J. Appl. Math. & Informatics Vol. 42(2024), No. 5, pp. 1039 - 1049
https://doi.org/10.14317/jami.2024.1039

PROLONGATION OF G-STRUCTURES IMMERSED IN THE GOLDEN STRUCTURE TO TANGENT BUNDLES OF HIGHER ORDER

MANISHA M. KANKAREJ* AND GEETA VERMA

Abstract. The aim of the present paper is to study the lifts of a golden structure to tangent bundles of order r. We proved that r-lift of the golden structure F in the tangent bundle of order r is also a golden structure. We have also proved some theorems on the projection tensor in the tangent bundle of order r. Later we have established prolongations of G-structures immersed in the golden structure to the tangent bundle of order r and 2. Finally, we constructed few examples of the golden structure that admit an almost para contact structure on the tangent bundle of order 3 and 4.

AMS Mathematics Subject Classification : 53C15, 53C25, 53C40, 58A30.
Key words and phrases : Lifts, Golden structure, integrability, projection tensor, prolongation.

1. Introduction

Let us consider the polynomial structure of degree n

$$Q(F) = F^n + a_n F^{n-1} + ... + a_2 F + a_1 I$$

where F is the tensor field of type (1,1) and I is an identity map on the Lie algebra of vector fields on the differentiable manifold M [20, 3]. Let us consider, 'an almost complex structure' and 'an almost contact structure' which are polynomial structures of degree 2 and degree 3 respectively. The polynomial structure satisfying $F^2 - F - I$ is called golden structure. The set of positive solutions of $F^2 - F - I = 0$, denoted by $\sigma = \frac{1}{2}(1 + \sqrt{5})$, is named Golden Mean [16, 31, 26]

Hretcanu & Crasmareanu [17] introduced the notion of golden structures satisfying $F^2 - F - I = 0$, on the Riemannian manifold. Bilen et. al. [15] studied a Kähler Norden Codazzi golden structures on pseudo-Riemannian manifolds.

Received April 4, 2023. Revised January 4, 2024. Accepted May 21, 2024. *Corresponding author.

Azami [1] introduced a tensor field of type (1,1) and showed that the 'complete' and 'horizontal' lifts of such tensor field are also metallic structure on the tangent bundle. Recently, Khan[25] studied metallic structures on the frame bundle FM and discussed diagonal lift of a Riemannian metric, derivative and co-derivative of 2-form F of metallic Riemannian structure on FM. The metallic structures are studied in the literature [27, 21].

In differential geometry, tangent bundle is a primary field to investigate the geometrical structures and to study their properties like integrability, curvature, lie derivative etc. Yano & Ishihara [33] introduced an almost complex structure with some basic properties induced in tangent bundles. Das and Khan [18] have researched these lifts of an almost product structure over an almost r-contact structure along with TM. The work of several scholars like Das & Nivas [19], Khan [22, 23, 24, 30], Omran at el [29], Tekkoyun [32] and Yano & Ishihara [33] on various geometric structures and connections have been extremely beneficial in differential geometry.

Lifts of tensor fields and connections to tangent bundle were studied in detail by Yano & Davis [11], Ledger & Yano [6] and Yano [12]. Theory of prolongations of these geometric objects to the tangent bundle of order 2 and their integrability conditions were investigated by Yano & Ishiharo [13]. Morimoto [8] has studied prolongations of tensor fields, connections and G-structures to the tangent bundle of higher order. Khan [4] has studied the prolongation of G-structure immersed in generalized almost r-contact structure on M to its tangent bundle TM of order 2.

[4, 10, 34] studied the prolongation of some classical G-structure tensor fields and connections immersed in an almost complex structure, f-structure, generalized almost r-contact structure, etc. to the tangent bundles of order r.

2. Main results

In the current research we define and study the prolongation of G-structures immersed in the golden structure to tangent bundles of higher order. The main contributions of the paper can be structured as follows:

(1) In Section 3, r-lift is applied to the golden structure F to prove that it is also a golden structure in $T_r(M)$.
(2) In section 4, it is shown that the golden structure $F^{(r)}$ in $T_r(M)$ is integrable if and only if F is an integrable golden structure.
(3) In Section 5, the projection tensors are defined for the golden structure and a related theorem is proved.
(4) Finally, to elaborate the work in section 5, examples are constructed on the golden structure that admits an almost para contact structure in $T_3(M)$ and $T_4(M)$.

3. Preliminaries

3.1. The tangent bundle of order r and lifts of tensor fields. In a n-dimensional differentiable manifold, let $r \geq 1$ be a fixed integer and $\mathbb{R}$ be the real line. Consider the following equivalence relation $\sim$ among all differentiable mappings. If the mappings $F : \mathbb{R} \to M$ and $G : \mathbb{R} \to M$ meet the following criteria

$$F^h(0) = G^h(0), \ \frac{dF^h(0)}{dt} = \frac{dG^h(0)}{dt}, ..., \frac{dF^r(0)}{dt} = \frac{dG^r(0)}{dt}, \tag{1}$$

where F and G are characterized respectively by $x^h = F^h(t)$ and $x^h = G^h(t)$, t is an element of $\mathbb{R}$ with respect to local coordinates (x^h) in a coordinate neighborhood of $\{U, x^h\}$ containing the point $P = F(0) = G(0)$, then we state that $F \sim G$.

Each equivalence relation is called r-jet of M and denoted by $j^r_P(F)$, if this class contains a mapping $F : \mathbb{R} \to M$ such that $F(0) = P$. The point P is called the target of the r-jet $j^r_P(F)$. The set of all r-jets of M is called the tangent bundle of order r and denoted by $T_r(M)$ [10].

Let π_r be the bundle projection such that $\pi_r = T_r(M) \to M$, i.e. $j^r_P(F) = P$. We define $\pi_{sr} : T_r(M) \to T_s(M)$ for $r > s$ by $\pi_{sr}(j^r_P(F)) = j^s_P(F)$. Then we have $\pi_r = \pi_s \circ \pi_{sr}$.

Consider an r-jet $j^r_P(F)$ belonging to $\pi^{-1}(U)$ and the set

$$y^{(\nu)h} = \frac{1}{\mu!}\frac{d^\mu F(0)}{dt^\nu}, \ \nu = 0, 1, ..., r, \tag{2}$$

where $x^h = F^h(t)$, t is an element of $\mathbb{R}$ of F in U and $P = F(0)$. The r-jet $j^r_P(F)$ is represented by the set $(y^{(\nu)h}, \ \nu = 0, 1, ..., r)$, where $(y^{(0)h}) = (x^h)$ are coordinates of P in U. Therefore, the system of coordinates $(y^{(\nu)h}; \ \nu = 0, 1, ..., r)$ is established in the open set $\pi^{-1}(U)$ of $T_r(M)$ and called induced coordinates in $\pi^{-1}(U)$ [7, 34].

Let $\Im^0_0(M), \ \Im^1_0(M), \ \Im^0_1(M), \ \Im^1_1(M)$ be the set of functions, vector fields, 1-forms and tensor fields of type (1,1) in M, respectively. Let f be a function in M. The λ-lift $f^{(\lambda)}$ of a function f to $T_r(M)$ is defined in [8] as

$$f^{(\lambda)}(j^r_P(F)) = \frac{1}{\lambda!}\frac{d^\lambda(f \circ F)}{dt^\lambda}, \ \lambda = 0, 1, ..., r. \tag{3}$$

By (3), we have

$$(fg)^{(\lambda)} = \sum_{\mu=0}^{\lambda} f^{(\mu)} g^{(\lambda-\mu)} \tag{4}$$

for all $f, g \in \Im^0_0(M), \ \mu = 1, 2, ..., r$.

In [2] for $T_r(M)$, we have:

For $r = 1$, $T_1(M) = T(M)$ (tangent bundle) $f^V = f^{(0)},\ f^C = f^{(1)}$.

For $r = 2$, $T_2(M)$, i.e. tangent bundle of order 2, $f^0 = f^{(0)},\ f^I = f^{(1)},\ f^{II} = f^{(2)}$ for any $f \in \Im_0^0(M)$.

We shall first state the following propositions ([34], p. 379, 383, 384).

Proposition 3.1. *For any vector field $X, Y \in \Im_0^1(M)$, $F \in \Im_1^1(M)$, $f \in \Im_0^0(M)$, $\omega \in \Im_1^0(M)$, the λ-lift X^λ of X to $T_r(M)$ is a known result:*

$$[X^{(\lambda)}, Y^{(\mu)}] = [X,Y]^{(\lambda+\mu-r)},\ X^{(\lambda)} f^{(\mu)} = (Xf)^{(\lambda+\mu-r)}, \tag{5}$$

$$(fX)^{(\lambda)} = \sum_{\mu=0}^{\lambda} f^{(\mu)} X^{(\lambda-\mu)},\ (f\omega)^{(\lambda)} = \sum_{\mu=0}^{\lambda} f^{(\mu)} \omega^{(\lambda-\mu)}, \tag{6}$$

$$\omega^{(\lambda)}(X^{(\mu)}) = (\omega(X))^{(\lambda+\mu-r)},\ F^{(\lambda)} X^{(\mu)} = (FX)^{(\lambda+\mu-r)}, \tag{7}$$

where $\lambda, \mu = 0, 1, ..., r$.

Proposition 3.2. *If for $\tilde{S}$ and $\tilde{T} \in \Im_s^0(T_r(M))$,*

$$\tilde{S}(X_s^{(r)}, ..., X_1^{(r)}) = \tilde{T}(X_s^{(r)}, ..., X_1^{(r)}),$$

for $X_1, ..., X_s \in \Im_0^1(M)$, then $\tilde{S} = \tilde{T}$.

3.2. Almost para contact structure. Let M be an n-dimensional differentiable manifold of class C^∞. Suppose that there is given a tensor field F of type (1,1) satisfying

$$F^2 = I - U \otimes \omega,\ \omega(U) = 1,\ FU = 0,\ \omega \circ F = 0, \tag{8}$$

where $U \in \Im_0^1(M),\ \omega \in \Im_1^0(M)$.

The structure (F, U, ω) of such fields $F,\ U,\ \omega$ is said to be an almost para contact structure ([34], p. 66).

4. Lifts of the golden structure

Let M be an n-dimensional differentiable manifold and $F \in \Im_0^1(M)$. Then F is called a golden structure on M satisfying [5]

$$F^2 - F - I = 0, \tag{9}$$

where I is the unit vector field.

Let F and G be tensor fields of type (1,1) in M. Then ([34], p. 393)

$$G^{(r)} F^{(r)} = (GF)^{(r)}.$$

Let $P(t)$ be a polynomial of t and $F^{(r)}$ its r-lift in $T_r(M)$. Then ([34], p. 393)

$$P(F^{(r)}) = (P(F))^r, \forall F \in \Im_0^1(M). \tag{10}$$

Theorem 4.1. *Let $F \in \Im_0^1(M)$. Then $F^{(r)}$ is a golden structure in $T_r(M)$ if and only if F is a golden structure in M.*

Proof. By operating r-lift on eq (9) and using eq (10), we obtain

$$\begin{aligned}(F^2 - F - I)^{(r)} &= 0,\\ (F^2)^{(r)} - F^{(r)} - I^{(r)} &= 0,\ I^{(r)} = I,\\ (F^{(r)})^2 - F^{(r)} - I &= 0.\end{aligned}$$

Hence, $F^{(r)}$ is a golden structure in $T_r(M)$.

Theorem 4.2. *Let F and $F^{(r)}$ be golden structures in M and $T_r(M)$, respectively. Then $F^{(r)}$ is integrable in $T_r(M)$ if and only if F is integrable in M.*

Proof. Let N_F and $N_{F^{(r)}}$ denote the Nijenhuis tensors of F and $F^{(r)}$ respectively. Then from ([34], p. 393) we have,

$$N_{F^{(r)}} = (N_F)^{(r)}, \tag{11}$$

since F is integrable if and only if $N_F = 0$. So, from (11), we obtained $N_{F^{(r)}} = 0$. Hence, completes the proof.

Let l and m be the projection tensors defined in [5]:

$$l = F^2 - F, \tag{12}$$

and

$$m = I - (F^2 - F), \tag{13}$$

where I denotes the identity operator in M.

Theorem 4.3. *Let l and m be the projection tensors. Then*

$$\begin{aligned}l + m &= 0,\\ l^2 &= l,\ m^2 = m,\ lm = ml = 0,\\ Fl = lF &= F,\ Fm = mF = 0.\end{aligned} \tag{14}$$

Proof. From eqs (12) and (13),

$$\begin{aligned}l + m &= F^2 - F + I - (F^2 - F) = I,\\ l^2 &= (F^2 - F)(F^2 - F),\ as\ F^2 - F = I,\\ &= l.\\ m^2 &= [I - (F^2 - F)]^2 = I - 2(F^2 - F) + (F^2 - F)^2\\ &= I - 2(F^2 - F) + (F^2 - F)I\\ &= I - (F^2 - F)\\ &= m.\\ lm &= (F^2 - F)(I - (F^2 - F)),\\ &= 0 = ml.\end{aligned}$$

Similarly, other identities can be proved.

Let D_l and D_m be the complementary distributions corresponding to l and m, respectively in M. Let $\operatorname{rank}(F) = s$, therefore the dimension of D_l is s and the dimension of D_m is $(n - s)$, where the dimension of M is n.

Theorem 4.4. *Let m be a projection tensor in M. Then the r-lift $m^{(r)}$ of m is a projection tensor in $T_r(M)$ and the distribution $\tilde{D}$ determined by $m^{(r)}$ in $T_r(M)$ is integrable if and only if the distribution D determined by m is so in M.*

Proof. Let D be a distribution in M. D is determined by a projection tensor m, i.e. m is an element of $\Im_1^1(M)$ such that $m^2 = m$. From eq (10) and $m^2 = m$, we have

$$(m^{(r)})^2 = m^{(r)},$$

that is the r-lift $m^{(r)}$ of m which is a projection tensor in $T_r(M)$. The distribution D is integrable if and only if

$$l[mX, mY] = 0, \tag{15}$$

where $l = I - m$ denotes a projection tensor complementary to m and $[,]$ is the Lie bracket.

By applying r-lift on (15), we obtain

$$l^{(r)}[m^{(r)}X^{(r)}, m^{(r)}Y^{(r)}] = 0, \tag{16}$$

where $l^{(r)} = (I - m)^{(r)} = I - m^{(r)}$.

Thus, conditions (15) and (16) are equivalent to each other.

This finishes the proof.

5. Prolongation of G-structures immersed in the golden structure to the tangent bundle of higher order

In this section, we study the tangent bundle of order r on some classical G-structures, which are defined by tensor fields immersed in a golden structure.

Let $P(M, \pi^*, G)$ be a G-structure over a manifold M, where G is a Lie subgroup of $GL(n, \mathbb{R})$, $\mathbb{R}$ is a real line. A G-structure on M is a G-subbundle $P(M, \pi^*, G)$ of the frame bundle FM over M [9]. Let $u = U$ be an open covering of M such that in each U there exists an n-frame $\{X_i\}$ which is adapted to the G-structure $P(M, \pi^*, G)$. The structure group $T_r(GL(n, \mathbb{R}))$ of $T(T_r(M))$ is reducible to the tangent group $T_r(G)$ of order r, that is, the tangent bundle $T_r(M)$ of order r admits a $T_r(G)$-structure $\tilde{P}$, which is called the prolongation of G-structure P in M to $T(T_r(M))$ [14, 28, 34].

Let $\overset{o}{F}$ be a tensor field of type (1,1) in $\mathbb{R}^n$, which is invariant by G. We consider that M admits a G-structure P. Consider a coordinate neighborhood $\{U, X^h\}$ of M and an n-frame $\{X_{(i)}\}$ in U. Thus, if we set

$$\overset{o}{F} = \overset{o}{F}{}_i^{(h)} X_{(h)}\theta^{(i)} \tag{17}$$

in $\{U,\theta^{(i)}\}$ being the n-coframe dual to $\{X_{(i)}\}$ in U and $\overset{o}{F}{}_i^{(h)}$ are the components of $\overset{o}{F}$ in R^n. The local tensor field F is defined by Eq (17) in M. Hence F defines a global tensor field, which is called the tensor field induced in M from $(\overset{o}{F},P)$ [34].

Now, we state the following proposition for later use ([34], p. 406).

Proposition 5.1. *The prolongation $\tilde{P}$ of a G-structure P given in M is integrable in the tangent bundle $T(M)$ if and only if the G-structure P is integrable in M.*

Theorem 5.2. *Let M denote a manifold that admits a golden structure P (as a G-structure) defined by a tensor field F of type (1,1) such that $F^2 = F + I$. Let the tangent bundle $T_r(M)$ of order r be $\tilde{P}$ of P is the golden structure defined by the r-lift $F^{(r)}$ of F to $T_r(M)$. Then the golden structure P is integrable in M if and only if $\tilde{P}$ of P to $T_r(M)$ is integrable.*

Proof. Let $\overset{o}{F}$ be a tensor of type (1,1) in $\mathbb{R}^{2n}$ such that $\overset{o}{F}{}^2 = \overset{o}{F} + I$ and denote by $GL(n,\mathbb{C})$ the group of all elements of $G = GL(2n,\mathbb{C})$ which leave $\overset{o}{F}$ invariant. Then the r-lift $\overset{o}{F}{}^{(r)}$ of $\overset{o}{F}$ to $T_r(R^{2n})$ is a tensor of type (1,1) satisfying $(\overset{o}{F}{}^{(r)})^2 = \overset{o}{F}{}^{(r)} + I$ and the tangent group $T_r(G)$ leaves $\overset{o}{F}{}^{(r)}$ invariant. Thus, we obtain $T_r(G) = GL(3n,\mathbb{C})$. By proposition 4.1, the golden structure P is integrable in M if and only if $\tilde{P}$ of P to $T_r(M)$ is integrable.
This finishes the proof.

Theorem 5.3. *If a manifold M of $2n$-dimension admits an almost para contact structure P (as a G-structure) determined by (F,U,ω) given in eq (8), then, on the tangent bundle $T_2(M)$, $\tilde{P}$ of P is the golden structure determined by the tensor field*

$$\overset{o}{J} = \frac{1}{2} - \left(\frac{2\sigma-1}{2}\right)\left(\overset{o}{F}{}^{II} + \overset{o}{\eta}{}^{0} \otimes \overset{o}{\xi}{}^{0} + \overset{o}{\eta}{}^{II} \otimes \overset{o}{\xi}{}^{II}\right),\ \overset{o}{U} = \overset{o}{\xi}{}^{I},\ \overset{o}{\omega} = \overset{o}{\eta}{}^{I}.$$

Proof. $G = GL(n,\mathbb{C}) \times I$. Let the rank of $\overset{o}{F}$ be $2s$. Let $\overset{o}{\xi}$ denote a contravariant vector field and $\overset{o}{\omega}$ a 1-form in $\mathbb{R}^{2n}$ such that

$$\overset{o}{F}{}^2 = I - \overset{o}{\eta} \otimes \overset{o}{\xi}, \tag{18}$$

where

$$\begin{aligned} &\text{(i)}\ \ \overset{o}{F} \circ \overset{o}{\xi} = 0,\\ &\text{(ii)}\ \ \overset{o}{\eta}(\overset{o}{F}) = 0,\\ &\text{(iii)}\ \ \overset{o}{\eta}(\overset{o}{\xi}) = 1. \end{aligned} \tag{19}$$

If we denote by G, the group of all the elements of $GL(2n, \mathbb{C})$, which leave $\overset{o}{F}, \overset{o}{\xi}, \overset{o}{\eta}$ invariant, then it is obvious that

$$G = GL(n, \mathbb{C}) \times I \subset GL(2n, \mathbb{R}),$$

where I denotes the trivial group.

We set

$$\overset{o}{J} = \frac{1}{2} - (\frac{2\sigma - 1}{2})(\overset{o}{F}{}^{II} + \overset{o}{\eta}{}^{0} \otimes \overset{o}{\xi}{}^{0} + \overset{o}{\eta}{}^{II} \otimes \overset{o}{\xi}{}^{II}),\ \overset{o}{U} = \overset{o}{\xi}{}^{I},\ \overset{o}{\omega} = \overset{o}{\eta}{}^{I}. \tag{20}$$

By operating 2-lift of both sides of (18) and (19), we get

$$\begin{aligned} (F^{II})^2 &= (F^2)^{II} = I - \eta^{II} \otimes \xi^0 - \eta^0 \otimes \xi^{II}, & (21)\\ \eta^0(\xi^{II}) &= \eta^{II}(\xi^0) = 1,\ \eta^0(\xi^0) = \eta^{II}(\xi^{II}) = 0, & (22)\\ F^{II}(\xi^0) &= F^{II}(\xi^{II}) = 0,\ \eta^0 \circ F^{II} = \eta^{II} \circ F^0 = 0. & (23) \end{aligned}$$

Then

$$\overset{o}{J}(\overset{o}{\xi}{}^{0}) = \frac{1}{2}\overset{o}{\xi}{}^{0} - (\frac{2\sigma - 1}{2})(\overset{o}{\xi}{}^{II}), \tag{24}$$

$$\overset{o}{J}(\overset{o}{\xi}{}^{II}) = \frac{1}{2}\overset{o}{\xi}{}^{II} - (\frac{2\sigma - 1}{2})(\overset{o}{\xi}{}^{0}), \tag{25}$$

$$\overset{o}{J}(F^{II}\tilde{X}) = \frac{1}{2}F^{II}\tilde{X} - (\frac{2\sigma - 1}{2})(\tilde{X} - \eta^0(\tilde{X})\xi^0 - \eta^{II}(\tilde{X})\xi^{II}) \tag{26}$$

and

$$\overset{o}{J}{}^{2}\tilde{X} = \overset{o}{J}\tilde{X} + \tilde{X}.$$

So, $(\overset{o}{J}, \overset{o}{\eta}, \overset{o}{\xi})$ is a golden structure in $T_2(r)$. Hence, $T_2(R^{2n})$ leaves $(\overset{o}{J}, \overset{o}{\eta}, \overset{o}{\xi})$ invariant. Thus, we obtain

$$T(G) \subset GL(3n, C) \times I \subset GL(6n, R).$$

This finishes the proof.

6. Examples of golden structures admitting an almost para contact structure

Let M be a $2n$-dimensional differential manifold and $T_3(M)$ and $T_4(M)$ be its tangent bundles of order 3 and 4. We construct the following examples on golden structures that admits an almost para contact structure.

Example 6.1. If a manifold M of $2n$-dimension admits an almost para contact structure P (as a G-structure) determined by (F, U, ω) given in eq (8). Then, on the tangent bundle $T_3(M)$, $\tilde{P}$ of P is the golden structure is determined by the tensor field

$$\tilde{J} = \frac{1}{2}I - (\frac{2\sigma - 1}{2})(F^{II} + \eta^0 \otimes \xi^0 + \eta^I \otimes \xi^I - \eta^{II} \otimes \xi^{II} - \eta^{III} \otimes \xi^{III}).$$

Example 6.2. If a manifold M of $2n$-dimension admits an almost para contact structure P (as a G-structure) determined by (F, U, ω) given in eq (8), then, on the tangent bundle $T_4(M)$, $\tilde{P}$ of P is the golden structure is determined by the tensor field

$$\begin{aligned}\tilde{J} &= \frac{1}{2}I - (\frac{2\sigma - 1}{2})(F^{(4)} + \eta^{(0)} \otimes \xi^{(0)} + \eta^{(1)} \otimes \xi^{(1)} - \eta^{(3)} \otimes \xi^{(3)}) - \eta^{(4)} \otimes \xi^{(4)}),\\ \tilde{U} &= \xi^{(2)},\ \tilde{\omega} = \eta^{(2)}.\end{aligned}$$

7. Conclusions

In this research, we have proved that if r-lift is applied to the golden structure F, then F is also a golden structure in $T_r(M)$. We also proved that the golden structure $F^{(r)}$ in $T_r(M)$ is integrable if and only if F is an integrable golden structure. We have defined the projection tensors for the golden structure and proved related theorems on it. Finally, we added examples on the golden structure that admits an almost para contact structure in $T_3(M)$ and $T_4(M)$. This research will open path for new researches and future studies could be done on the polynomial structure $Q(F) = F^n + a_n F^{n-1} + ... + a_2 F + a_1 I$, where F is the tensor field of type (1,1) on the differentiable manifold M.

Conflict of interest : The authors declare no conflicts of interest in this paper.

Data availability : Not applicable

References

1. S. Azami, *General natural metallic structure on tangent bundle*, Iran. J. Sci. Technol. Trans. A Sci. **42** (2018), , 81-88.
2. A. Gezer, A. Magden, *Geometry of the second-order tangent bundles of Riemannian manifolds*, Chin. Ann. Math. Ser. B **38** (2017), 985-998.
3. S.I. Goldberg, N.C. Petridis, *Differentiable solutions of algebraic equations on manifolds*, Kodai Math. Sem. Rep. **25** (1973), 111-128.
4. M.N.I. Khan, J.B. Jun, *Prolongations of G-structures immersed in generalized almost r-contact structure to tangent bundle of order 2*, J. Chungcheong Math. Soc. **31** (2018), 421-427.
5. M.N.I. Khan, *Novel theorems for metallic structures on the frame bundle of the second order*, Filomat **36** (2022), 4471-4482.
6. A.J. Ladger, K. Yano, *Almost complex structures on tensor bundles*, J. Differ. Geom. **1** (1967), 355-368.
7. A. Mağden, A. Gezer, K. Karaca, *Some problems concerning with Sasaki metric on the second-order tangent bundles*, Int. Electron. J. Geom. **13** (2020), 75-86.
8. A. Morimoto, *Lifting of tensor fields and connections to tangent bundles of higher order*, Nagoya Math. J. **40** (1970), 99-120.
9. A. Morimoto, *Prolongations of G-structures to tangent bundles*, Nagoya Math. J. **32** (1968), 67-108.
10. M. Özkan, F. Yilmaz, *Prolongations of golden structures to tangent bundles of order r*, Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat. **65** (2016), 35-48.

11. K. Yano, E.T. Davis, *Metrics and connections in the tangent bundle*, Kodai Math. Sem. Rep. **23** (1971), 493-504.
12. K. Yano, *Tensor fields and connections on cross-sections in the tangent bundle of a differentiable manifold*, Proc. Roy. Soc. Edinburgh Sect. A **67** (1967), 277-288.
13. K. Yano, S. Ishihara, *Differentiable geomtry of tangent bundles of order 2*, Kodai Math. Sem. Rep. **20** (1968), 318-354.
14. M. Altunbas, L. Bilen, A. Gezer, *Remarks about the Kaluza-Klein metric on the tangent bundle*, International Journal of Geometric Methods in Modern Physics **16** (2019), 1-11.
15. L. Bilen, S. Turanli and A. Gezer, *On Kähler–Norden–Codazzi golden structures on pseudo-Riemannian manifolds*, International Journal of Geometric Methods in Modern Physics **15** (2018), 1-10.
16. A.M. Blaga, M. Crasmareanu, *Golden-statistical structures*, Comptes rendus de l'Académie bulgare des Sciences **69** (2016), 1113-1120.
17. M. Crasmareanu, C.E. Hretcanu, *Golden differential geometry*, Chaos, Solitons and Fractals **38** (2008), 1229-1238.
18. L.S. Das, M.N.I. Khan, *Almost r-contact structures on the tangent bundle*, Differential Geometry-Dynamical Systems **7** (2005), 34-41.
19. L.S. Das, R. Nivas, M.N.I. Khan, *On submanifolds of codimension 2 immersed in a hsu–quarternion manifold*, Acta Mathematica Academiae Paedagogicae Nyíregyháziensis **25** (2009), 129-135.
20. S.I. Goldberg, K. Yano, *Polynomial structures on manifolds*, Kodai Math Sem Rep. **22** (1970), 199-218.
21. M.N.I. Khan and U.C. De, *Liftings of metallic structures to tangent bundles of order r*, AIMS Mathematics **7** (2022), 7888-7897.
22. M.N.I. Khan and U.C. De, L.S. Velimirovic, *Lifts of a quarter-symmetric metric connection from a Sasakian manifold to its tangent bundle*, Mathematics **11** (2023), 53.
23. M.N.I. Khan, *Lifts of hypersurfaces with quarter-symmetric semi-metric connection to tangent bundles*, Afr. Mat. **25** (2014), 475-482.
24. M.N.I. Khan, *Proposed theorems for lifts of the extended almost complex structures on the complex manifold*, Asian-European Journal of Mathematics **15** (2022), 13 pages.
25. M.N.I. Khan, *Novel theorems for the frame bundle endowed with metallic structures on an almost contact metric manifold*, Chaos, Solitons & Fractals **146** (2021), 110872.
26. J. Kappraff, *Connections: The geometric bridge between Art and Science*, World Scientific, 2001.
27. M.N.I. Khan, M.A. Choudhary, S.K. Chaubey, *Alternative Equations for Horizontal Lifts of the Metallic Structures from Manifold onto Tangent Bundle*, Journal of Mathematics **2022** (2022), 8 pages, Article ID 5037620. https://doi.org/10.1155/2022/5037620
28. A. Mağden, N. Cengiz, A.A. Salimov, *Horizontal lift of affinor structures and its applications*, Applied Mathematics and Computation **156** (2004), 455-461.
29. T. Omran, A. Sharffuddin, S.I. Husain, *Lift of structures on manifold*, Publications De L'institut Mathematique (Beograd) (N.S.) **36** (1984), 93-97.
30. A. Sardar, M.N.I. Khan, U.C. De, *$\eta - *$-Ricci Solitons and Almost co-Kähler Manifolds*, Mathematics **9** (2021), 3200.
31. V.W. de Spinadel, *From the golden mean to chaos*, Ed. Nueva Libreria, Buenos Aires, 1998.
32. M. Tekkoyun, *On lifts of paracomplex structures*, Turk. J. Math. **30** (2006), 197-210.
33. K. Yano, S. Ishihara, *Almost complex structures induced in tangent bundles*, Kodai Math. Sem. Rep. **19** (1967), 1-27.
34. K. Yano, S. Ishihara, *Tangent and cotangent bundles*, Marcel Dekker, 1973.

Manisha M. Kankarej received her Ph.D. at Deen Dayal Upadhyaya Gorakhpur University, India. Currently she is working as at Rochester Institute of Technology, Dubai, UAE. Her research interests include differential geometry and application of math in hydrogeology.

College of Science and Liberal Arts, Rochester Institute of Technology, Dubai, UAE.
e-mail: manisha.kankarej@gmail.com

Geeta Verma received her Ph.D. from University of Lucknow. She is currently an Associate Professor at Shri Ramswaroop Memorial Group Of Professional Colleges, India. Her research interests includes study of geometry of manifolds.

Department of Mathematics, Shri Ramswaroop Memorial Group Of Professional Colleges, India.
e-mail: geeta.verma15@gmail.com