon')$. Since $MO(\varepsilon)$ is the smallest open set containing ε, then $MO(\varepsilon) \subseteq MO(\varepsilon')$. Similar, $MO(\varepsilon') \subseteq MO(\varepsilon)$. □

Theorem 3.11. Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be any simple graph. Then the adjacent topological space $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ is discrete if and only if $\mathcal{A}(\varepsilon') \nsubseteq \mathcal{A}(\varepsilon)$ and $\mathcal{A}(\varepsilon) \nsubseteq \mathcal{A}(\varepsilon')$.

Proof. Let $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ be a discrete. Then $MO(\varepsilon) = \{\varepsilon\}$ for all $\varepsilon \in \mathcal{E}(\mathcal{G})$. Then for two different edges $\varepsilon, \varepsilon' \in \mathcal{E}(\mathcal{G})$, $\varepsilon' \in MO(\varepsilon)$ and $\varepsilon \in MO(\varepsilon')$. By Theorem(3.9), $\mathcal{A}(\varepsilon') \nsubseteq \mathcal{A}(\varepsilon)$ and $\mathcal{A}(\varepsilon) \nsubseteq \mathcal{A}(\varepsilon')$.
Conversely, let ε be any edge in $\mathcal{E}(\mathcal{G})$. It is clear that $\varepsilon \in MO(\varepsilon)$. If $\varepsilon' \neq \varepsilon \in \mathcal{E}(\mathcal{G})$ and $\varepsilon' \in MO(\varepsilon)$, then by Theorem(3.9), $\mathcal{A}(\varepsilon) \subseteq \mathcal{A}(\varepsilon')$ and this is contradiction with the hypothesis. Hence $MO(\varepsilon) = \{\varepsilon\}$ for all $\varepsilon \in \mathcal{E}(\mathcal{G})$. $\{\varepsilon\}$ is open set for all $\varepsilon \in \mathcal{E}(\mathcal{G})$. That is, $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ be a discrete. □

Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be any simple graph. For the adjacent topological space $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ and $F \subseteq \mathcal{E}(\mathcal{G})$, by $\overline{F}$ we mean the closure set of F which is defined as the intersection of all closed sets containing F, that is, $\overline{F}$ is the smallest closed set containing F. Recall [8] that $x \in \overline{F}$ if and only if for every open set G containing x, $G \cap F \neq \emptyset$.

Theorem 3.12. Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be any simple graph and $\varepsilon \in \mathcal{E}(\mathcal{G})$. Then $MO(\varepsilon) \subseteq \mathcal{A}(\varepsilon')$ for all $\varepsilon' \in (\varepsilon)$ and $\overline{MO(\varepsilon)} \subseteq \overline{\mathcal{A}(\varepsilon')}$ for all $\varepsilon' \in (\varepsilon)$.

Proof. For the first part, by Theorem(3.8),

$$MO(\varepsilon) = \cap_{\varepsilon' \in \mathcal{A}(\varepsilon)} \mathcal{A}(\varepsilon').$$

That is, $MO(\varepsilon) \subseteq \mathcal{A}(\varepsilon')$ for all $\varepsilon' \in \mathcal{A}(\varepsilon)$.
For the second part, let $\theta \in \overline{MO(\varepsilon)}$. Then for every open set G containing θ, $G \cap MO(\varepsilon) \neq \emptyset$. Since $MO(\varepsilon) \subseteq \mathcal{A}(\varepsilon')$ for all $\varepsilon' \in (\varepsilon)$, then $G \cap \mathcal{A}(\varepsilon') \neq \emptyset$ for all $\varepsilon' \in (\varepsilon)$. Then $\theta \in \overline{\mathcal{A}(\varepsilon')}$ for all $\varepsilon' \in (\varepsilon)$. Hence $\overline{MO(\varepsilon)} \subseteq \overline{\mathcal{A}(\varepsilon')}$ for all $\varepsilon' \in (\varepsilon)$. ☐

Corollary 3.13. Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be any simple graph and $\varepsilon \in \mathcal{E}(\mathcal{G})$. Then $\overline{\{\varepsilon\}} \subseteq \overline{MO(\varepsilon)} \subseteq \overline{\mathcal{A}(\varepsilon')}$ for all $\varepsilon' \in (\varepsilon)$.

Proof. Let $\theta \in \overline{\{\varepsilon\}}$. Then for every open set G containing θ, $G \cap \{\varepsilon\} \neq \emptyset$. Since $\varepsilon \in MO(\varepsilon)$, then $G \cap MO(\varepsilon) \neq \emptyset$. Then $\theta \in \overline{MO(\varepsilon)}$. Hence $\overline{\{\varepsilon\}} \subseteq \overline{MO(\varepsilon)})$. For the second part, it is clear from Theorem(3.12). ☐

Remark 3.4. *Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be any simple graph and $\varepsilon, \varepsilon' \in \mathcal{E}(\mathcal{G})$. Then it is clear from Corollary(3.13) and Theorem(3.12), $\varepsilon \in \overline{\{\varepsilon'\}}$ if and only if $\mathcal{A}(\varepsilon) \subseteq \mathcal{A}(\varepsilon')$.*

Remark 3.5. *Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be any 2-simple graph. It is clear from Theorem(3.4) that the set of all edges ε with $\partial(\varepsilon) = 2$ in $\mathcal{E}(\mathcal{G})$ is an open set in the adjacent topological space $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ where it will be the union of single sets of its elements.*

Theorem 3.14. Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be any simple graph. The set of all edges ε with $\partial(\varepsilon) = 1$ in $\mathcal{E}(\mathcal{G})$ is a closed set in the adjacent topological space $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$.

Proof. Let F be set of all edges ε with $\partial(\varepsilon) = 1$ in $\mathcal{E}(\mathcal{G})$ and $\varepsilon \in \overline{F}$. Recall [8] that in the theory of topological spaces $\overline{A \cup B} = \overline{A} \cup \overline{B}$. So

$$\varepsilon \in \overline{F} = \overline{\cup_{\varepsilon' \in F}\{\varepsilon'\}} = \cup_{\varepsilon' \in F}\overline{\{\varepsilon'\}}.$$

This implies $\varepsilon \in \overline{\{\varepsilon'\}}$ for some $\varepsilon' \in F$. Hence from Remark(3.5), $\mathcal{A}(\varepsilon) \subseteq \mathcal{A}(\varepsilon')$. Since $\partial(\varepsilon') = 1$ then $\partial(\varepsilon) = 1$, that is, $\varepsilon \in F$. Hence F is a closed set. ☐

4. Homeomorphically relation

Recall [8] that the map $f : (X_1, \tau_1) \to (X_2, \tau_2)$ of a topological space (X_1, τ_1) into a topological space (X_1, τ_1) is continuous if and only if $f(\overline{G}) \subseteq \overline{f(G)}$ for all $G \subseteq X_1$. A map $f : (X_1, \tau_1) \to (X_2, \tau_2)$ is called closed map if $f(G)$ is closed set in X_2 for all closed set $G \subseteq X_1$. A map $f : (X_1, \tau_1) \to (X_2, \tau_2)$ is homoeomorphism if it is bijective, closed map and continuous map.

Theorem 4.1. Let $\mathcal{G}_1 = (\mathcal{V}(\mathcal{G}_1), \mathcal{E}(\mathcal{G}_1))$ and $\mathcal{G}_2 = (\mathcal{V}(\mathcal{G}_2), \mathcal{E}(\mathcal{G}_2))$ be two simple graphs and $\Phi : \mathcal{E}(\mathcal{G}_1) \to \mathcal{E}(\mathcal{G}_2)$ be any map. Then the following statements are equivalent:

(1) A map Φ is continuous as a map of adjacent topological spaces $(\mathcal{E}(\mathcal{G}_1), T_{\mathcal{A}\mathcal{G}_1})$ into $(\mathcal{E}(\mathcal{G}_2), T_{\mathcal{A}\mathcal{G}_2})$.
(2) For all $\varepsilon, \varepsilon' \in \mathcal{E}(\mathcal{G}_1)$,

$$\mathcal{A}(\varepsilon) \subseteq \mathcal{A}(\varepsilon') \text{ implies } \mathcal{A}(\Phi(\varepsilon)) \subseteq \mathcal{A}(\Phi(\varepsilon')).$$

Proof. Suppose that a map Φ is conyinuous as a map of adjacent topological spaces $(\mathcal{E}(\mathcal{G}_1), T_{\mathcal{A}\mathcal{G}_1})$ into $(\mathcal{E}(\mathcal{G}_2), T_{\mathcal{A}\mathcal{G}_2})$ and $\varepsilon, \varepsilon' \in \mathcal{E}(\mathcal{G}_1)$ be arbitrary edges such that $\mathcal{A}(\varepsilon) \subseteq \mathcal{A}(\varepsilon')$. Then by Remark(3.5) we get $\varepsilon \in \overline{\{\varepsilon'\}}$. By the continuity of Φ this implies

$$\Phi(\varepsilon) \in \Phi(\overline{\{\varepsilon'\}}) \subseteq \overline{\{\Phi(\varepsilon')\}}.$$

Hence by Remark(3.5) we get $\mathcal{A}(\Phi(\varepsilon)) \subseteq \mathcal{A}(\Phi(\varepsilon'))$.
Conversely, suppose that for all $\varepsilon, \varepsilon' \in \mathcal{E}(\mathcal{G}_1)$,

$$\mathcal{A}(\varepsilon) \subseteq \mathcal{A}(\varepsilon') \text{ implies } \mathcal{A}(\Phi(\varepsilon)) \subseteq \mathcal{A}(\Phi(\varepsilon')).$$

Let G be any subset of $\mathcal{E}(\mathcal{G}_1)$ and $\varepsilon \in G$. If $\varepsilon \in \overline{G}$ then $\varepsilon \in \overline{\{\varepsilon'\}}$ for some $\varepsilon' \in G$. Hence $\mathcal{A}(\varepsilon) \subseteq \mathcal{A}(\varepsilon')$. By the hypothesis we get $\mathcal{A}(\Phi(\varepsilon)) \subseteq \mathcal{A}(\Phi(\varepsilon'))$. This implies

$$\Phi(\varepsilon) \in \overline{\{\Phi(\varepsilon')\}} \subseteq \overline{\Phi(G)}.$$

Hence Φ is continuous. □

Theorem 4.2. Let $\mathcal{G}_1 = (\mathcal{V}(\mathcal{G}_1), \mathcal{E}(\mathcal{G}_1))$ and $\mathcal{G}_2 = (\mathcal{V}(\mathcal{G}_2), \mathcal{E}(\mathcal{G}_2))$ be two simple graphs and $\Phi : \mathcal{E}(\mathcal{G}_1) \to \mathcal{E}(\mathcal{G}_2)$ be any map. Then Φ is a closed map if Φ is onto and for all $\varepsilon, \varepsilon' \in \mathcal{E}(\mathcal{G}_1)$,

$$\mathcal{A}(\Phi(\varepsilon)) \subseteq \mathcal{A}(\Phi(\varepsilon')) \text{ implies } \mathcal{A}(\varepsilon) \subseteq \mathcal{A}(\varepsilon').$$

Proof. Let G be any closed set in $\mathcal{E}(\mathcal{G}_1)$. Since Φ is onto then there is a map $\Theta : \mathcal{E}(\mathcal{G}_2) \to \mathcal{E}(\mathcal{G}_1)$ such that $\Phi \circ \Theta = id_{\mathcal{E}(\mathcal{G}_2)}$. We will prove that Θ is a continuous. Let $\varepsilon, \varepsilon' \in \mathcal{E}(\mathcal{G}_2)$ be arbitrary edges such that $\mathcal{A}(\varepsilon) \subseteq \mathcal{A}(\varepsilon')$. Hence $\mathcal{A}(\Phi(\Theta(\varepsilon)) \subseteq \mathcal{A}(\Phi(\Theta(\varepsilon'))$. By the hypothesis we get $\mathcal{A}(\Theta(\varepsilon) \subseteq \mathcal{A}(\Theta(\varepsilon')$. Then by Theorem(4.1) Θ is a continuous. Hence $\Phi(G) = \Theta^{-1}(G)$ is closed set and so Φ is a closed map. □

Theorem 4.3. Let $\mathcal{G}_1 = (\mathcal{V}(\mathcal{G}_1), \mathcal{E}(\mathcal{G}_1))$ and $\mathcal{G}_2 = (\mathcal{V}(\mathcal{G}_2), \mathcal{E}(\mathcal{G}_2))$ be two simple graphs and $\Phi : \mathcal{E}(\mathcal{G}_1) \to \mathcal{E}(\mathcal{G}_2)$ be any map. If Φ is a closed map and 1-1 then for all $\varepsilon, \varepsilon' \in \mathcal{E}(\mathcal{G}_1)$,

$$\mathcal{A}(\Phi(\varepsilon)) \subseteq \mathcal{A}(\Phi(\varepsilon')) \text{ implies } \mathcal{A}(\varepsilon) \subseteq \mathcal{A}(\varepsilon').$$

Proof. Let $\varepsilon, \varepsilon' \in \mathcal{E}(\mathcal{G}_2)$ be arbitrary edges such that $\mathcal{A}(\Phi(\varepsilon)) \subseteq \mathcal{A}(\Phi(\varepsilon'))$. Since Φ is 1-1 then there is a map $\Theta : \mathcal{E}(\mathcal{G}_2) \to \mathcal{E}(\mathcal{G}_1)$ such that $\Theta \circ \Phi = id_{\mathcal{E}(\mathcal{G}_1)}$. Since Φ is 1-1 and closed map then it is clear to see that Θ is continuous. This implies that $\mathcal{A}(\Theta(\Phi(\varepsilon))) \subseteq \mathcal{A}(\Theta(\Phi(\varepsilon')))$. That is, $\mathcal{A}(\varepsilon) \subseteq \mathcal{A}(\varepsilon')$. □

Lemma 4.4. Let $\mathcal{G}_1 = (\mathcal{V}(\mathcal{G}_1), \mathcal{E}(\mathcal{G}_1))$ and $\mathcal{G}_2 = (\mathcal{V}(\mathcal{G}_2), \mathcal{E}(\mathcal{G}_2))$ be two simple graphs. A bijective map Φ is homeomorphism if and only if for all $\varepsilon, \varepsilon' \in \mathcal{E}(\mathcal{G}_1)$,

$$\mathcal{A}(\varepsilon) \subseteq \mathcal{A}(\varepsilon') \text{ if and only if } \mathcal{A}(\Phi(\varepsilon)) \subseteq \mathcal{A}(\Phi(\varepsilon')).$$

Proof. It is clear from Theorems(4.1), (4.2) and (4.3). □

Let $\mathcal{G}_1 = (\mathcal{V}(\mathcal{G}_1), \mathcal{E}(\mathcal{G}_1))$ and $\mathcal{G}_2 = (\mathcal{V}(\mathcal{G}_2), \mathcal{E}(\mathcal{G}_2))$ be two simple graphs without isolated vertices. We say that two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic and we write $\mathcal{G}_1 \cong \mathcal{G}_2$ if there is a bijective map $\omega : \mathcal{V}(\mathcal{G}_1) \to \mathcal{V}(\mathcal{G}_2)$ such that if $\varepsilon_1 \in \mathcal{E}(\mathcal{G}_1)$ with $J_{\varepsilon_1} = \{x, y\}$ then there is $\varepsilon_2 \in \mathcal{E}(\mathcal{G}_2)$ with $J_\varepsilon = \{\Omega(x), \Omega(y)\}$.

Theorem 4.5. Let $\mathcal{G}_1 = (\mathcal{V}(\mathcal{G}_1), \mathcal{E}(\mathcal{G}_1))$ and $\mathcal{G}_2 = (\mathcal{V}(\mathcal{G}_2), \mathcal{E}(\mathcal{G}_2))$ be two simple graphs without isolated vertices. If $\mathcal{G}_1 \cong \mathcal{G}_2$ then the two adjacent topological spaces $(\mathcal{E}(\mathcal{G}_1), T_{\mathcal{A}\mathcal{G}_1})$ and $(\mathcal{E}(\mathcal{G}_1), T_{\mathcal{A}\mathcal{G}_1})$ are homeomorphic.

Proof. Let $\Omega : \mathcal{V}(\mathcal{G}_1) \to \mathcal{V}(\mathcal{G}_2)$ be a bijective map such that if $\varepsilon_1 \in \mathcal{E}(\mathcal{G}_1)$ with $J_{\varepsilon_1} = \{x, y\}$ then there is $\varepsilon_2 \in \mathcal{E}(\mathcal{G}_2)$ with $J_\varepsilon = \{\Omega(x), \Omega(y)\}$. Define a map $\Omega_E : \mathcal{E}(\mathcal{G}_1) \to \mathcal{E}(\mathcal{G}_2)$ by $\Omega_E(\varepsilon) = \varepsilon'$ where $J_\varepsilon = x, y$ and $J_{\varepsilon'} = \{\Omega(x), \Omega(y)\}$. For any $\varepsilon, \varepsilon \in \mathcal{E}(\mathcal{G}_1)$ if $\varepsilon = \varepsilon'$ then $J_\varepsilon = J_{\varepsilon'}$. Then $\Omega(J_\varepsilon) = \Omega(J_{\varepsilon'})$ and so $\Omega(\varepsilon) = \Omega(\varepsilon')$. That is, Ω_E is well define map. Let ε' be any edge in $\mathcal{E}(\mathcal{G}_2)$ and $J_{\varepsilon'} = \{x, y\}$. Since $x, y \in \mathcal{V}(\mathcal{G}_2)$ and Ω is onto then there are $a, b \in \mathcal{V}(\mathcal{G}_1)$ such that $\Omega(a) = x$ and $\Omega(b) = y$. By definition of Ω there is $\varepsilon \in \mathcal{E}(\mathcal{G}_1)$ with $J_\Omega = \{a, b\}$ such that $\Omega_E(\varepsilon) = \varepsilon'$. That is, Ω_E is onto. Now we will prove that Ω_E is 1-1. Let $\varepsilon, \varepsilon' \in \mathcal{E}(\mathcal{G}_1)$ such that $\Omega_E(\varepsilon) = \Omega_E(\varepsilon')$. Then $J_{\Omega_E(\varepsilon)} = J_{\Omega_E(\varepsilon')}$ and this implies $J_\varepsilon = J_{\varepsilon'}$. Since $\mathcal{G}_1$ is simple then $\varepsilon = \varepsilon'$. Hence Ω_E is objective and by Lemma(4.4) the proof is complete. □

The converse of above Theorem no need to be true, for example, see in Figure(3), the graph $\mathcal{G}_1 = (\mathcal{V}(\mathcal{G}_1), \mathcal{E}(\mathcal{G}_1))$ is given by

$$\mathcal{V}(\mathcal{G}_1) = \{x_1, x_2, x_3, x_4\}$$

and

$$\mathcal{E}(\mathcal{G}_1) = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}.$$

The subbasis $\mathcal{A}\mathcal{G}_1$ is given by

$$\mathcal{A}\mathcal{G}_1 = \{\mathcal{A}(\varepsilon_1), \mathcal{A}(\varepsilon_2), \mathcal{A}(\varepsilon_3), \mathcal{A}(\varepsilon_4)\}$$

where

$$\mathcal{A}(\varepsilon_1) = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}, \ \ \mathcal{A}(\varepsilon_2) = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}, \ \ \mathcal{A}(\varepsilon_3) = \{\varepsilon_1, \varepsilon_3, \varepsilon_4\}$$
$$\mathcal{A}(\varepsilon_4) = \{\varepsilon_1, \varepsilon_3, \varepsilon_4\}.$$

Then the adjacent topology space of a graph $\mathcal{G}_1$ is discrete. The graph $\mathcal{G}_2 = (\mathcal{V}(\mathcal{G}_2), \mathcal{E}(\mathcal{G}_2))$ is given by

$$\mathcal{V}(\mathcal{G}_2) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8\}$$

and

$$\mathcal{E}(\mathcal{G}_2) = \{\varepsilon'_1, \varepsilon'_2, \varepsilon'_3, \varepsilon'_4\}.$$

The subbasis $\mathcal{A}\mathcal{G}_2$ is given by

$$\mathcal{A}\mathcal{G}_2 = \{\mathcal{A}(\varepsilon'_1), \mathcal{A}(\varepsilon'_2), \mathcal{A}(\varepsilon'_3), \mathcal{A}(\varepsilon'_4)\}$$

where $\mathcal{A}(\varepsilon'_i) = \{\varepsilon'_i\}$ $(i = 1, 2, 3, 4)$. Then the adjacent topology space of a graph $\mathcal{G}_2$ is discrete. Note that $\mathcal{G}_1$ and $\mathcal{G}_2$ are not isomorphic since $|\mathcal{V}(\mathcal{G}_1)|$ and

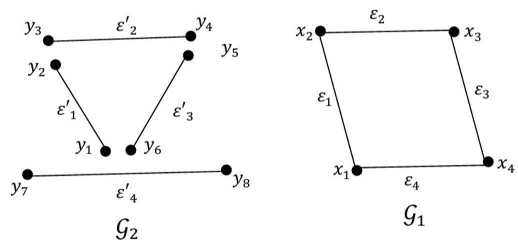

FIGURE 3

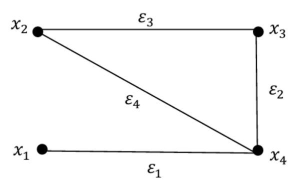

FIGURE 4

$|\mathcal{V}(\mathcal{G}_1)|$ are finite and $|\mathcal{V}(\mathcal{G}_1)| \neq |\mathcal{V}(\mathcal{G}_1)|$ but the two adjacent topological spaces $(\mathcal{E}(\mathcal{G}_1), T_{\mathcal{AG}_1})$ and $(\mathcal{E}(\mathcal{G}_1), T_{\mathcal{AG}_1})$ are homeomorphic.

Let (X, τ) be any topological space. If $\mathcal{G}$ is a simple graph with edge set X such that $\tau = T_{\mathcal{AG}}$ then we say that (X, τ) is adjacent topological space induced by the graph $\mathcal{G}$. Note that if (X, τ) is an indiscrete then (X, τ) is adjacent topological space induced by the complete graph and if (X, τ) is discrete then (X, τ) is adjacent topological space induced by the cyclic graph. Let $X = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ be any set and

$$\tau = \{\emptyset, X, \{\varepsilon_2, \varepsilon_4\}, \{\varepsilon_1, \varepsilon_2, \varepsilon_4\}, \{\varepsilon_2, \varepsilon_3, \varepsilon_4\}\}$$

$$\rho = \{\emptyset, X, \{\varepsilon_1\}, \{\varepsilon_2\}, \{\varepsilon_1, \varepsilon_2\}, \{\varepsilon_3, \varepsilon_4\}, \{\varepsilon_1, \varepsilon_3, \varepsilon_4\}, \{\varepsilon_2, \varepsilon_3, \varepsilon_4\}\}.$$

Note that (X, ρ) is not adjacent topological space induced by any graph since $\{\varepsilon_1, \varepsilon_2\}$ and $\{\varepsilon_3, \varepsilon_4\}$ don't form subbasis for (X, ρ) while (X, τ) is adjacent topological space induced by the graph $\mathcal{G}$ in Figure(4).

Theorem 4.6. Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be simple graph without isolated edges with adjacent topological space $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ and (X, τ) be any topological space. If $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ and (X, τ) are homeomorphic then (X, τ) is adjacent topological space.

Proof. Let $H : \mathcal{E}(\mathcal{G}) \to X$ be a homeomorphism. Since $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ is an Alexandroff space then (X, τ) is an Alexandroff space. Construct graph $\mathcal{G}' =$

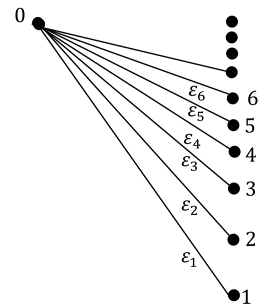

FIGURE 5

$(\mathcal{V}(\mathcal{G}'), \mathcal{E}(\mathcal{G}'))$ by $X = \mathcal{E}(\mathcal{G}')$ and

$$\mathcal{V}(\mathcal{G}') = \cup_{\varepsilon \in \mathcal{E}(\mathcal{G})} J_{H(\varepsilon)}$$

and so $\tau = T_{\mathcal{AG}'}$. □

5. On some properties

Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be any simple graph. Recall [1] that the incidence topological space $(\mathcal{V}(\mathcal{G}), T_{IG})$ is a compact space if and only if $\mathcal{V}(\mathcal{G})$ is a finite. For the adjacent topological space $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$, it is a compact space if $\mathcal{E}(\mathcal{G})$ is a finite. But if $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ is a compact space, then $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ no need to be a finite. For example, take a simple graph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ in Figure(5), where

$$\mathcal{V}(\mathcal{G}) = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, ...\}$$

and

$$\mathcal{E}(\mathcal{G}) = \{\varepsilon_n : n = 1, 2, 3, ... \text{ and } J_{\varepsilon_n} = \{0, n\}\}.$$

Note that

$$\mathcal{AG} = \{\mathcal{A}(\varepsilon_n) : n = 1, 2, 3, ...\}$$

where

$$\mathcal{A}(\varepsilon_n) = \mathcal{E}(\mathcal{G}) \text{ for all } n = 1, 2, 3, ...$$

Then the adjacent topological space $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ is an indiscrete, where

$$T_{\mathcal{AG}} = \{\emptyset, \mathcal{E}(\mathcal{G})\}.$$

In this case, $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ is a compact space but $\mathcal{E}(\mathcal{G})$ is infinite.

As we know that a graph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ is called connected graph if we can travel a long the edges from any vertex into any other vertex. A topological space (X, τ) is called disconnected space if there are two nonempty proper open subsets G and G' of X such that $X = G \cup G'$ and $G \cap G' = \emptyset$. Otherwise it is called connected space.

Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be any simple graph and $\varepsilon \in \mathcal{E}(\mathcal{G})$. If we remove ε from the graph $\mathcal{G}$ and then we get the number of components (connected subgraphs) of $\mathcal{G}$ is increasing then we say that ε is a cutedge. If ε is a cutedge then $\partial(\varepsilon) \geq 2$, that is, $\{\varepsilon\}$ is an open set in the adjacent topological space $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ by Theorem(3.4). Let $\mathcal{CE} \subseteq \mathcal{E}(\mathcal{G})$ be a component of a connected graph $\mathcal{G}$. If $\mathcal{G} - \mathcal{CE}$ has more that one component then $\mathcal{CE}$ is called edeges-cut. If every proper subset of $\mathcal{CE}$ is not edeges-cut then $\mathcal{CE}$ is called m-edegescut.

Theorem 5.1. Every m-edegescut in connected 2-simple graph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ is an open set in the adjacent topological space $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$.

Proof. Let $\mathcal{CE}$ be m-edegescut in $\mathcal{G}$. Then every $\varepsilon \in \mathcal{CE}$ must be adjacent with at least two different components. Hence $\partial(\varepsilon) = 2$ for all $\varepsilon \in \mathcal{CE}$. Hence by Theorem(3.4), $\mathcal{CE}$ an open set in the adjacent topological space $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$. □

Theorem 5.2. Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be simple graph without isolated vertix with adjacent topological space $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$. If $\mathcal{G}$ is disconnected graph then $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ is disconnected space.

Proof. If $\mathcal{G}$ is disconnected graph then take $C := \{\mathcal{G}_i : i \in I\}$ is the family of all components in $\mathcal{G}$ where $\mathcal{G}_i = (\mathcal{V}(\mathcal{G}_i), \mathcal{E}(\mathcal{G}_i))$ for all $i \in I$. Now for all $i \in I$, $\mathcal{E}(\mathcal{G}_i) = \cup_{\varepsilon \in \mathcal{E}(\mathcal{G}_i)} \mathcal{A}(\varepsilon)$. Then $G := \mathcal{E}(\mathcal{G}_k)$ is nonempty proper open subset of $\mathcal{E}(\mathcal{G})$ where $k \in I$. Then

$$G' := G^c = [\mathcal{E}(\mathcal{G}_i)]^c = \cup_{i \in I-k} \mathcal{E}(\mathcal{G}_i)$$

is also nonempty proper open subset of $\mathcal{E}(\mathcal{G})$. That is, $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ is disconnected space. □

Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be two simple graphs. It is clear that if $\mathcal{G}$ is a connected graph and $|\mathcal{E}(\mathcal{G})| \leq 3$ then $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ is connected space.

Example 5.3. *A simple graph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ in Figure(6) is a connected graph and $|\mathcal{E}(\mathcal{G})| > 3$. The adjacent topological space $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ is given by*

$$\mathcal{AG} = \{\mathcal{A}(\varepsilon_1), \mathcal{A}(\varepsilon_2), \mathcal{A}(\varepsilon_3), \mathcal{A}(\varepsilon_4), \mathcal{A}(\varepsilon_5), \mathcal{A}(\varepsilon_6)\}$$

where

$$\mathcal{A}(\varepsilon_1) = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\},\ \ \mathcal{A}(\varepsilon_2) = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\},\ \ \mathcal{A}(\varepsilon_3) = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_6\}$$

$$\mathcal{A}(\varepsilon_4) = \{\varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\},\ \ \mathcal{A}(\varepsilon_5) = \{\varepsilon_2, \varepsilon_4, \varepsilon_5, \varepsilon_6\},\ \ \mathcal{A}(\varepsilon_6) = \{\varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\}.$$

Note that $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$ is not discrete space and connected space.

Let (X, τ) be any topological space. A subset A of X is called dense in X if $\overline{A} = X$.

Theorem 5.4. Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be simple connected graph with $\partial(\varepsilon) = 1$ for some $\varepsilon \in \mathcal{E}(\mathcal{G})$ and $|\mathcal{E}(\mathcal{G})| > 2$. Then the set $A = \{\varepsilon \in \mathcal{E}(\mathcal{G}) : \partial(\varepsilon) > 1\}$ is a dense in adjacent topological space $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$.

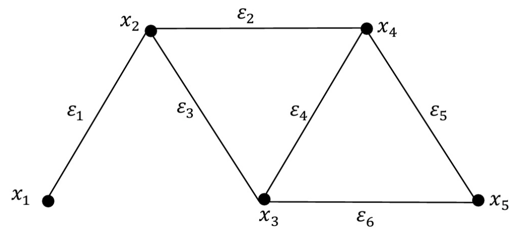

FIGURE 6

Proof. It is clear that $\overline{A} \subseteq \mathcal{E}(\mathcal{G})$. Let $\varepsilon \in \mathcal{E}(\mathcal{G})$. Suppose that $\varepsilon \notin \overline{A}$. Then there is open set G containing ε such that $G \cap A = \emptyset$. Hence $\varepsilon \in G \subseteq A^c$. Then for every $\varepsilon \in G$, $\partial(\varepsilon) = 1$. Then G doesn't equal a union of finitely intersection of elements of $\mathcal{AG}$, that is, G is not open set and this is contradiction. Hence $\varepsilon \in \overline{A}$. □

Theorem 5.5. Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be simple connected graph with $\partial(\varepsilon) = 1$ for some $\varepsilon \in \mathcal{E}(\mathcal{G})$ and $|\mathcal{E}(\mathcal{G})| > 2$. Then the set $A = \{\varepsilon \in \mathcal{E}(\mathcal{G}) : \partial(\varepsilon) > 1\}$ is a dense in adjacent topological space $(\mathcal{E}(\mathcal{G}), T_{\mathcal{AG}})$.

Proof. It is clear that $\overline{A} \subseteq \mathcal{E}(\mathcal{G})$. Let $\varepsilon \in \mathcal{E}(\mathcal{G})$. Suppose that $\varepsilon \notin \overline{A}$. Then there is open set G containing ε such that $G \cap A = \emptyset$. Hence $\varepsilon \in G \subseteq A^c$. Then for every $\varepsilon \in G$, $\partial(\varepsilon) = 1$. Then G doesn't equal a union of finitely intersection of elements of $\mathcal{AG}$, that is, G is not open set and this is contradiction. Hence $\varepsilon \in \overline{A}$. □

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