Korean J. Math. **32** (2024), No. 3, pp. 545–560
https://dx.doi.org/10.11568/kjm.2024.32.3.545

A NEW BANACH SPACE DEFINED BY ABSOLUTE JORDAN TOTIENT MEANS

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Abstract. In the present study, we have constructed a new Banach series space $|\Upsilon^r|_p^u$ by using concept of absolute Jordan totient summability $|\Upsilon^r, u_n|_p$ which is derived by the infinite regular matrix of the Jordan's totient function. Also, we prove that the series space $|\Upsilon^r|_p^u$ is linearly isomorphic to the space of all p-absolutely summable sequences ℓ_p for $p \geq 1$. Moreover, we compute the α-, β- and γ- duals of this space and construct Schauder basis for the series space $|\Upsilon^r|_p^u$. Finally, we characterize the classes of infinite matrices $\left(|\Upsilon^r|_p^u, X\right)$ and $\left(X, |\Upsilon^r|_p^u\right)$, where X is any given classical sequence spaces ℓ_∞, c, c_0 and ℓ_1.

1. Introduction and Preliminaries

The theory of sequence spaces has always been of great interest as it is involved in various fields in analysis, especially summability. Also, it has many applications in numerical analysis, approximation theory, operator theory, and orthogonal series theory. Classical summability theory concerns on the generalization convergence of sequences or series of real or complex numbers. In order to do so, it aims to assign a limit of some sort to divergent sequences or series by considering a transformation of a sequence rather than the original sequence or series. Recent studies have focused on generating new Banach sequence and series spaces by means of the matrix domain of infinite triangular matrices. Recently, matrices corresponding to arithmetic functions in number theory have been widely used to construct these triangular matrices. Interesting studies using arithmetic functions in the summability theory are found in [4–8, 10, 17, 20, 24, 26]. Recently, researchers have focused on studies on Euler totient and Jordan totient matrices [4–7]. Also, some details for theory and applications of Banach space, the reader can refer to [12, 14–16, 21].

The Euler's totient function φ is one of the most famous arithmetic functions with many applications in number theory. Recall that Euler totient function $\varphi(m)$ is defined as the number of positive integers less than m that coprime with m for every

Received May 23, 2023. Revised July 12, 2024. Accepted July 14, 2024.
2010 Mathematics Subject Classification: 11A25, 40C05, 46B45, 40F05, 46A45.
Key words and phrases: Sequence spaces, Jordan totient function, Dual spaces, Matrix operators, BK spaces.

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$m \in \mathbb{N}$ and $m > 1$, and $\varphi(1) = 1$. Euler totient matrix operator $\Phi = (\phi_{nk})$ is defined as

$$\phi_{nk} = \begin{cases} \frac{\varphi(k)}{n} & , \quad \text{if } k \mid n \\ 0 & , \quad \text{if } k \nmid n \end{cases}$$

which is a regular matrix [5]. Also the new sequence spaces have been introduced by using this matrix in [4–6].

The Euler totient function has been generalized in many ways because of its applications in various branches of number theory. Among the generalizations, the most significant is probably the Jordan's totient function. The Jordan's function of order r is an arithmetic function which is generalizing the well-known Euler totient function φ, where r is a positive integer. This function is denoted by J_r and it is the number of r tuples $(a_1, ..., a_r)$ with the properties $1 \leq a_i \leq n, i = 1, 2, ..., r$ and $\gcd(a_1, ..., a_r, n) = 1$ for a fixed positive integer r. It is obvious that $J_1 = \varphi$. Recent history of Jordan's function is given in [3]. The function J_r has some interesting properties and many applications. In what follows we recall some of the important ones [9, 13].

The function J_r is multiplicative, that is, the relation $J_r(mn) = J_r(m) J_r(n)$ is satisfied for any positive integers m, n with $\gcd(m, n) = 1$. Also, the Gauss type formula J_r reads as follows:

$$\sum_{d \mid n} J_k(d) = n^k$$

from which, the Möbius inversion formula gives

$$J_r(n) = \sum_{d \mid n} \mu(d) \left(\frac{n}{d}\right)^r$$

where μ is the Möbius function defined as

$$\mu(n) = \begin{cases} (-1)^m & \text{if } n = p_1 p_2 ... p_m, \text{ where } p_1, p_2, ..., p_m \text{ are} \\ & \text{non-equivalent prime numbers,} \\ 1 & \text{if } n = 1, \\ 0 & \text{if } p^2 \mid n \text{ for some prime number } p, \end{cases}$$

for $n \in \mathbb{N}$.

Properties of the Jordan's totient function take place in Sándor et al [19] and Andrica and Piticari [2]. Also, for more results and applications of this function, we refer the reader to [1, 2, 13, 23].

Then, the Jordan totient matrix operator denoted by $\Upsilon^r = (\upsilon^r_{nk})$ was defined in [7] as follows:

$$\upsilon^r_{nk} = \begin{cases} \frac{J_r(k)}{n^r} & , \quad \text{if } k \mid n \\ 0 & , \quad \text{if } k \nmid n \end{cases}$$

for each $r \in \mathbb{N}$ and its inverse $(\Upsilon^r)^{-1} = \left((\upsilon^r_{nk})^{-1}\right)$ is introduced by

$$(\upsilon^r_{nk})^{-1} = \begin{cases} \frac{\mu\left(\frac{n}{k}\right)}{J_r(n)} k^r & , \quad \text{if } k \mid n \\ 0 & , \quad \text{if } k \nmid n. \end{cases}$$

The remainder of this work is organized as follows: In section 2, a new Banach series space $|\Upsilon^r|^u_p$ is defined by using concept of the Jordan totient matrix operator and absolute summability. Also, some topological properties of the space $|\Upsilon^r|^u_p$ are given and α-, β- and γ- duals of this space are computed. In section 3, the characterizations of matrix transformations in the related spaces are presented. The results presented in this paper are motivated by those of [6] and [7].

In what follows, we recall some basic definition or notations that are needful for this paper.

Let ω denote the space of all sequences (real or complex). ℓ_∞, c and c_0 denote the spaces of all bounded, convergent and null sequences, respectively. Also, by bs, cs, ℓ_1 and ℓ_p, we denote the spaces of all bounded, convergent, absolutely and p-absolutely convergent series, respectively. Let e and (e_n), $(n \in \mathbb{N})$ be the sequences with $e_k = 1$ for all $k \in \mathbb{N}$, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$, respectively.

A subspace X of ω is said to be a BK- space if it is a Banach space with continuous coordinates $P_n : X \to \mathbb{C}$,$(n \in \mathbb{N})$, where $P_n(x) = x_n$ for all $x \in X$. For example, the spaces ℓ_p $(1 \leq p < \infty)$ and ℓ_∞, c are BK- spaces with norms $\|x\|_{\ell_p} = \left(\sum_{v=0}^{\infty} |x_v|^p\right)^{1/p}$ and $\|x\|_\infty = \sup_v x_v$, respectively. The set $S(X, Y)$ is defined by

$$S(X,Y) = \{a = (a_k) \in \omega : ax = (a_k x_k) \in Y \text{ for all } x = (x_k) \in X\}, \tag{1.1}$$

which is called the multiplier space of the spaces X and Y. With the notation of (1.1), the α-, β- and γ- duals of a sequence space X, which are denoted by X^α, X^β and X^γ, are defined by

$$X^\alpha = S(X, \ell_1), \ X^\beta = S(X, cs) \ \text{ and } X^\gamma = S(X, bs),$$

respectively.

Let $A = (a_{nk})$ be an infinite matrix of complex numbers for all $n, k \in \mathbb{N}$. We write $A_n = (a_{nk})_{k \in \mathbb{N}}$ for the sequence in the n–th row of A. If $x = (x_k) \in \omega$, then we define the A-transform of x as the sequence $A(x) = (A_n(x))$, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k$$

provided the series on the right converges for each $n \in \mathbb{N}$.

For arbitrary subsets X and Y of ω, we write (X, Y) for the class of all infinite matrices that map X into Y. So, $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for all $n \in \mathbb{N}$ and $A(x) \in Y$ for all $x \in X$. Moreover, the matrix domain of an infinite matrix A in X is defined by

$$X_A = \{x \in w : A(x) \in X\}. \tag{1.2}$$

We assume throughout unless stated otherwise that $p, q > 1$ with $p^{-1} + q^{-1} = 1$ and use the convention that any term with negative subscript is equal to zero.

A sequence (b_n) in a normed space X is called a Schauder base (or briefly base) for X, if for every $x \in X$ there exists an unique sequence (α_n) of scalars such that $\|x - \sum_{n=0}^{m} \alpha_n b_n\| \to 0$ $(m \to \infty)$, and we write $x = \sum_{n=0}^{\infty} \alpha_n b_n$. For example, $\left(e^{(n)}\right)$ is a Schauder base of the space ℓ_p $(1 \leq p < \infty)$ under the norm $\|x\|_p = \left(\sum_{v=0}^{\infty} |x_v|^p\right)^{1/p}$.

2. A New Series Space using Absolute Jordan Totient Means

In the present section, we introduce the series space $|\Upsilon^r|_p^u$ by using concept of the Jordan totient matrix operator and absolute summability, where $1 \leq p < \infty$.

Moreover, we examine some topological and algebraic properties of this space and also, we compute the α-, β- and γ- duals of the series space $|\Upsilon^r|_p^u$.

Let Σx_v be a given infinite series with nth partial sums (s_n). We give the Jordan totient transform $\Upsilon^r(s)$ of the sequence (s_n) by

$$\Upsilon_n^r(s) = \frac{1}{n^r}\sum_{k|n} J_r(k)\, s_k \tag{2.1}$$

for $n \geq 1$ and $\Upsilon_0^r(s) = 0$.

Let (u_n) be a sequence of nonnegative terms. We define the new absolute summability method $|\Upsilon^r, u_n|_p$ using Jordan totient matrix. A series Σx_v is called summable $|\Upsilon^r, u_n|_p$, if

$$\sum_{n=1}^{\infty} u_n^{p-1} |\Delta\Upsilon_n^r(s)|^p < \infty, \tag{2.2}$$

for $1 \leq p < \infty$, where $\Delta\Upsilon_n^r(s) = \Upsilon_n^r(s) - \Upsilon_{n-1}^r(s)$, and $\Upsilon_n^r(s)$ is defined by (2.1). This definition is motivated by [18].

Now, we introduce the new series space $|\Upsilon^r|_p^u$ as the set of all series summable by absolute Jordan totient summability method $|\Upsilon^r, u_n|_p$ using (2.2) as follows:

$$|\Upsilon^r|_p^u = \left\{ x = (x_v) \in \omega : \sum_{n=1}^{\infty} u_n^{p-1} |\Delta\Upsilon_n^r(s)|^p < \infty \right\}.$$

Note that since (s_n) is the sequence of partial sums of the series Σx_v, then we deduce that

$$\Upsilon_n^r(s) = \frac{1}{n^r}\sum_{j=1}^{n}\left(\sum_{\substack{k=j\\ k|n}}^{n} J_r(k)\right) x_j$$

for $n \geq 1$ and $\Upsilon_0^r(s) = 0$.

Thus we obtain that

$$\Delta\Upsilon_n^r(s) = \Upsilon_n^r(s) - \Upsilon_{n-1}^r(s) = \sum_{j=1}^{n-1} x_j \left(\sum_{\substack{k=j\\ k|n}}^{n} \frac{J_r(k)}{n^r} - \sum_{\substack{k=j\\ k|n-1}}^{n-1} \frac{J_r(k)}{(n-1)^r}\right) + x_n \frac{J_r(n)}{n^r}$$

for $n \geq 2$, and $\Delta\Upsilon_n^r(s) = x_1$ for $n = 1$.

If we define the matrices $E^{(p)} = \left(e_{nk}^{(p)}\right)$, $1 \leq p < \infty$ and $F = (f_{nk})$ by

$$e_{nk}^{(p)} = \begin{cases} -u_n^{1/q}, k = n-1, \\ u_n^{1/q}, k = n, \\ 0, \text{ otherwise} \end{cases} \tag{2.3}$$

and

$$f_{nk} = \begin{cases} \frac{1}{n^r}\sum\limits_{\substack{j=k\\ j|n}}^{n} J_r(j), 1 \leq k \leq n \\ 0, \qquad k > n \end{cases} \tag{2.4}$$

respectively, then we may restate $|\Upsilon^r|_p^u = (\ell_p)_{E^{(p)} \circ F}$ according to the notation matrix domain (1.2).

We compute inverse matrices by $\left(E^{(p)}\right)^{-1} = \hat{E}^{(p)}$ and $F^{-1} = \hat{F}$ of the matrices $E^{(p)}$ and F, respectively,

$$\hat{e}_{nk}^{(p)} = \begin{cases} u_k^{-1/q}, 1 \leq k \leq n \\ 0, \; k > n \end{cases} \tag{2.5}$$

and

$$\hat{f}_{nk} = \begin{cases} \dfrac{\mu\left(\frac{n}{k}\right) k^r}{J_r(n)}, \; k \mid n \text{ and } n \geq 1 \\ -\dfrac{\mu\left(\frac{n-1}{k}\right) k^r}{J_r(n-1)}, \; k \mid n-1 \text{ and } n \geq 2 \\ \dfrac{\mu(n)}{J_r(n)} - \dfrac{\mu(n-1)}{J_r(n-1)}, \; k = 1 \text{ and } n \geq 2 \\ 0, \; \text{otherwise.} \end{cases} \tag{2.6}$$

We may state topological properties and the α-, β- γ- duals of spaces of $|\Upsilon^r|_p^u$ for $1 \leq p < \infty$.

THEOREM 2.1. *Let $1 \leq p < \infty$ and (u_n) be a sequence of nonnegative numbers. Then, the space $|\Upsilon^r|_p^u$ is a BK-space with the norm*

$$\|x\|_{|\Upsilon^r|_p^u} = \left\|E^{(p)} o F(x)\right\|_{\ell_p}$$

and norm isomorphic to the space ℓ_p, that is, $|\Upsilon^r|_p^u \cong \ell_p$, where the matrices $E^{(p)}$and F are defined by (2.3) *and* (2.4) , *respectively.*

Proof. Let $1 \leq p < \infty$ and define operators $F : |\Upsilon^r|_p^u \to (\ell_p)_{E^{(p)}}$ and $E^{(p)} : (\ell_p)_{E^{(p)}} \to \ell_p$ by (2.3) and (2.4), respectively. It is clear that the composite function $E^{(p)} oF$ is a linear operator, since $E^{(p)}$ and F are linear operators. Also, since ℓ_p is the BK-space with its usual norm, $|\Upsilon^r|_p^u = (\ell_p)_{E^{(p)} oF}$ and $E^{(p)} oF$ is a triangle matrix, then we get that $|\Upsilon^r|_p^u$ is a BK-space for $p \geq 1$ from Theorem 4.3.2 of Wilansky [25].

Also, we should show the existence of a linear bijection between the spaces $|\Upsilon^r|_p^u$ and ℓ_p. For this, it is easy to see that the composite function $E^{(p)} oF$ is a linear bijective operator, since F and $E^{(p)}$ are linear bijective operators. In fact, it is trivial that $x = \theta$ whenever $E^{(p)} oF(x) = \theta$, then $E^{(p)} oF$ is injective. Also, to show surjective, given $z = (z_n) \in \ell_p$. Then, if we get

$$y = (y_n) = (\Sigma_{k=1}^n u_k^{-1/q} z_k) \in (\ell_p)_{E^{(p)}} \, ,$$

and define the sequence $x = (x_n)$ as

$$x_n = \sum_{k|n} \frac{\mu\left(\frac{n}{k}\right) k^r}{J_r(n)} y_k - \sum_{k|n-1} \frac{\mu\left(\frac{n-1}{k}\right) k^r}{J_r(n-1)} y_k \text{ for } n \geq 2 \text{ and } \; x_1 = y_1$$

then, $x = (x_n) \in |\Upsilon^r|_p^u$. Thus, we have $z = E^{(p)} oF(x)$, as asserted. Further, it preserves the norm, since

$$\left\|E^{(p)} oF(x)\right\|_{\ell_p} = \|x\|_{|\Upsilon^r|_p^u} \, .$$

Consequently, $E^{(p)} oF$ is a linear bijection and norm preserving, which completes the proof. □

Lemma 2.2. [22]

a-) *$A = (a_{nk}) \in (\ell_1, c)$ if and only if*

$$\lim_n a_{nk} \text{ exists for each } k \geq 1 \tag{2.7}$$

and

$$\sup_{n,k} |a_{nk}| < \infty. \tag{2.8}$$

b-) *$A = (a_{nk}) \in (\ell_1, \ell_\infty)$ if and only if (2.8) holds.*

c-) *Let $1 < p < \infty$. $A = (a_{nk}) \in (\ell_p, c)$ if and only if (2.7) holds and*

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}|^q < \infty. \tag{2.9}$$

d-) *Let $1 < p < \infty$. $A = (a_{nk}) \in (\ell_p, \ell_\infty)$ if and only if (2.9) holds.*

e-) *Let $1 < p < \infty$. $A = (a_{nk}) \in (\ell_p, \ell_1)$ if and only if*

$$\sup_{N \in \mathcal{N}} \sum_k \left| \sum_{n \in N} a_{nk} \right|^q < \infty,$$

$\mathcal{N}$ denotes the family of all finite subsets of $\mathbb{N}$.

Lemma 2.3. [11] *Let $1 \leq p < \infty$. $A = (a_{nk}) \in (\ell_1, \ell_p)$ if and only if*

$$\sup_k \sum_{n=1}^{\infty} |a_{nk}|^p < \infty.$$

Using following notations and Lemmas 2.2-2.3, we state following theorem related to α-, β- and γ-duals of the series space $|\Upsilon^r|_p^u$.

$$\Lambda_1 = \left\{ a = (a_j) \in \omega : \lim_m \left(\sum_{j=v}^{m} \sum_{k=v}^{j} a_j \hat{f}_{jk} \right) \text{ exists, for } v \geq 1 \right\},$$

$$\Lambda_2 = \left\{ a = (a_j) \in \omega : \sup_{m,v} \left| \sum_{j=v}^{m} \sum_{k=v}^{j} a_j \hat{f}_{jk} \right| < \infty \right\},$$

$$\Lambda_3 = \left\{ a = (a_j) \in \omega : \sup_m \sum_{v=1}^{m} \left| u_v^{-1/q} \sum_{j=v}^{m} \sum_{k=v}^{j} a_j \hat{f}_{jk} \right|^q < \infty \right\},$$

$$\Lambda_4 = \left\{ a = (a_j) \in \omega : \sup_j \sum_{n=j}^{\infty} \left| \sum_{k=j}^{n} a_n \hat{f}_{nk} \right| < \infty \right\},$$

$$\Lambda_5 = \left\{ a = (a_j) \in \omega : \sup_{N \in \mathcal{N}} \sum_j \left| u_j^{-1/q} \sum_{n \in N} \sum_{k=j}^{n} a_n \hat{f}_{nk} \right|^q < \infty \right\}.$$

Theorem 2.4. *Let $\hat{F} = \left(\hat{f}_{nk} \right)$ be defined by (2.6). Then, we have:*

a-) $\left(|\Upsilon^r|_p^u \right)^\beta = \Lambda_1 \cap \Lambda_3$ *for* $1 < p < \infty$ *and* $\left(|\Upsilon^r|_1 \right)^\beta = \Lambda_1 \cap \Lambda_2$ *for* $p = 1$.

b-) $\left(|\Upsilon^r|_p^u \right)^\gamma = \Lambda_3$ *for* $1 < p < \infty$ *and* $\left(|\Upsilon^r|_1 \right)^\gamma = \Lambda_2$ *for* $p = 1$.

c-) $\left(|\Upsilon^r|_p^u \right)^\alpha = \Lambda_5$ *for* $1 < p < \infty$ *and* $\left(|\Upsilon^r|_1 \right)^\alpha = \Lambda_4$ *for* $p = 1$.

Proof. **a-)** Let $1 < p < \infty$. $a = (a_j) \in \left(|\Upsilon^r|_p^u\right)^\beta$ if and only if $ax \in cs$ for every $x \in |\Upsilon^r|_p^u$. Let $y = F(x)$. Then, $z \in \ell_p$, where $z_n = u_n^{1/q}(y_n - y_{n-1})$ for $n \geq 1$, $y_0 = 0$, and also we have $y_n = \sum_{k=1}^n u_k^{-1/q} z_k$. Since we have

$$x_n = \sum_{k=1}^n \hat{f}_{nk} y_k,$$

we obtain that

$$\begin{aligned}\sum_{j=1}^m a_j x_j &= \sum_{j=1}^m a_j \sum_{k=1}^j \hat{f}_{jk} y_k = \sum_{k=1}^m \left\{ \sum_{j=k}^m a_j \hat{f}_{jk} \right\} y_k \\ &= \sum_{v=1}^m u_v^{-1/q} \left(\sum_{j=v}^m \sum_{k=v}^j a_j \hat{f}_{jk} \right) z_v = \sum_{v=1}^m h_{mv} z_v\end{aligned}$$

where the matrix $H = (h_{mv})$ is given by

$$h_{mv} = \begin{cases} u_v^{-1/q} \sum\limits_{j=v}^m \sum\limits_{k=v}^j a_j \hat{f}_{jk}, & 1 \leq v \leq m \\ 0, & v > m. \end{cases} \tag{2.10}$$

So it is written by part c) of Lemma 2.2 that $a \in \left(|\Upsilon^r|_p^u\right)^\beta$ iff $H \in (\ell_p, c)$, or equivalently, $a \in \Lambda_1 \cap \Lambda_3$, which completes the proof.

Since the proof for $p = 1$ is similar by using part a) of Lemma 2.2, the desired result is obtained.

b-) Let $1 < p < \infty$. Then, $a = (a_j) \in \left(|\Upsilon^r|_p^u\right)^\gamma$ if and only if $ax \in bs$ for every $x \in |\Upsilon^r|_p^u$. Also, $x \in |\Upsilon^r|_p^u$ iff $z \in \ell_p$, where $z_n = u_n^{1/q}(y_n - y_{n-1})$, $y_0 = 0$ and $y_n = \sum_{j=1}^n \left(\sum_{\substack{k=j \\ k|n}}^n \frac{J_r(k)}{n^r} \right) x_j$ for $n \geq 1$. Thus, since we have

$$\sum_{j=1}^m a_j x_j = \sum_{v=1}^m h_{mv} z_v$$

where $H = (h_{mv})$ is defined by (2.10), this implies that $a \in \left(|\Upsilon^r|_p^u\right)^\gamma$ iff $H \in (\ell_p, \ell_\infty)$. Hence, it follows from part d) of Lemma 2.2 that $a \in \Lambda_3$ as asserted.

Since the proof for $p = 1$ is similar by using part b) of Lemma 2.2, we omit the detail.

c-) Let $1 < p < \infty$. Then, $a \in \left(|\Upsilon^r|_p^u\right)^\alpha$ if and only if $ax \in \ell_1$ for every $x \in |\Upsilon^r|_p^u$. Then, we get

$$\begin{aligned}a_n x_n &= a_n \sum_{k=1}^n \hat{f}_{nk} y_k = a_n \sum_{k=1}^n \hat{f}_{nk} \sum_{j=1}^k u_j^{-1/q} z_j \\ &= a_n \sum_{j=1}^n u_j^{-1/q} \sum_{k=j}^n \hat{f}_{nk} z_j = \delta_n(z)\end{aligned}$$

where $\delta_n = (\delta_{nj})$ is defined by

$$\delta_{nj} = a_n u_j^{-1/q} \sum_{k=j}^{n} \hat{f}_{nk}.$$

So, $ax \in \ell_1$ for every $x \in |\Upsilon^r|_p^u$ if and only if $\delta(z) \in \ell_1$ for every $z \in \ell_p$, or equivalently, $a \in \left(|\Upsilon^r|_p^u\right)^\alpha$ iff $\delta \in (\ell_p, \ell_1)$, which gives $a \in \Lambda_5$ from Lemma 2.2, as desired. □

Since the proof for $p = 1$ is similar by using Lemma 2.3, we omit the detail.

THEOREM 2.5. *Let* $1 \le p < \infty$, $\hat{F} = \left(\hat{f}_{nk}\right)$ *and* $\tau^{(j)} = \left(\tau_v^{(j)}\right)$ *be defined by* (2.6) *and*

$$\tau_v^{(j)} = \begin{cases} u_j^{-1/q} \sum\limits_{k=j}^{v} \hat{f}_{vk}, & j \le v \\ 0, & j > v \end{cases}$$

respectively. Then, the sequence $\left(\tau_v^{(j)}\right)$ *is the Schauder base of the space* $|\Upsilon^r|_p^u$.

Proof. It is known that the sequence $\left(e^{(n)}\right)$ is a Schauder base for the space ℓ_p, where $e^{(n)}$ is a sequence with 1 in n-th place and zeros elsewhere. Because of the transformation $E^{(p)} \circ F$ defined in the proof of Theorem 2.1 is an isomorphism, the inverse image $\left(E^{(p)} \circ F\right)^{-1}$ of $\left(e^{(n)}\right)$ is a Schauder basis for $|\Upsilon^r|_p^u$. In fact, if $x \in |\Upsilon^r|_p^u$, then there exists $z \in \ell_p$ such that $z = \left(E^{(p)} \circ F\right)(x)$, so we can deduce from Theorem 2.1 that

$$\left\| x - \sum_{j=1}^{m} x_j \tau^{(j)} \right\|_{|\Upsilon^r|_p^u} = \left\| z - \sum_{j=1}^{m} z_j e^{(j)} \right\|_{\ell_p} \to 0 \text{ as } m \to \infty,$$

where $\left(E^{(p)} \circ F\right)^{-1}\left(e^{(j)}\right) = \tau^{(j)}, j \ge 1$. Furthermore, every $x \in |\Upsilon^r|_p^u$ has an unique representation of the form $x = \sum\limits_{j=1}^{\infty} x_j \tau^{(j)}$. □

3. Matrix Transformations Related to Space $|\Upsilon^r|_p^u$

In this section we give the characterizations of classes $\left(|\Upsilon^r|_p^u, X\right)$ and $\left(X, |\Upsilon^r|_p^u\right)$, where X is any given of classical sequence spaces ℓ_∞, c, c_0 and ℓ_1. Also, we point out that these characterizations reduce to results on absolute Euler totient series spaces which are given recently by İlkhan and Hazar [6].

LEMMA 3.1. [22]

a-) $A = (a_{nk}) \in (\ell_1, c_0)$ *if and only if* (2.8) *holds and*

(3.1) $$\lim_n a_{nk} = 0 \text{ for each } k \ge 1.$$

b-) *Let* $1 < p < \infty$. $A = (a_{nk}) \in (\ell_p, c_0)$ *if and only if* (2.9) *and* (3.1) *hold.*

THEOREM 3.2. *Let define the matrices* $\hat{F} = (\hat{f}_{nj})$ *and* $D = (d_{nj})$ *with* (2.6) *and*

$$d_{nj} = \sum_{v=j}^{\infty} a_{nv} \sum_{k=j}^{v} \hat{f}_{vk}$$

respectively, for all $n, j \in \mathbb{N}$. *Then, we have*

1. $A = (a_{nk}) \in (|\Upsilon^r|_1, \ell_\infty)$ *if and only if*

$$\lim_{m\to\infty} \sum_{j=v}^{m} a_{nj} \sum_{k=v}^{j} \hat{f}_{jk}\ ,\ \textit{exists for all } n, v \in \mathbb{N}, \tag{3.2}$$

$$\sup_{m,v} \left| \sum_{j=v}^{m} a_{nj} \sum_{k=v}^{j} \hat{f}_{jk} \right| < \infty \textit{ for each } n \in \mathbb{N}, \tag{3.3}$$

$$\sup_{n,j} |d_{nj}| < \infty. \tag{3.4}$$

2. $A = (a_{nk}) \in |\Upsilon^r|_1, c)$ *if and only if* (3.2), (3.3) *and* (3.4) *hold, and*

$$\lim_{n\to\infty} d_{nj} \textit{ exists for each } j \in \mathbb{N}.$$

3. $A = (a_{nk}) \in (|\Upsilon^r|_1, c_0)$ *if and only if* (3.2), (3.3) *and* (3.4) *hold, and*

$$\lim_{n\to\infty} d_{nj} = 0 \textit{ for each } j \in \mathbb{N}.$$

4. $A = (a_{nk}) \in (|\Upsilon^r|_1, \ell_1)$ *if and only if* (3.2) *and* (3.3) *hold, and*

$$\sup_j \sum_n |d_{nj}| < \infty.$$

Proof. The proof is given only for the first case. The proofs in the other cases are similar.

$A \in (|\Upsilon^r|_1, \ell_\infty)$ if and only if $Ax \in \ell_\infty$ for all $x \in |\Upsilon^r|_1$. Then the series $\sum_{k=0}^{\infty} a_{nk}x_k$ is convergent. So we have that $(a_{nk}) \in (|\Upsilon^r|_1)^\beta$ for each fixed $n \in \mathbb{N}$. By Theorem 2.4 , we obtain that

$$\lim_{m\to\infty} \sum_{j=v}^{m} a_{nj} \sum_{k=v}^{j} \hat{f}_{jk}$$

exists for each $n, v \in \mathbb{N}$ and

$$\sup_{m,v} \left| \sum_{j=v}^{m} a_{nj} \sum_{k=v}^{j} \hat{f}_{jk} \right| < \infty \text{ for each } n \in \mathbb{N}.$$

That is, (3.2) and (3.3) hold. Now, to prove the necessity and sufficiency of (3.4), let $x \in |\Upsilon^r|_1$ and consider the linear operator $E^{(1)} \circ F : |\Upsilon^r|_1 \to \ell_1$. Let $y = Fx$ and $z = \Delta y = (E^{(1)} \circ F)x$ for any $x \in |\Upsilon^r|_1$. Then we have $y_n = \sum_{j=1}^{n} z_j$ and

$x_v = \sum\limits_{k=1}^{v} \hat{f}_{vk} y_k$. Hence we can write that

$$\begin{aligned}\sum_{v=1}^{m} a_{nv}x_v &= \sum_{v=1}^{m} a_{nv} \sum_{k=1}^{v} \hat{f}_{vk} y_k = \sum_{v=1}^{m} a_{nv} \sum_{k=1}^{v} \hat{f}_{vk} \sum_{j=1}^{k} z_j \\ &= \sum_{v=1}^{m} a_{nv} \sum_{j=1}^{v} \sum_{k=j}^{v} \hat{f}_{vk} z_j \\ &= \sum_{j=1}^{m} \left(\sum_{v=j}^{m} a_{nv} \sum_{k=j}^{v} \hat{f}_{vk} \right) z_j \\ &= \sum_{j=1}^{m} d_{mj}^{(n)} z_j = D_m^{(n)}(z)\end{aligned}$$

where $D_m^{(n)} = \left(d_{mj}^{(n)} \right)$ is defined by

$$d_{mj}^{(n)} = \begin{cases} \sum_{v=j}^{m} a_{nv} \sum_{k=j}^{v} \hat{f}_{vk} & , \quad 1 \leq j \leq m \\ 0 & , \quad j > m \end{cases}$$

for each $n \in \mathbb{N}$. Also, it follows from (3.2) and (3.3) that $D_m^{(n)} = (d_{mj}^{(n)}) \in (\ell_1, c)$. Then the series $D_m^{(n)}(z) = \sum_{j=1}^{\infty} d_{mj}^{(n)} z_j$ converges uniformly in m for all $z \in \ell_1$ and so it can be written that $\lim_{m\to\infty} D_m^{(n)}(z) = \sum_{j=1}^{\infty} \lim_{m\to\infty} d_{mj}^{(n)} z_j$. Thus, we obtain that

$$A_n(x) = \lim_{m\to\infty} D_m^{(n)}(z) = \sum_{j=1}^{\infty} (\lim_{m\to\infty} d_{mj}^{(n)}) z_j = \sum_{j=1}^{\infty} d_{nj} z_j = D_n(z),$$

where $d_{nj} = \lim_{m\to\infty} d_{mj}^{(n)}$.

This yields that $Ax \in \ell_\infty$ for $x \in |\Upsilon^r|_1$ if and only if $Dz \in \ell_\infty$ for $z \in \ell_1$. We conclude that $A \in (|\Upsilon^r|_1, \ell_\infty)$ if and only if (3.2) and (3.3) hold and also $D \in (\ell_1, \ell_\infty)$ which means (3.4). This completes the proof. □

If we take $r = 1$ in the Theorem 3.2, we have that this characterization reduces result in the [6].

Theorem 3.3. *Let $1 < p < \infty$ and define the matrices $\hat{F} = (\hat{f}_{nj})$ and $D^p = (d_{nj}^p)$ with* (2.6) *and*

$$d_{nj}^p = u_j^{-1/q} \lim_{m\to\infty} \sum_{v=j}^{m} a_{nv} \sum_{k=j}^{v} \hat{f}_{vk},$$

respectively, for all $n, j \in \mathbb{N}$.

1. $A = (a_{nk}) \in (|\Upsilon^r|_p^u, \ell_\infty)$ *if and only if* (3.2) *holds and*

$$\sup_m \sum_{v=1}^{m} \left| u_v^{-1/q} \sum_{j=v}^{m} a_{nj} \sum_{k=v}^{j} \hat{f}_{jk} \right|^q < \infty \text{ for each } n \in \mathbb{N}, \tag{3.5}$$

$$\sup_n \sum_{j=1}^{\infty} \left| d_{nj}^p \right|^q < \infty. \tag{3.6}$$

2. $A=(a_{nk})\in(|\,|\Upsilon^r|_p^u, c)$ *if and only if* (3.2), (3.5) *and* (3.6) *hold, and*

$$\lim_{n\to\infty} d_{nj}^p \text{ exists for each } j\in\mathbb{N}.$$

3. $A=(a_{nk})\in(|\Upsilon^r|_p^u, c_0)$ *if and only if* (3.2), (3.5) *and* (3.6) *hold, and*

$$\lim_{n\to\infty} d_{nj}^p = 0 \text{ for each } j\in\mathbb{N}.$$

4. $A=(a_{nk})\in(|\Upsilon^r|_p^u, \ell_1)$ *if and only if* (3.2) *and* (3.5) *hold, and*

$$\sup_{N\in\mathcal{N}}\sum_j\left|\sum_{n\in N} d_{nj}^p\right|^q<\infty.$$

Proof. The proof is given only for the first case. The proofs in the other cases are similar.

$A\in(|\Upsilon^r|_p^u, \ell_\infty)$ if and only if $Ax\in\ell_\infty$ for all $x\in|\Upsilon^r|_p^u$. Then, the series $\sum_{k=1}^\infty a_{nk}x_k$ is convergent. So, we have that $(a_{nk})\in(|\Upsilon^r|_p^u)^\beta$ for each fixed $n\in\mathbb{N}$. From Theorem 2.4, we can see that (3.2) holds and

$$\sup_m\sum_{v=1}^m\left|u_v^{-1/q}\sum_{j=v}^m a_{nj}\sum_{k=v}^j \hat{f}_{jk}\right|^q<\infty$$

for each $n\in\mathbb{N}$, which says that (3.5) holds.

Now, to prove the necessity and sufficiency of (3.6), let $x\in|\Upsilon^r|_p^u$ and consider the linear operator $E^{(p)}\circ F:|\Upsilon^r|_p^u\to\ell_p$ defined by $(E^{(p)}\circ F)_n(x)=u_n^{1/q}(F_n(x)-F_{n-1}(x))$, $n\geq 1$ and $F_0(x)=0$. Let $y=Fx$ and $z=(E^{(p)}\circ F)x$ for any $x\in|\Upsilon^r|_p^u$. Then we have $y_k=\sum_{j=1}^k u_j^{-1/q}z_j$ and $x_v=\sum\limits_{k=1}^v \hat{f}_{vk}y_k$. Hence we can write that

$$\begin{aligned}
\sum_{v=1}^m a_{nv}x_v &= \sum_{v=1}^m a_{nv}\sum_{k=1}^v \hat{f}_{vk}y_k=\sum_{v=1}^m a_{nv}\sum_{k=1}^v \hat{f}_{vk}\sum_{j=1}^k u_j^{-1/q}z_j\\
&= \sum_{v=1}^m a_{nv}\sum_{j=1}^v\sum_{k=j}^v \hat{f}_{vk}u_j^{-1/q}z_j\\
&= \sum_{j=1}^m\left(u_j^{-1/q}\sum_{v=j}^m a_{nv}\sum_{k=j}^v \hat{f}_{vk}\right)z_j\\
&= \sum_{j=1}^m d_{mj}^{(n)}z_j=D_m^{(n)}(z)
\end{aligned}$$

where $D_m^{(n)}=\left(d_{mj}^{(n)}\right)$ is defined by

$$d_{mj}^{(n)}=\begin{cases} u_j^{-1/q}\sum_{v=j}^m a_{nv}\sum_{k=j}^v \hat{f}_{vk} & ,\quad 1\leq j\leq m\\ 0 & ,\quad j>m\end{cases}$$

for each $n\in\mathbb{N}$. Also, it follows from (3.2) and (3.5) that $D_m^{(n)}=(d_{mj}^{(n)})\in(\ell_p, c)$. Then the series $D_m^{(n)}(z)=\sum_{j=1}^\infty d_{mj}^{(n)}z_j$ converges uniformly in m for all $z\in\ell_p$ and so it can be written that $\lim_{m\to\infty} D_m^{(n)}(z) = \sum_{j=1}^{\infty} \lim_{m\to\infty} d_{mj}^{(n)} z_j$. Thus, we obtain that

$$A_n(x) = \lim_{m\to\infty} D_m^{(n)}(z) = \sum_{j=1}^{\infty} (\lim_{m\to\infty} d_{mj}^{(n)}) z_j = \sum_{j=1}^{\infty} d_{nj}^{(p)} z_j = D_n^{(p)}(z),$$

where $d_{nj}^{(p)} = \lim_{m\to\infty} d_{mj}^{(n)}$.

This yields that $Ax \in \ell_\infty$ for $x \in |\Upsilon^r|_p^u$ if and only if $D^{(p)} z \in \ell_\infty$ for $z \in \ell_p$. We conclude that $A \in (|\Upsilon^r|_p^u, \ell_\infty)$ if and only if (3.2) and (3.5) hold and also $D^{(p)} \in (\ell_p, \ell_\infty)$ which means (3.6). This completes the proof.

If we take $r = 1$ in the Theorem 3.3, we have that this characterization reduces result in the [6].

Now, we give the characterizations of the matrix classes from the classical spaces ℓ_∞, c, c_0 and ℓ_1 to the spaces $|\Upsilon^r|_p^u$ for $1 \le p < \infty$. We need the following lemma to prove our results. □

LEMMA 3.4. [22]

a) *$A = (a_{nk}) \in (\ell_\infty, \ell_1) = (c, \ell_1) = (c_0, \ell_1)$ if and only if*

$$\sup_{K\in\mathcal{N}} \sum_{n=1}^{\infty} \left| \sum_{k\in K} a_{nk} \right| < \infty.$$

b) *Let $p > 1$. $A = (a_{nk}) \in (\ell_\infty, \ell_p) = (c, \ell_p) = (c_0, \ell_p)$ if and only if*

$$\sup_{K\in\mathcal{N}} \sum_{n=1}^{\infty} \left| \sum_{k\in K} a_{nk} \right|^p < \infty.$$

THEOREM 3.5. *Let $A = (a_{nk})$ be an infinite matrix. Then we have:*

1. *$A \in (\ell_\infty, |\Upsilon^r|_1) = (c, |\Upsilon^r|_1) = (c_0, |\Upsilon^r|_1)$ if and only if*

$$\sup_{K\in\mathcal{N}} \sum_{n=1}^{\infty} \left| \sum_{v\in K} \left\{ \sum_{k=1}^{n} \left(\sum_{j=k, j|n}^{n} \frac{J_r(j)}{n^r} - \sum_{j=k, j|n-1}^{n-1} \frac{J_r(j)}{(n-1)^r} \right) a_{kv} \right\} \right| < \infty.$$

2. *$A \in (\ell_1, |\Upsilon^r|_1)$ if and only if*

$$\sup_{v} \sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} \left(\sum_{j=k, j|n}^{n} \frac{J_r(j)}{n^r} - \sum_{j=k, j|n-1}^{n-1} \frac{J_r(j)}{(n-1)^r} \right) a_{kv} \right| < \infty.$$

Proof. The proof is given only for the matrix class $(\ell_\infty, |\Upsilon^r|_1)$. One can see that the proof of the other cases are similarly. Consider the matrix $H^1 = (h_{nv}^1)$ defined as

$$h_{nv}^1 = \sum_{k=1}^{n} \left(\sum_{j=k, j|n}^{n} \frac{J_r(j)}{n^r} - \sum_{j=k, j|n-1}^{n-1} \frac{J_r(j)}{(n-1)^r} \right) a_{kv}$$

for $n \geq 2$ and $h^1_{nv} = \sum_{k=1}^{n} \sum_{j=k,j|n}^{n} \frac{J_r(j)}{n^r} a_{kv}$ for $n = 1$. Let $x = (x_n) \in \ell_\infty$. Further, we have the following equality:

$$\begin{aligned}\sum_{v=1}^{\infty} h^1_{nv} x_v &= \sum_{v=1}^{\infty} \left(\sum_{k=1}^{n} \left(\sum_{j=k,j|n}^{n} \frac{J_r(j)}{n^r} - \sum_{j=k,j|n-1}^{n-1} \frac{J_r(j)}{(n-1)^r} \right) a_{kv} \right) x_v \\ &= \sum_{k=1}^{n} \sum_{v=1}^{\infty} a_{kv} x_v \sum_{j=k,j|n}^{n} \frac{J_r(j)}{n^r} - \sum_{k=1}^{n-1} \sum_{v=1}^{\infty} a_{kv} x_v \sum_{j=k,j|n-1}^{n-1} \frac{J_r(j)}{(n-1)^r} \\ &= F_n(Ax) - F_{n-1}(Ax)\end{aligned}$$

for $n \geq 1$ and $F_0(Ax) = 0$.

This implies that $H^1_n(x) = (E^{(1)} \circ F)_n(Ax)$ for all $n \in \mathbb{N}$. Hence, it follows that $Ax \in |\Upsilon^r|_1$ for any $x \in \ell_\infty$ if and only if $H^1 x \in \ell_1$ for any $x \in \ell_\infty$.

Since we have $H^1 \in (\ell_\infty, \ell_1)$, we conclude that

$$\sup_{K \in \mathcal{N}} \sum_{n=1}^{\infty} \left| \sum_{v \in K} \left\{ \sum_{k=1}^{n} \left(\sum_{j=k,j|n}^{n} \frac{J_r(j)}{n^r} - \sum_{j=k,j|n-1}^{n-1} \frac{J_r(j)}{(n-1)^r} \right) a_{kv} \right\} \right| < \infty.$$

□

Theorem 3.6. *Let $A = (a_{nk})$ be an infinite matrix and $1 < p < \infty$.*

1. *$A \in (\ell_\infty, |\Upsilon^r|^u_p) = (c, |\Upsilon^r|^u_p) = (c_0, |\Upsilon^r|^u_p)$ if and only if*

$$\sup_{K \in \mathcal{N}} \sum_{n=1}^{\infty} \left| \sum_{v \in K} \left\{ \sum_{k=1}^{n} u_n^{1/q} \left(\sum_{j=k,j|n}^{n} \frac{J_r(j)}{n^r} - \sum_{j=k,j|n-1}^{n-1} \frac{J_r(j)}{(n-1)^r} \right) a_{kv} \right\} \right|^p < \infty.$$

2. *$A \in (\ell_1, |\Upsilon^r|^u_p)$ if and only if*

$$\sup_{v} \sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} u_n^{1/q} \left(\sum_{j=k,j|n}^{n} \frac{J_r(j)}{n^r} - \sum_{j=k,j|n-1}^{n-1} \frac{J_r(j)}{(n-1)^r} \right) a_{kv} \right|^p < \infty.$$

Proof. The proof is given only for the matrix class $(\ell_1, |\Upsilon^r|^u_p)$ since the other cases can be proved similarly. Let $p > 1$. Consider the matrix $H^p = (h^p_{nk})$ defined as

$$h^p_{nv} = \sum_{k=1}^{n} u_n^{1/q} \left(\sum_{j=k,j|n}^{n} \frac{J_r(j)}{n^r} - \sum_{j=k,j|n-1}^{n-1} \frac{J_r(j)}{(n-1)^r} \right) a_{kv}$$

for $n \geq 2$ and $h^p_{nv} = u_n^{1/q} \sum_{k=1}^{n} \sum_{j=k,j|n}^{n} \frac{J_r(j)}{n^r} a_{kv}$ for $n = 1$. Let $x = (x_n) \in \ell_1$. We obtain the following equality:

$$\begin{aligned}\sum_{v=1}^{\infty} h^p_{nv} x_v &= \sum_{v=1}^{\infty} \left(\sum_{k=1}^{n} u_n^{1/q} \left(\sum_{j=k,j|n}^{n} \frac{J_r(j)}{n^r} - \sum_{j=k,j|n-1}^{n-1} \frac{J_r(j)}{(n-1)^r} \right) a_{kv} \right) x_v \\ &= u_n^{1/q} \left(\sum_{k=1}^{n} \sum_{v=1}^{\infty} a_{kv} x_v \sum_{j=k,j|n}^{n} \frac{J_r(j)}{n^r} - \sum_{k=1}^{n-1} \sum_{v=1}^{\infty} a_{kv} x_v \sum_{j=k,j|n-1}^{n-1} \frac{J_r(j)}{(n-1)^r} \right) \\ &= u_n^{1/q} \left(F_n(Ax) - F_{n-1}(Ax) \right),\end{aligned}$$

for $n \geq 1$ and $F_0(Ax) = 0$.

This implies that $H_n^p(x) = (E^{(p)} \circ F)_n(Ax)$ for all $n \in \mathbb{N}$. Hence, it follows that $Ax \in |\Upsilon^r|_p^u$ for any $x \in \ell_1$ if and only if $H^p x \in \ell_p$ for any $x \in \ell_1$. Since we have $H^p \in (\ell_1, \ell_p)$,we conclude that

$$\sup_v \sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} u_n^{1/q} \left(\sum_{j=k, j|n}^{n} \frac{J_r(j)}{n^r} - \sum_{j=k, j|n-1}^{n-1} \frac{J_r(j)}{(n-1)^r} \right) a_{kv} \right|^p < \infty.$$

□

Conclusions

Summability theory deals with generalizing the concept of convergence of sequence or series by assigning a limit for non-convergent sequence or series. In order to do so, infinite triangular special matrices are used. Matrices corresponding to arithmetic functions in number theory have been widely used to construct these triangular matrices. The most well-known arithmetic functions corresponding to these triangular matrices are Jordan totient and Euler totient functions. In this paper, a new Banach series space $|\Upsilon^r|_p^u$ is defined by using concept of the Jordan totient matrix operator and absolute summability. Also, some topological properties of the space $|\Upsilon^r|_p^u$ are given and α-, β- and γ- duals of this space are computed. The characterizations of matrix transformations in the related spaces are presented. By using the new series space defined the Jordan totient matrix, many impressive results can be obtained in the theory of series spaces and matrix transformations.

Acknowledgments

We would like to thank the anonymous reviewers and the editor for their very helpful comments and valuable suggestions.

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