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# ON GENERALIZED SHEN'S SQUARE METRIC

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Abstract. In this paper, following the pullback approach to global Finsler geometry, we investigate a coordinate-free study of Shen square metric in a more general manner. Precisely, for a Finsler metric $(M, L)$ admitting a concurrent $\pi$-vector field, we study some geometric objects associated with $\widetilde{L}(x, y) = \frac{(L+\mathfrak{B})^2}{L}$ in terms of the objects of $L$, where $\mathfrak{B}$ is the associated 1-form. For example, we find the geodesic spray, Barthel connection and Berwald connection of $\widetilde{L}(x, y)$. Moreover, we calculate the curvature of the Barthel connection of $\widetilde{L}$. We characterize the non-degeneracy of the metric tensor of $\widetilde{L}(x, y)$.

## 1. Introduction

For a Finsler manifold $(M, L)$ and a 1-form $\beta$ on $M$, there is a very rich class of special Finsler spaces called the $(\alpha, \beta)$-metrics. Numerous research articles and applications on these spaces can bound in the literatire, for example, we refer to [7,10,11,14]. One of these spaces or metrics is the Shen square metric $L(x, y) = \frac{(\alpha+\beta)^2}{\alpha}$ which plays an important role in Finsler geometry (see [2,13,14]).

Generally, the theory of special Finsler spaces is a rich area of research and has many applications, for example, in Physics and Biology. The $\pi$-tensor fields (torsions and curvatures) related to the Cartan connection satisfy special forms, which is the source of many special Finsler spaces in Finsler geometry. Several researchers have studied special Finsler spaces locally, that is, using local coordinates. For instance, M. Masumoto [1,8,9,18] and others. On the other hand, to the best or our knowledge, there are few intrinsic investigations of such spaces. A. Tamim, L. Youssef, and others who made some contributions in this direction (see [15–17,21,24]).

In the study of the $(\alpha, \beta)$-metrics, we used to use the notation $\beta$ for the 1-form giving on the manifold $M$. But since in the pullback approach to coordinate-free Finsler geometry, we use the notation $\beta$ in the settings of the geometry of that approach, so in this work we have use another notation for the 1-form. In this paper, we investigate an intrinsic study of Shen square metric which is a specific $(\alpha, \beta)$-metric with replcing

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the Riemannian metric with a Fineslerian one and we call it a generalized Shen square metric.

In this paper, following the pullback formalism to Finsler geoemtry, we provide a coordinate-free study of generalized Shen square metric with special one $\pi$-form. First, by the concept of generalized Shen square metric we mean the deformation of a Finsler metric $L$ (not necessarily Riemannian) by a one form $B$, that is, $\widetilde{L} = \frac{(L+B)^2}{L}$ and $L$ is Finslerian. Now, in this work, we consider a Finsler space $(M, L)$ that admits a concurrent $\pi$-vector field $\overline{p}$, and then we compute the corresponding $\pi$-form $\mathbf{B} := i_{\overline{p}}\, g$, where $g$ is the metric tensor of $L$, and hence the attached one form $\mathfrak{B}(x, y) := \mathbf{B}(\overline{\eta})$. Then, we we consider the generalized Shen square deformation

$$\widetilde{L}(x, y) = \frac{(L(x, y) + \mathfrak{B}(x, y))^2}{L(x, y)}. \tag{1}$$

Within the generalized Shen square metric (1), we calculate intrinsically some geometric objects attached to $\widetilde{L}$. Namely, the supporting form $\widetilde{\ell}$, the angular metric tensors $\widetilde{\hbar}$, the Finsler metric $\widetilde{g}$, and the Cartan torsion $\widetilde{\mathbf{T}}$. Hence, we characterize the non-degenerate property of the metric tensor $\widetilde{g}$, that is, $\widetilde{g}$ is non-degenerate if and only if

$$L^2(1 + 2p^2) - 3\mathfrak{B}^2 \neq 0,$$

where $p^2 := g(\overline{p}, \overline{p})$.

On the other hand, we obtsin the relationship that relates the two associated Barthel connections $\Gamma$ and $\widetilde{\Gamma}$, as well as the corresponding canonical sprays to this change. Moreover, the curvature tensor for the Barthel connection $\widetilde{\Gamma}$ is investigated. Also, the associated canonical sprays $G$ and $\widetilde{G}$ are related by

$$\widetilde{G} = G - \frac{2L^2(2\mathfrak{B} - L)}{L^2(1 + 2p^2) - 3\mathfrak{B}^2}\,\mathcal{C} + \frac{2L^4}{L^2(1 + 2p^2) - 3\mathfrak{B}^2}\,\gamma\overline{p},$$

where, $\mathcal{C}$ is the Liouville vector field.

As an example of a Finsler metric $(M, L)$ that admits a concurrent vector field, let $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 \neq 0\}$ and $L$ be a conic Finsler metric given by

$$L = \sqrt{x_2^2\left(\sqrt{y_1^2 + x_1^2 y_3^2} + \ y_3\right)^2 + y_2^2}.$$

Moreover, the components of the corresponding $\pi$-form $\mathbf{B}$ are given by $\mathbf{B}_1 = 0$, $\mathbf{B}_2 = x_2$, $\mathbf{B}_3 = 0$, and hence the associated one form $\mathfrak{B}(x, y)$ becomes $\mathfrak{B}(x, y) = x_2 y_2$. Therefore, we have

$$\widetilde{L}(x, y) = \frac{(L(x, y) + \mathfrak{B}(x, y))^2}{L(x, y)} = \frac{\left\{\sqrt{x_2^2\left(\sqrt{y_1^2 + x_1^2 y_3^2} + \ y_3\right)^2 + y_2^2} + x_2 y_2\right\}^2}{\sqrt{x_2^2\left(\sqrt{y_1^2 + x_1^2 y_3^2} + \ y_3\right)^2 + y_2^2}},$$

which defines a special generalized Shen square metric over $M = \mathbb{R}^3$.

## 2. Notations and Preliminaries

Let $M$ be an $n$-dimensional differentiable manifold. Assume that the tangent bundle $(TM, \pi, M)$ and its differential $(TTM, d\pi, TM)$. The vertical bundle $V(TM)$ of $TM$ is denoted by $\ker(d\pi)$, and the pullback bundle of the tangent bundle is denoted by $\pi^{-1}(TM)$. The short exact sequence of vector bundle morphisms is given by [4] as follows

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} TTM \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

where $\mathcal{T}M$ is the slit tangent bundle, $\gamma$ is the natural injection and $\rho := (\pi_{TM}, d\pi)$.

The tangent structure $J$ of $TM$ or the vertical endomorphism defined by $J = \gamma \circ \rho$. Moreover, $C^\infty(TM)$ denotes the algebra of $C^\infty$ functions on $TM$ and $\mathfrak{X}(\pi(M))$ denotes the $C^\infty(TM)$-module of smooth sections of the pullback bundle $\pi^{-1}(TM)$. The elements of $\mathfrak{X}(\pi(M))$ will referred as $\pi$-vector fields and marked by barred letters $\overline{X}$.

The Liouville vector field $\mathcal{C}$ is given by $\mathcal{C} := \gamma\,\overline{\eta}$, where $\overline{\eta}(u) = (u, u)$, for all $u$ in the slit tangent bundle $\mathcal{T}M := TM \backslash \{0\}$, and called the fundamental $\pi$-vector field .

We recall some basics and facts about the Klein-Grifone formalism to coordinate-free Finsler geometry. We refer to [4–6], for further information.

A nonlinear connection $\Gamma$ on a manifold $M$ is a vector 1-form on $TM$, $C^\infty$ on $\mathcal{T}M$, and $C^0$ on $TM$, wherein

$$J\Gamma = J, \;\; \Gamma J = -J.$$

In this case, the horizontal projector $h$ and the vertical projector $v$ of $\Gamma$ are defined, receptively, by

$$h := \frac{1}{2}(I + \Gamma), \quad v := \frac{1}{2}(I - \Gamma).$$

Also, the torsion $t$ and the curvature $\mathfrak{R}$ of the connection $\Gamma$ are given, respectively, by

$$t := \frac{1}{2}[J, \Gamma], \quad \mathfrak{R} := -\frac{1}{2}[h, h].$$

Now, for a linear connection $D$ on $\pi^{-1}(TM)$, the attached connection map $K$ is given by $K : TTM \longrightarrow \pi^{-1}(TM) : X \longmapsto D_X\overline{\eta}$. In addition, the horizontal space $H_u(TM)$ to $M$ at $u$ is $H_u(TM) := \{X \in T_u(TM) : K(X) = 0\}$. The connection $D$ is referred to be regular if

$$T_u(TM) = V_u(TM) \oplus H_u(TM) \;\; \forall\, u \in TM.$$

Let $D$ be a regular connection on $M$, then the maps $\rho|_{H(TM)}$ and $K|_{V(TM)}$ are vector bundle isomorphisms. In this case, the map $\beta := (\rho|_{H(TM)})^{-1}$ is called the horizontal map of $D$.

Let $D$ be a regular connection with the horizontal map $\beta$ and the attached classical torsion (resp. curvature) tensor field $\mathbf{T}$ (resp. $\mathbf{K}$). Then, the associated covariant derivatives as well as the torsion and curvature tensors are defined or given as follows:

1. For a $\pi$-tensor field $A$ of type $(0, p)$, the $h$- and $v$-covariant derivatives $\overset{h}{D}$ and $\overset{v}{D}$ are defined, respectively, by:

$$(\overset{h}{D} A)(\overline{X}, \overline{X}_1, ..., \overline{X}_p) \;\; := \;\; (D_{\beta\overline{X}}A)(\overline{X}_1, ..., \overline{X}_p).$$

$$(\overset{v}{D} A)(\overline{X}, \overline{X}_1, ..., \overline{X}_p) \;\; := \;\; (D_{\gamma\overline{X}}A)(\overline{X}_1, ..., \overline{X}_p).$$

2. The (h)h-torsion $Q(\overline{X},\overline{Y}) := \mathbf{T}(\beta\overline{X},\beta\overline{Y})$, the (h)hv-torsion $T(\overline{X},\overline{Y}) := \mathbf{T}(\gamma\overline{X},\beta\overline{Y})$, and the (h)v-torsion $V(\overline{X},\overline{Y}) := \mathbf{T}(\gamma\overline{X},\gamma\overline{Y})$.
3. The horizontal curvature $R(\overline{X},\overline{Y})\overline{Z} := \mathbf{K}(\beta\overline{X},\beta\overline{Y})\overline{Z}$, the mixed curvature $R(\overline{X},\overline{Y})\overline{Z} := \mathbf{K}(\beta\overline{X},\beta\overline{Y})\overline{Z}$, and the vertical curvature $S(\overline{X},\overline{Y})\overline{Z} := \mathbf{K}(\gamma\overline{X},\gamma\overline{Y})\overline{Z}$.
4. The (v)h-torsion $\widehat{R}(\overline{X},\overline{Y}) := R(\overline{X},\overline{Y})\overline{\eta}$, the (v)hv-torsion $\widehat{P}(\overline{X},\overline{Y}) := P(\overline{X},\overline{Y})\overline{\eta}$, and the (v)v-torsion $\widehat{S}(\overline{X},\overline{Y}) := S(\overline{X},\overline{Y})\overline{\eta}$.

We define a Finsler manifold as follows.

Definition 2.1. A Finsler manifold (or, Finsler space ) of dimension $n$ is a pair $(M, L)$, where $M$ is a $n$-dimensional smooth manifold and $L$ is a map

$$L : TM \longrightarrow \mathbb{R},$$

such that the following conditions hold:

**(a):** $L(u) > 0$ for all $u \in \mathcal{T}M$ and $L(0) = 0$,
**(b):** $L$ is $C^\infty$ on $\mathcal{T}M$, $C^0$ on $TM$,
**(c):** $L$ is homogenous of degree 1 in the directional variable $y$: $\mathcal{L}_\mathcal{C} L = L$,
**(d):** The exterior 2-form $\Omega := dd_J E$ has maximal rank (non-degenerate), where $E := L^2/2$. The Finsler metric $g$ attached to $L$ on $\pi^{-1}(TM)$ is defined as follows

$$g(\rho X, \rho Y) := \Omega(JX, Y), \ \forall X, Y \in \mathfrak{X}(TM). \tag{2}$$

$L$ is called the *Finsler structure* and $E$ is the *energy function* corresponding with $L$. We will use the notation $(M, L)$ for a Finsler manifold.

Remark 2.2. When $L$ is defined on a conic subset $U$ of $\mathcal{T}M$ (that is, if $p \in U$ and $\lambda > 0$, then $\lambda p \in U$), then $(M, L)$ is called *conic Finsler manifold.*

A semi-spray is a vector field $G$ on $TM$ that is $C^\infty$ on $\mathcal{T}M$, and $C^1$ on $TM$, as well as $JG = \mathcal{C}$. A spray is a homogeneous semispray $G$ of degree 2 in the directional argument ($[\mathcal{C}, G] = G$).

Proposition 2.3. [5,6] *For a Finsler space $(M, L)$, we associated*

***(a):*** *The canonical spray $G$: $i_G\, dd_J E = -dE$.*
***(b):*** *The Barthel connection $\Gamma$: $\Gamma = [J, G]$.*

Now, we present the following theorem that provides the existence and uniqueness of Cartan connection.

Theorem 2.4. [22] *Let $(M, L)$ be a Finsler space with the attached metric tensor $g$ to the Finsler function $L$. Then, $(M, L)$ admits a unique regular connection $\nabla$ with the properties:*

**(i):** *$\nabla g = 0$, that is, $\nabla$ is metrical.*
**(ii):** *The (h)h-torsion of $\nabla$ vanishes, that is, $Q = 0$,*
**(iii):** *$g(T(\overline{X},\overline{Y}),\overline{Z}) = g(T(\overline{X},\overline{Z}),\overline{Y})$, where the (h)hv-torsion $T$ of $\nabla$.*

*The connection $\nabla$ is referred as the Cartan connection attached to the Finsler space $(M, L)$.*

Let's provide the following lemma which is required for subsequent use.

Lemma 2.5. [22] *Let $(M, L)$ be a Finsler space and $\beta$ be the horizontal map of the Cartan connection $\nabla$. Then, the metricity of the Cartan and Berwald connections is characterized by:*

*(a)* $(D^{\circ}_{\gamma\overline{X}}\, g)(\overline{Y},\overline{Z}) = 2\mathbf{T}(\overline{X},\overline{Y},\overline{Z})$, $\nabla_{\gamma\overline{X}}\, g = 0$.

*(b)* $(D^{\circ}_{\beta\overline{X}} g)(\overline{Y},\overline{Z}) = -2\widehat{\mathbf{P}}(\overline{X},\overline{Y},\overline{Z})$, $\nabla_{\beta\overline{X}}\, g = 0$.

*Where* $\widehat{\mathbf{P}}$ *is the* $(v)hv$*-torsion of type* $(0,3)$ *defined by* $\widehat{\mathbf{P}}(\overline{X},\overline{Y},\overline{Z}) := g(\widehat{P}(\overline{X},\overline{Y}),\overline{Z})$ *and* $\widehat{P}$ *is the* $(v)hv$*-torsion tensor of Cartan connection.*

For more details about pullback formalism to coordinate-free Finsler geometry, we refer for example, to [12, 15, 19, 25, 26].

## 3. Generalized Shen square metric

In this section, we introduce an intrinsic study of Shen square metric in a more general settings. The study of such kind of Finsler metric in coordinate-free fashion and to avoid the complications of the coordinate-free formalae, we restrict ourselves to a special 1-form. Precisely, we present the following definition.

Definition 3.1. Assume that the Finsler space $(M, L)$ provides a concurrent $\pi$-vector field $\overline{p}(x)$ with the associated $\pi$-form $\mathbf{B}$. Consider the following deformation

$$\widetilde{L}(x,y) = \frac{(L(x,y) + \mathfrak{B}(x,y))^2}{L(x,y)}, \tag{3}$$

with $\mathfrak{B}(x,y) := g(\overline{p},\overline{\eta}) =: \mathbf{B}(\overline{\eta})$, and $g$ is the metric tensor attached to $L$. Assuming that $\widetilde{L}$ is Finsler structure on $M$,then it will be called a generalized Shen square metric.

In [23], Nabil et al. investigated the concept of a concurrent $\pi$-vector field, intrinsically, in Finsler geometry. Furthermore, some geometric consequences and properties of concurrent $\pi$-vector fields are established. We review the definitions and features of the concurrent $\pi$-vector field and its corresponding $\pi$-form.

Definition 3.2 ( [23]). Assume $(M, L)$ is a Finsler space. A concurrent $\pi$-vector field is a $\pi$-vector field $\overline{p} \in \mathfrak{X}(\pi(M))$ such that

$$\nabla_{\beta\overline{X}}\, \overline{p} = -\overline{X} = D^{\circ}_{\beta\overline{X}}\, \overline{p}, \quad \nabla_{\gamma\overline{X}}\, \overline{p} = 0 = D^{\circ}_{\gamma\overline{X}}\, \overline{p}. \tag{4}$$

Moreover, if $\mathbf{B}$ is the $\pi$-form attached to $\overline{p}$ obtained by the metric tensor $g$: $\mathbf{B} = i_{\overline{p}}\, g$, then the $\pi$-form $\mathbf{B}$ satisfies the properties

$$(\nabla_{\beta\overline{X}}\mathbf{B})(\overline{Y}) = -g(\overline{X},\overline{Y}) = (D^{\circ}_{\beta\overline{X}}\mathbf{B})(\overline{Y}), \quad (\nabla_{\gamma\overline{X}}\mathbf{B})(\overline{Y}) = 0 = (D^{\circ}_{\gamma\overline{X}}\mathbf{B})(\overline{Y}).$$

Definition 3.3 ( [23]). Assume that $(M, L)$ is a Finsler space with the Berwald connection $D^{\circ}$ on $\pi^{-1}(TM)$. Then, a $\pi$ vector field $\overline{Y}$ does not dependent on the directional variable $y$ if and only if $D^{\circ}_{\gamma\overline{X}}\overline{Y} = 0$ for all $\overline{X} \in \mathfrak{X}(\pi(M))$. Furthermore, a scalar *(*vector*)* $\pi$-form $\omega$ does not dependent on the directional variable $y$ if, and only if, $D^{\circ}_{\gamma\overline{X}}\, \omega = 0$ for all $\overline{X} \in \mathfrak{X}(\pi(M))$.

Theorem 3.4 ( [23]). *The concurrent* $\pi$*-vector field* $\overline{p}$ *and the attached* $\pi$*-form* $\boldsymbol{B}$ *do not depend on the direction* $y$*.*

**3.1. The metric and Cartan tensors of $\widetilde{L}$.** In this subsection, we calculate some geometric objects associated to $\widetilde{L}(x,y)$ in terms of the objects attached to the Finsler structure $L$. We need the following lemma.

Lemma 3.5. *Under every change $L \mapsto \widetilde{L}$, the vertical counterpart for Berwald connection $D^{\circ}_{\gamma\overline{X}}\overline{Y}$ is invariant. i.e. $\widetilde{D}^{\circ}_{\gamma\overline{X}}\overline{Y} = D^{\circ}_{\gamma\overline{X}}\overline{Y}$.*

*Proof.* Under every change $L \mapsto \widetilde{L}$, the difference between the horizontal maps $\widetilde{\beta}$ and $\beta$ is vertical, that is $\widetilde{\beta} = \beta + \gamma\overline{\mu}$, for some $\pi$-vector field $\overline{\mu}$. Hence, the proof follows from the facts that [22]

$$D^{\circ}_{\gamma\overline{X}}\overline{Y} = \rho[\gamma\overline{X}, \beta\overline{Y}],$$

together with the fact that $\rho\circ\gamma$ vanishes identically and that the vertical distribution is integrable.
In more details

$$\widetilde{D}^{\circ}_{\gamma\overline{X}}\overline{Y} = \rho[\gamma\overline{X}, \widetilde{\beta}\overline{Y}] = \rho[\gamma\overline{X}, \beta\overline{Y}] + \rho[\gamma\overline{X}, \gamma\overline{\mu}] = \rho[\gamma\overline{X}, \beta\overline{Y}] = D^{\circ}_{\gamma\overline{X}}\overline{Y}.$$

Hence, the result follows. □

Lemma 3.6. *Let $(M, L)$ be a Finsler space providing a concurrent $\pi$-vector field $\overline{p}(x)$ with the associated $\pi$-form $\mathbf{B}$. Then, we have*

- **(a):** $d_J\mathfrak{B}(\gamma\overline{X}) = 0, D^{\circ}_{\gamma\overline{X}}\mathfrak{B} = d\mathfrak{B}(\gamma\overline{X}) = d_J\mathfrak{B}(\beta\overline{X}) = \mathbf{B}(\overline{X})$.
- **(b):** $d_J\, L(\gamma\overline{X}) = 0, D^{\circ}_{\gamma\overline{X}}\, L = dL(\gamma\overline{X}) = d_J\, L(\beta\overline{X}) = \ell(\overline{X})$.
- **(c):** $d_h\,\mathfrak{B}(\beta\overline{X}) = D^{\circ}_{\beta\overline{X}}\,\mathfrak{B} = d\mathfrak{B}(\beta\overline{X}) = -L\,\ell(\overline{X})$, $d\mathfrak{B}(G) = -L^2$
- **(d):** $d_h\, L(\beta\overline{X}) = D^{\circ}_{\beta\overline{X}}\, L = dL(\beta\overline{X}) = 0$.
- **(e):** $(D^{\circ}_{\gamma\overline{X}}\,\ell)(\overline{Y}) = (\nabla_{\gamma\overline{X}}\,\ell)(\overline{Y}) = L^{-1}\hbar(\overline{X},\overline{Y})$.
- **(f):** $dd_JE(\gamma\overline{X}, \beta\overline{Y}) = g(\overline{X},\overline{Y})$.

*Where $g$ is the Finser metric defined by the Finsler structure $L$, and $\ell$ is the normalized supporting element defined by $\ell := L^{-1}i_{\overline{\eta}}\, g$ .*

*Proof.* We prove only the items (a) and (c) as follows: According to the facts that $\rho\circ\gamma$ and $K\circ\beta$ vanish identically, $\rho\circ\beta = id_{\mathfrak{X}(\pi(M))}$, $i_{\overline{\eta}}\mathbf{T} = 0 = i_{\overline{\eta}}\widehat{\mathbf{P}}$, taking into account Definition 3.2 and Lemma 2.5, we obtain

**(a)**

$$\begin{aligned}
d_J\mathfrak{B}(\gamma\overline{X}) &= (J\circ\gamma\overline{X})\cdot\mathfrak{B} = \gamma\,(\rho\circ\gamma)\overline{X}\cdot\mathfrak{B} = 0.\\
d_J\mathfrak{B}(\beta\overline{X}) &= J\,(\beta\overline{X}\cdot\mathfrak{B}) = \gamma\,(\rho\circ\beta)\overline{X}\cdot\mathfrak{B} = \gamma\overline{X}\cdot\mathfrak{B}\\
&= \gamma\overline{X}\cdot g(\overline{p},\overline{\eta}) = (D^{\circ}_{\gamma\overline{X}}\, g)(\overline{p},\overline{\eta}) + g(D^{\circ}_{\gamma\overline{X}}\,\overline{p},\overline{\eta}) + g(\overline{p}, D^{\circ}_{\gamma\overline{X}}\overline{\eta})\\
&= 2\mathbf{T}(\overline{X},\overline{p},\overline{\eta}) + 0 + g(\overline{P},\overline{X})\\
&= \mathbf{B}(\overline{X}).
\end{aligned}$$

**(c)**

$$\begin{aligned} d_h\mathfrak{B}(\beta\overline{X}) &= (\beta\circ\rho\circ\beta\overline{X})\cdot\mathfrak{B} = \beta\overline{X}\cdot\mathfrak{B} = d\mathfrak{B}(\beta\overline{X}) \\ &= \beta\overline{X}\cdot g(\overline{p},\overline{\eta}) = (D^{\circ}_{\beta\overline{X}}\, g)(\overline{p},\overline{\eta}) + g(D^{\circ}_{\beta\overline{X}}\,\overline{p},\overline{\eta}) + g(\overline{p}, D^{\circ}_{\beta\overline{X}}\overline{\eta}) \\ &= -2\widehat{\mathbf{P}}(\overline{X},\overline{p},\overline{\eta}) - g(\overline{X},\overline{\eta}) + 0 \\ &= -L\,\ell(\overline{X}). \\ d\mathfrak{B}(G) &= -L\,\ell(\overline{\eta}) = -L^2. \end{aligned}$$

This completes the proof. □

Calculating the change of the normalized supporting element $\ell$ as well as the change of the angular metric tensor $\hbar$, under the (3), we obtain the following formulas.

Proposition 3.7. *Under the generalized Shen square metric* (3)*, we have*

*1. The supporting element* $\widetilde{\ell}$ *has the formula*

$$\widetilde{\ell}(\overline{X}) = (1-\frac{\mathfrak{B}^2}{L^2})\,\ell(\overline{X}) + \frac{2(L+\mathfrak{B})}{L}\,\mathbf{B}(\overline{X}). \tag{5}$$

*2. The angular metric tensors* $\widetilde{\hbar}$ *is determined by*

$$\begin{aligned} \widetilde{\hbar}(\overline{X},\overline{Y}) &= \frac{(L-\mathfrak{B})(L+\mathfrak{B})^3}{L^4}\hbar(\overline{X},\overline{Y}) + \frac{2(L+\mathfrak{B})^2}{L^2}\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y}) \\ &\quad + \frac{2\mathfrak{B}^2(L+\mathfrak{B})^2}{L^3}\,\ell(\overline{X})\,\ell(\overline{Y}) \\ &\quad - \frac{2\mathfrak{B}(L+\mathfrak{B})^2}{L^3}\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}. \end{aligned} \tag{6}$$

*Proof.* Under the generalized Shen square metric (3), and taking into account Lemma 3.6, we have
1). Using the facts that $\rho\circ\gamma = 0$ and that $\rho\circ\beta = \rho\circ\widetilde{\beta} = id_{\mathfrak{X}(\pi(M))}$, we get

$$\begin{aligned} \widetilde{\ell}(\overline{X}) &= d_J\widetilde{L}(\widetilde{\beta}\overline{X}) = d_J\widetilde{L}(\beta\overline{X}) \\ &= \frac{\partial\widetilde{L}}{\partial L}\,d_JL(\beta\overline{X}) + \frac{\partial\widetilde{L}}{\partial\mathfrak{B}}\,d_J\mathfrak{B}(\beta\overline{X}) \\ &= (1-\frac{\mathfrak{B}^2}{L^2})\,\ell(\overline{X}) + \frac{2(L+\mathfrak{B})}{L}\,\mathbf{B}(\overline{X}). \end{aligned}$$

2). In view of item 1.) above, Lemma 3.6(e), together with Lemma 3.5, and Definition

3.2, one obtain

$$\begin{aligned}
\widetilde{\hbar}(\overline{X},\overline{Y}) &= \widetilde{L}(\widetilde{D}^{\circ}_{\gamma\overline{X}}\,\widetilde{\ell})(\overline{Y}) = \widetilde{L}(D^{\circ}_{\gamma\overline{X}}\,\widetilde{\ell})(\overline{Y})\\
&= \widetilde{L}\,D^{\circ}_{\gamma\overline{Y}}\left\{(1-\frac{\mathfrak{B}^2}{L^2})\,\ell(\overline{Y})+\frac{2(L+\mathfrak{B})}{L}\,\mathbf{B}(\overline{Y})\right\}\\
&= \widetilde{L}\left\{(D^{\circ}_{\gamma\overline{X}}\,(1-\frac{\mathfrak{B}^2}{L^2}))\,\ell(\overline{Y})+(D^{\circ}_{\gamma\overline{X}}\,\frac{2(L+\mathfrak{B})}{L})\,\mathbf{B}(\overline{Y})\right\}\\
&\quad+\widetilde{L}\left\{(1-\frac{\mathfrak{B}^2}{L^2})\,(D^{\circ}_{\gamma\overline{X}}\,\ell)(\overline{Y})+\frac{2(L+\mathfrak{B})}{L}(D^{\circ}_{\gamma\overline{X}}\,\mathbf{B})(\overline{Y})\right\}\\
&= \frac{(L+\mathfrak{B})^2}{L}\{(\frac{2\mathfrak{B}^2}{L^3}\,\ell(\overline{X})-\frac{2\mathfrak{B}}{L^2}\,\mathbf{B}(\overline{X}))\,\ell(\overline{Y})\\
&\quad+(-\frac{2\mathfrak{B}}{L^2}\,\ell(\overline{X})+\frac{2}{L}\,\mathbf{B}(\overline{X}))\,\mathbf{B}(\overline{Y})\}\\
&\quad+\frac{(L+\mathfrak{B})^2}{L}\left\{(1-\frac{\mathfrak{B}^2}{L^2})\,(L^{-1}\,\hbar(\overline{X},\overline{Y})+0\right\}.
\end{aligned}$$

Hence, the result follows. □

Now, we provide the relationship between the metric tensors $g$ and $\widetilde{g}$ attached to the Finsler structures $L$ and $\widetilde{L}$, respectively.

Proposition 3.8. *The Finsler metric $\widetilde{g}$ associated with the special generalized Shen square metric* (3) *is given by the following relation:*

$$\begin{aligned}
\widetilde{g}(\overline{X},\overline{Y}) &= \frac{(L-\mathfrak{B})(L+\mathfrak{B})^3}{L^4}g(\overline{X},\overline{Y})+\frac{6(L+\mathfrak{B})^2}{L^2}\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\
&\quad+\frac{2\mathfrak{B}(2\mathfrak{B}-L)(L+\mathfrak{B})^2}{L^4}\,\ell(\overline{X})\,\ell(\overline{Y})+\frac{6(L+\mathfrak{B})^2}{L^2}\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y})+\mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.
\end{aligned}$$

*Consequently, the Cartan torsion $\widetilde{\mathbf{T}}$ of the special generalized Shen square metric has the form*

$$\begin{aligned}
2\widetilde{\mathbf{T}}(\overline{X},\overline{Y},\overline{Z}) &= 2\frac{(L-\mathfrak{B})(L+\mathfrak{B})^3}{L^4}\,T(\overline{X},\overline{Y},\overline{Z})\\
&\quad+\frac{2\mathfrak{B}(2\mathfrak{B}-L)(L+\mathfrak{B})^2}{L^5}\,\{\hbar(\overline{X},\overline{Z})\,\ell(\overline{Y})+\hbar(\overline{Y},\overline{Z})\,\ell(\overline{X})\}\\
&\quad+\frac{6(L+\mathfrak{B})^2}{L^3}\left\{\mathbf{B}(\overline{X})\,\hbar(\overline{Y},\overline{Z})+\mathbf{B}(\overline{Y})\,\hbar(\overline{X},\overline{Z})\right\}\\
&\quad+\left(D^{\circ}_{\gamma\overline{Z}}\,\frac{(L-\mathfrak{B})(L+\mathfrak{B})^3}{L^4}\right)g(\overline{X},\overline{Y})+(D^{\circ}_{\gamma\overline{Z}}\,\frac{6(L+\mathfrak{B})^2}{L^2})\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\
&\quad+\left(D^{\circ}_{\gamma\overline{Z}}\,\frac{2\mathfrak{B}(2\mathfrak{B}-L)(L+\mathfrak{B})^2}{L^4}\right)\,\ell(\overline{X})\,\ell(\overline{Y})\\
&\quad+\left(D^{\circ}_{\gamma\overline{Z}}\,\frac{6(L+\mathfrak{B})^2}{L^2}\right)\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y})+\mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.
\end{aligned}$$

*where $D^{\circ}_{\gamma\overline{X}}f(L,\mathfrak{B})=d_Jf(\beta\overline{X})=\frac{\partial f}{\partial L}\,\ell(\overline{X})+\frac{\partial f}{\partial\mathfrak{B}}\,\mathbf{B}(\overline{X})$ and $T$ is the $(h)hv$-torsion of the attached Cartan connection to the metric $L$.*

*Proof.* In view of the generalized Shen square metric *(3)*, and using Proposition 3.7, we obtain

$$\begin{aligned}\widetilde{\ell}(\overline{X}) &= (1-\frac{\mathfrak{B}^2}{L^2})\,\ell(\overline{X}) + \frac{2(L+\mathfrak{B})}{L}\,\mathbf{B}(\overline{X}).\\ \widetilde{\hbar}(\overline{X},\overline{Y}) &= \frac{(L-\mathfrak{B})(L+\mathfrak{B})^3}{L^4}\hbar(\overline{X},\overline{Y}) + \frac{2(L+\mathfrak{B})^2}{L^2}\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\ &\quad+\frac{2\mathfrak{B}^2(L+\mathfrak{B})^2}{L^3}\,\ell(\overline{X})\,\ell(\overline{Y}) - \frac{2\mathfrak{B}(L+\mathfrak{B})^2}{L^3}\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.\end{aligned}$$

Hence, using the definition of the angular metric tensor $\widetilde{\hbar} := \widetilde{g} - \widetilde{\ell}\otimes\widetilde{\ell}$, we have

$$\begin{aligned}\widetilde{g}(\overline{X},\overline{Y}) &= \frac{(L-\mathfrak{B})(L+\mathfrak{B})^3}{L^4}\,\hbar(\overline{X},\overline{Y}) + \frac{2(L+\mathfrak{B})^2}{L^2}\,\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\ &\quad+\frac{2\mathfrak{B}^2(L+\mathfrak{B})^2}{L^3}\,\ell(\overline{X})\,\ell(\overline{Y}) - \frac{2\mathfrak{B}(L+\mathfrak{B})^2}{L^3}\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}\\ &\quad+\left\{(1-\frac{\mathfrak{B}^2}{L^2})\,\ell(\overline{X}) + \frac{2(L+\mathfrak{B})}{L}\,\mathbf{B}(\overline{X})\right\}\left\{(1-\frac{\mathfrak{B}^2}{L^2})\,\ell(\overline{Y}) + \frac{2(L+\mathfrak{B})}{L}\,\mathbf{B}(\overline{Y})\right\}\\ &= \frac{(L-\mathfrak{B})(L+\mathfrak{B})^3}{L^4}g(\overline{X},\overline{Y}) + \frac{6(L+\mathfrak{B})^2}{L^2}\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\ &\quad+\frac{2\mathfrak{B}(2\mathfrak{B}-L)(L+\mathfrak{B})^2}{L^4}\,\ell(\overline{X})\,\ell(\overline{Y}) + \frac{6(L+\mathfrak{B})^2}{L^2}\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.\end{aligned}$$

Consequently, using the expression of the metric $\widetilde{g}$, taking into account Lemma 2.5 (a), it follows the expression of the Cartan torsion $\widetilde{\mathbf{T}}$ of the generalized Shen square metric. □

Theorem 3.9. *The metric tensor $\widetilde{g}$ of $\widetilde{L}$ is non-degenerate if and only if*

$$L^2(1+2p^2) - 3\mathfrak{B}^2 \neq 0. \tag{7}$$

*That is, the generalized Shen square metric is a Finsler structure (or, conic Finsler structure) if and only if the condition* (7) *is satisfied.*

*Proof.* Assume that the Finsler metric $\widetilde{g}$ associated with the generalized Shen square metric, defined by (3), is non-degenerate. Now, let $\widetilde{g}(\overline{X},\overline{Y}) = 0$ for all $\overline{X} \in \mathfrak{X}(\pi(M))$. By using Proposition 3.8, we obtain

$$\begin{aligned}0 &= \frac{(L-\mathfrak{B})(L+\mathfrak{B})^3}{L^4}g(\overline{X},\overline{Y}) + \frac{6(L+\mathfrak{B})^2}{L^2}\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\ &\quad+\frac{2\mathfrak{B}(2\mathfrak{B}-L)(L+\mathfrak{B})^2}{L^4}\,\ell(\overline{X})\,\ell(\overline{Y}) + \frac{6(L+\mathfrak{B})^2}{L^2}\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.\end{aligned}$$

From which, by substituting $\overline{X} = \overline{p}$, noting that $\ell(\overline{p}) = \frac{\mathfrak{B}}{L}$ and $\mathbf{B}(\overline{p}) = g(\overline{p},\overline{p}) =: p^2$, one can show that

$$A_1\,\ell(\overline{Y}) + B_1\,\mathbf{B}(\overline{Y}) = 0, \tag{8}$$

where

$$\begin{aligned}A_1 &:= \frac{2(2\mathfrak{B}-L)(\mathfrak{B}+L)^2(\mathfrak{B}-Lp)(\mathfrak{B}+Lp)}{L^5}\\ B_1 &:= \frac{(\mathfrak{B}+L)^2\left(-5\mathfrak{B}^2+2\mathfrak{B}L+L^2\left(6p^2+1\right)\right)}{L^4}.\end{aligned}$$

Similarly, by substituting $\overline{X} = \overline{\eta}$, taking the facts that $\ell(\overline{\eta}) = L$ and $\mathbf{B}(\overline{\eta}) = \mathfrak{B}$ into account, we obtain

$$A_2\, \ell(\overline{Y}) + B_2\, \mathbf{B}(\overline{Y}) = 0, \tag{9}$$

with

$$\begin{aligned} A_2 &:= -\frac{(\mathfrak{B} - L)(\mathfrak{B} + L)^3}{L^3}, \\ B_2 &:= \frac{2(\mathfrak{B} + L)^3}{L^2}. \end{aligned}$$

Now, the system of the algebraic equations (8) and (9) has non-trivial solution if and only if

$$\frac{(\mathfrak{B} + L)^6\,(L^2(1 + 2p^2) - 3\mathfrak{B}^2)}{L^7} = 0.$$

Hence, as $\widetilde{L} \neq 0$ over $\mathcal{T}M$, then we conclude that $L^2(1 + 2p^2) - 3\mathfrak{B}^2 = 0$. Consequently,

$$\overline{Y} \neq 0 \quad \Longleftrightarrow \quad L^2(1 + 2p^2) - 3\mathfrak{B}^2 = 0.$$

Therefore, $\overline{Y} = 0$ if and only if the Finsler structure $L$ and the $\pi$-form $\mathfrak{B}$ satisfy the condition

$$L^2(1 + 2p^2) - 3\mathfrak{B}^2 \neq 0.$$

This means that the generalized Shen square metric tensor $\widetilde{g}$ is non-degenerate if and only if the condition (7) is satisfied. Hence, the proof is completed. □

Form now on, we consider that the generalized Shen square metric $\widetilde{L}$ satisfies the condition (7).

## 4. Geodesic spray and Berwald connection

In this section, we find the relationship between the canonical (geodesic) spray $\widetilde{G}$ of $\widetilde{L}$ in terms of the geodesic spray $G$ of $L$. Precisely, we have one of the main results in this work.

THEOREM 4.1. *The canonical spray $\widetilde{G}$ associated with the generalized Shen square metric* (3) *is given by*

$$\widetilde{G} = G - \frac{2L^2(2\mathfrak{B} - L)}{L^2(1 + 2p^2) - 3\mathfrak{B}^2}\,\mathcal{C} + \frac{2L^4}{L^2(1 + 2p^2) - 3\mathfrak{B}^2}\,\gamma\overline{p},$$

*where, $\mathcal{C}$ is the Liouville vector field defined by $\mathcal{C} := \gamma\,\overline{\eta}$ and $p^2 := \mathbf{B}(\overline{p}) = g(\overline{p}, \overline{p})$.*

*Proof.* Due to the generalized Shen square metric *(3)*, taking into account the expression of the exterior $\pi$-form $\widetilde{\Omega} := \frac{1}{2} dd_J\,\widetilde{L}^2$, the fact that the difference between two sprays is vertical (i.e. $\widetilde{G} = G + \gamma\overline{\mu}$, for some $\pi$-vector field $\overline{\mu}$) and using Proposition 2.3, one can show that

$$\begin{aligned} -d\widetilde{E}(X) &= i_{\widetilde{G}}\,\widetilde{\Omega}(X) = i_{G+\gamma\overline{\mu}}\,(\tfrac{1}{2}\,dd_J\widetilde{L}^2)(X) \\ &= \tfrac{1}{2} i_G\,dd_J\widetilde{L}^2(X) + \tfrac{1}{2} i_{\gamma\overline{\mu}}\,dd_J\widetilde{L}^2(X). \end{aligned} \tag{10}$$

Therefore, after some computation and using the fact that $\beta\overline{\eta} = G$ and $X = hX + vX = \beta\rho X + \gamma KX$, together with Lemma 3.6, we have

$$\begin{aligned} d\widetilde{E}(X) &= \frac{1}{2} d\widetilde{L}^2(X) = \widetilde{L}\, d\widetilde{L}(X) \\ &= \frac{(L+\mathfrak{B})^2}{L}\left\{(1-\frac{\mathfrak{B}^2}{L^2})\, dL(X) + \frac{2(L+\mathfrak{B})}{L}\, d\mathfrak{B}(X)\right\} \\ &= \frac{(L+\mathfrak{B})^2(L^2-\mathfrak{B}^2)}{L^3}\, dL(X) + \frac{2(L+\mathfrak{B})^3}{L^2}\, d\mathfrak{B}(X), \end{aligned}$$

$$\begin{aligned} \frac{1}{2} i_G\, dd_J\widetilde{L}^2(X) &= \frac{1}{2}\{dd_J\widetilde{L}^2(\beta\overline{\eta}, X)\} \\ &= \frac{1}{2}\left\{G\cdot d_J\widetilde{L}^2(X) - X\cdot d_J\widetilde{L}^2(G) - d_J\widetilde{L}^2[G,X]\right\} \\ &= \frac{1}{2}\left\{G\cdot(2\widetilde{L}\widetilde{\ell}(\rho X)) - X\cdot(2\widetilde{L}\widetilde{\ell}(\overline{\eta})) - 2\widetilde{L}\ell(\rho[G,X])\right\} \\ &= ((G\cdot\widetilde{L})\,\widetilde{\ell}(\rho X) + \widetilde{L}\, G\cdot\widetilde{\ell}(\rho X)) - (X\cdot\widetilde{L}^2) - \widetilde{L}\,\ell(\rho[G,X]). \end{aligned} \tag{11}$$

Moreover, we have

$$\begin{aligned} G\cdot\widetilde{L} &= d\widetilde{L}(G) = (1-\frac{\mathfrak{B}^2}{L^2})\, dL(G) + \frac{2(L+\mathfrak{B})}{L}\, d\mathfrak{B}(G) = -2L(L+\mathfrak{B}) \\ X\cdot\widetilde{L} &= d\widetilde{L}(X) = (1-\frac{\mathfrak{B}^2}{L^2})\, dL(X) + \frac{2(L+\mathfrak{B})}{L}\, d\mathfrak{B}(X) \\ \widetilde{\ell}(\overline{X}) &= (1-\frac{\mathfrak{B}^2}{L^2})\, \ell(\overline{X}) + \frac{2(L+\mathfrak{B})}{L}\, \mathbf{B}(\overline{X}), \\ \rho[G,X] &= \rho[G, hX+vX] = D^\circ_G\rho X - KX, \\ (D^\circ_G\mathbf{B})(\overline{X}) &= -g(\overline{X},\overline{\eta}) = -L\,\ell(\overline{X}), \\ (D^\circ_G\ell)(\overline{X}) &= (\nabla_G\ell)(\overline{X}) = 0, \\ d\mathfrak{B}(X) &= \mathbf{B}(KX) - L\ell(\rho X), \\ dL(X) &= dL(\gamma KX) = \ell(KX), \end{aligned}$$

Now, using the above facts and Lemma 3.6, (11) reduces to

$$\begin{aligned} \frac{1}{2} i_G\, dd_J\widetilde{L}^2(X) &= -2L(L+\mathfrak{B})\,(1-(\frac{\mathfrak{B}^2}{L^2})\,\ell(\rho X) + \frac{2(L+\mathfrak{B})}{L}\,\mathbf{B}(\rho X)) \\ &\quad + \frac{(L+\mathfrak{B})^2}{L}\, G\cdot\left(\frac{(L^2-\mathfrak{B}^2)}{L^2}\,\ell(\rho X) + \frac{2(L+\mathfrak{B})}{L}\,\mathbf{B}(\rho X)\right) \\ &\quad - 2\frac{(L+\mathfrak{B})^2}{L}\left(\frac{(L^2-\mathfrak{B}^2)}{L^2}\, dL(X) + \frac{2(L+\mathfrak{B})}{L} d\mathfrak{B}(X)\right) \\ &\quad - \frac{(L+\mathfrak{B})^2}{L}\left(\frac{(L^2-\mathfrak{B}^2)}{L^2}\,\ell(\rho[G,X]) + \frac{2(L+\mathfrak{B})}{L}\,\mathbf{B}(\rho[G,X])\right) \\ &= 2(2\mathfrak{B}-L)(L+\mathfrak{B})^2\ell(\rho X) - 6(L+\mathfrak{B})^2\,\mathbf{B}(\rho X) \\ &\quad - \frac{(L+\mathfrak{B})^2}{L}\left(\frac{(L^2-\mathfrak{B}^2)}{L^2}\, dL(X) + \frac{2(L+\mathfrak{B})}{L}\, d\mathfrak{B}(X)\right). \end{aligned}$$

On the other hand, using Proposition 3.8, we have

$$\begin{aligned}\frac{1}{2}\, i_{\gamma\overline{\mu}}\, dd_J\widetilde{L}^2(X) &= \widetilde{g}(\overline{\mu},\rho X)\\ &= \frac{(L-\mathfrak{B})(L+\mathfrak{B})^3}{L^4} g(\overline{X},\overline{Y}) + \frac{6(L+\mathfrak{B})^2}{L^2}\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\ &\quad + \frac{2\mathfrak{B}(2\mathfrak{B}-L)(L+\mathfrak{B})^2}{L^4}\,\ell(\overline{X})\,\ell(\overline{Y}) + \frac{6(L+\mathfrak{B})^2}{L^2}\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.\end{aligned}$$

Plugging the last two relations into Equation (10), after some calculation, it follows that

$$\begin{aligned}&-\frac{(L+\mathfrak{B})^2}{L}\left\{\frac{(L^2-\mathfrak{B}^2)}{L^2}\,dL(X) + \frac{2(L+\mathfrak{B})}{L}\,d\mathfrak{B}(X)\right\}\\ =\ & 2(2\mathfrak{B}-L)(L+\mathfrak{B})^2\ell(\rho X) - 6(L+\mathfrak{B})^2\,\mathbf{B}(\rho X)\\ &-\frac{(L+\mathfrak{B})^2}{L}\left(\frac{(L^2-\mathfrak{B}^2)}{L^2}\,dL(X) + \frac{2(L+\mathfrak{B})}{L}\,d\mathfrak{B}(X)\right)\\ &+\frac{(L-\mathfrak{B})(L+\mathfrak{B})^3}{L^4} g(\overline{X},\overline{Y}) + \frac{6(L+\mathfrak{B})^2}{L^2}\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\ &+\frac{2\mathfrak{B}(2\mathfrak{B}-L)(L+\mathfrak{B})^2}{L^4}\,\ell(\overline{X})\,\ell(\overline{Y}) + \frac{6(L+\mathfrak{B})^2}{L^2}\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}\end{aligned}$$

Using the non-degeneracy of the Finsler metric $g$, the above relation takes the form

$$\begin{aligned}\frac{(L-\mathfrak{B})(L+\mathfrak{B})^3}{L^4}\,\overline{\mu} &= \{\frac{2(L-2\mathfrak{B})(L+\mathfrak{B})^2}{L} - \frac{2(L-2\mathfrak{B})(L+\mathfrak{B})^2}{L^3}\,\mathbf{B}(\overline{\mu})\\ &+\frac{2\mathfrak{B}(L-2\mathfrak{B})(L+\mathfrak{B})^2}{L^5}\,\ell(\overline{\mu})\}\overline{\eta} + \{6(L+\mathfrak{B})^2 - \frac{6(L+\mathfrak{B})^2}{L^2}\,\mathbf{B}(\overline{\mu})\\ &+\frac{2(2\mathfrak{B}-L)(L+\mathfrak{B})^2}{L^2}\,\ell(\overline{\mu})\}\overline{p}.\end{aligned} \tag{12}$$

where $\ell(\overline{\mu})$ and $\mathbf{B}(\overline{\mu})$ are geometric quantities determined by the following two equations

$$\begin{aligned}&\frac{(L-\mathfrak{B})(L+\mathfrak{B})^3}{L^4}\,\ell(\overline{\mu}) + \frac{2(L+\mathfrak{B})^3}{L^3}\,\mathbf{B}(\overline{\mu}) = \frac{2(L+\mathfrak{B})^3}{L},\\ &\frac{2(2\mathfrak{B}-L)(L+\mathfrak{B})^2(\mathfrak{B}^2-L^2\,p^2)}{L^5}\,\ell(\overline{\mu}) + \frac{(L+\mathfrak{B})^2(-5\mathfrak{B}^2+2\mathfrak{B}\,L+L^2(1+6p^2))}{L^4}\,\mathbf{B}(\overline{\mu})\\ &\qquad = \frac{2(L+\mathfrak{B})^2(\mathfrak{B}(L-2\mathfrak{B})+3L^2\,p^2)}{L2},\end{aligned} \tag{13}$$

where $p^2 := \mathbf{B}(\overline{p})$. Solving the above system, we get

$$\begin{aligned}\ell(\overline{\mu}) &= \frac{2L^3(\mathfrak{B}-L)}{3\mathfrak{B}^2-L^2\,(2p^2+1)},\\ \mathbf{B}(\overline{\mu}) &= \frac{2L^2\,(-2\mathfrak{B}^2+\mathfrak{B}L+L^2p^2)}{L^2\,(2p^2+1)-3\mathfrak{B}^2}.\end{aligned}$$

Consequently, in view of Equation (12) taking the fact that $\widetilde{G} = G + \gamma\overline{\mu}$ into account, it follows that the canonical sprays $G$ and $\widetilde{G}$, are related by

$$\widetilde{G} = G - \frac{2L^2(2\mathfrak{B}-L)}{L^2(1+2p^2)-3\mathfrak{B}^2}\,\mathcal{C} + \frac{2L^4}{L^2(1+2p^2)-3\mathfrak{B}^2}\,\gamma\overline{p}.$$

Hence, the proof is completed. □

Now, we are in aposition to find the relationship that relates the two attached Barthel connections $\Gamma$ and $\widetilde{\Gamma}$. That is, we have the following theorem.

THEOREM 4.2. *The Barthel connection $\widetilde{\Gamma}$ associated with the generalized Shen square metric* (3) *is given by*

$$\widetilde{\Gamma} = \Gamma - \lambda_1 J - d_J\lambda_1 \otimes \gamma\overline{\eta} + d_J\lambda_2 \otimes \gamma\overline{p},$$

*where*

$$\begin{aligned}\lambda_1 &:= \frac{2L^2(2\mathfrak{B} - L)}{L^2(1+2p^2) - 3\mathfrak{B}^2},\\ \lambda_2 &:= \frac{2L^4}{L^2(1+2p^2) - 3\mathfrak{B}^2}.\end{aligned}$$

*Consequently, the horizontal map $\widetilde{\beta}$ associated with the generalized Shen square metric has the form*

$$\widetilde{\beta}\overline{X} = \beta\overline{X} - \frac{1}{2}\left\{\lambda_1\,\gamma\overline{X} + d_J\lambda_1(\beta\overline{X})\,\gamma\overline{\eta} - d_J\lambda_2(\beta\overline{X})\,\gamma\overline{p}\right\}.$$

*Proof.* From Theorem 4.1 and the fact that

$$[fX, J] = f[X, J] + df \wedge i_X J - d_J f \otimes X,$$

one can show that

$$\begin{aligned}\widetilde{\Gamma} &= [J, \widetilde{G}] = [J, G - \lambda_1\,\gamma\overline{\eta} + \lambda_2\gamma\overline{p}] = [J, G] + [\lambda_1\,\gamma\overline{\eta} - \lambda_2\gamma\overline{p}, J]\\ &= [J, G] + \lambda_1[\gamma\overline{\eta}, J] + d\lambda_1 \wedge i_{\gamma\overline{\eta}}\,J - d_J\lambda_1 \otimes \gamma\overline{\eta}\\ &\quad -\lambda_2[\gamma\overline{p}, J] - d\lambda_2 \wedge i_{\gamma\overline{p}}\,J + d_J\lambda_2 \otimes \gamma\overline{p}.\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}d_J p^2 &= 0\\ i_{\gamma\overline{\eta}}\,J &= 0 = i_{\gamma\overline{p}}\,J, \qquad\qquad (\text{as } J\circ\gamma = 0),\end{aligned}$$

whereas

$$\begin{aligned}[\gamma\overline{p}, J]X &= [\gamma\overline{p}, JX] - J[\gamma\overline{p}, X]\\ &= \gamma\{\nabla_{\gamma\overline{p}}\,\rho X - \nabla_{JX}\,\overline{p}\} - \gamma\{\nabla_{\gamma\overline{p}}\,\rho X - T(\overline{p}, \rho X)\} = 0.\end{aligned}$$

$$[\gamma\overline{\eta}, J]X = -JX.$$

Therefore,

$$\widetilde{\Gamma} = \Gamma - \lambda_1 J - d_J\lambda_1 \otimes \gamma\overline{\eta} + d_J\lambda_2 \otimes \gamma\overline{p}.$$

Consequently, using the fact that $\Gamma = 2\beta\circ\rho - I$, the horizontal map $\widetilde{\beta}$ associated with the special generalized Shen square metric has the form

$$\widetilde{\beta}\overline{X} = \beta\overline{X} - \frac{1}{2}\left\{\lambda_1\,\gamma\overline{X} + d_J\lambda_1(\beta\overline{X})\,\gamma\overline{\eta} - d_J\lambda_2(\beta\overline{X})\,\gamma\overline{p}\right\}.$$

This completes the proof. □

Theorem 4.3. *The Barthel curvature tensor* $\widetilde{\Re}$ *associated with the generalized Shen square metric* (3) *is determined by*

$$\widetilde{\Re} = \Re - [h, \mathbb{L}] - N_{\mathbb{L}},$$

*where* $N_{\mathbb{L}} := \frac{1}{2}[\mathbb{L}, \mathbb{L}]$ *is the Nijenhuis torsion of a vector 1-form* $\mathbb{L}$ *defined by*

$$\mathbb{L} := -\frac{1}{2}\left\{\lambda_1\, J + d_J\lambda_1 \otimes \gamma\overline{\eta} - d_J\lambda_2 \otimes \gamma\overline{p}\right\}. \tag{14}$$

*Proof.* By using Theorem 4.2, we conclude that the horizontal projection $\widetilde{h}$ and vertical projection $\widetilde{v}$ associated with the special generalized Shen square metric has the form

$$\widetilde{h} = h + \mathbb{L}, \quad \widetilde{v} = v - \mathbb{L},$$

where $\mathbb{L}$ is defined by (14). Hence, the Nijenhuis torsion [3] of a vector 1-form $\mathbb{L}$ is given by

$$N_{\mathbb{L}} := \frac{1}{2}[\mathbb{L}, \mathbb{L}](X, Y) = [\mathbb{L}X, \mathbb{L}Y] + \mathbb{L}^2[X, Y] - \mathbb{L}[\mathbb{L}X, Y] - \mathbb{L}[X, \mathbb{L}Y].$$

Now, the proof is attained by the fact that $\widetilde{\Re} = -\frac{1}{2}[\widetilde{h}, \widetilde{h}]$, and taking into account the properties of the Frölicher-Nijenhuis bracket. □

The Berwald vertical counterpart is given by Lemma 3.5 and the Berwald horizontal counterpart is given by the following proposition.

Proposition 4.4. *For the generalized Shen square metric* (3), *the Berwald horizontal counterpart is given by*

$$\begin{aligned}\widetilde{D^{\circ}}_{\widetilde{\beta}\overline{X}}\overline{Y} \;=\; & D^{\circ}{}_{\beta\overline{X}}\overline{Y} - \frac{1}{2}\{\lambda_1\, D^{\circ}_{\gamma\overline{X}}\,\overline{Y} + d_J\lambda_1(\beta\overline{X})\, D^{\circ}_{\gamma\overline{\eta}}\,\overline{Y} \\ & - d_J\lambda_1(\beta\overline{X})\,\overline{Y} - d_J\lambda_1(\beta\overline{Y})\,\overline{X} - d_J\lambda_2(\beta\overline{X})\, D^{\circ}_{\gamma\overline{p}}\,\overline{Y}\} \\ & + \frac{1}{2}\left\{dd_J\lambda_1(\gamma\overline{Y}, \beta\overline{X})\,\overline{\eta} - dd_J\lambda_2(\gamma\overline{Y}, \beta\overline{X}))\,\overline{p}\right\}.\end{aligned}$$

*Proof.* The proof can obtained by using the fact that $v := \gamma \circ K$, $h := \beta \circ \rho$, $\gamma D^{\circ}{}_{hX}\,\overline{Y} := v[hX, JY]$ and $D^{\circ}{}_{\gamma\overline{X}}\,\rho Y := \rho[\gamma\overline{X}, \beta\overline{Y}]$ ( [22, Proposition 4.4]), taking into account Theorem 4.3, and the facts that the map $\gamma : \pi^{-1}(TM) \to VTM$ is an isomorphism, the Berwlad (v)v-curvature $\widetilde{S}^{\circ} = 0$ , $[JX, JY] = J[X, JY] + J[JX, Y]$, $vJ = J$ and $Jv = 0$.

In more details.

$$\begin{aligned}
\gamma\widetilde{D^\circ}_{hX}\,\rho Y &= \widetilde{v}[\overline{h}X, JY] = (v-\mathbb{L})[hX+\mathbb{L}X, JY]\\
&= v[hX, JY] + v[\mathbb{L}X, JY] - \mathbb{L}[hX, JY] - \mathbb{L}[\mathbb{L}X, JY]\\
&= \gamma D^\circ{}_{hX}\overline{Y} - \frac{\gamma}{2}\{\lambda_1\, K[\,JX, JY] + d_J\lambda_1(X)\, K[\,\gamma\overline{\eta}, JY] - d_J\lambda_2(X)\, K[\,\gamma\overline{p}, JY]\}\\
&\quad +\frac{\gamma}{2}\{(JY\cdot\lambda_1)\,\rho X + (JY\cdot d_J\lambda_1(X))\,\overline{\eta} - (JY\cdot d_J\lambda_2(X))\,\overline{p}\}\\
&\quad +\frac{\gamma}{2}\{\lambda_1\,\rho([hX, JY]) + d_J\lambda_1([hX, JY])\,\overline{\eta} - d_J\lambda_2([hX, JY])\,\overline{p}\}\\
&= \gamma D^\circ{}_{hX}\rho Y - \frac{\gamma}{2}\{\lambda_1\, D^\circ_{JX}\,\rho Y + d_J\lambda_1(X)\, D^\circ_{\gamma\overline{\eta}}\,\rho Y\\
&\quad - d_J\lambda_1(X)\,\rho Y - d_J\lambda_1(Y)\,\rho X - d_J\lambda_2(X)\, D^\circ_{\gamma\overline{p}}\,\rho Y\}\\
&\quad +\frac{\gamma}{2}\{dd_J\lambda_1(JY, X)\,\overline{\eta} - dd_J\lambda_2(JY, X))\,\overline{p}\}\,.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\widetilde{D^\circ}_{\widetilde{\beta X}}\overline{Y} &= D^\circ{}_{\beta\overline{X}}\overline{Y} - \frac{1}{2}\{\lambda_1\, D^\circ_{\gamma\overline{X}}\,\overline{Y} + d_J\lambda_1(\beta\overline{X})\, D^\circ_{\gamma\overline{\eta}}\,\overline{Y}\\
&\quad - d_J\lambda_1(\beta\overline{X})\,\overline{Y} - d_J\lambda_1(\beta\overline{Y})\,\overline{X} - d_J\lambda_2(\beta\overline{X})\, D^\circ_{\gamma\overline{p}}\,\overline{Y}\}\\
&\quad +\frac{1}{2}\left\{dd_J\lambda_1(\gamma\overline{Y}, \beta\overline{X})\,\overline{\eta} - dd_J\lambda_2(\gamma\overline{Y}, \beta\overline{X}))\,\overline{p}\right\}.
\end{aligned}$$

This ends the proof. □

We end our work by giving an example of a Finsler space that providing a concurrent $\pi$-vector field, and computing the $\pi$-form that corresponds and it should be remarked that the presence of a concurrent vector field on Finsler spaces has been established first by Tachibana [20]. Also, a generalization of a concurrent vector field, called a semi-concurrent vector field, has been introduced in [28].

Example 4.5. *Let $M = \{(x_1, x_2, x_3)\in\mathbb{R}^3 :\ x_2\neq 0\}$ and $L$ be given by*

$$L = \sqrt{x_2^2\left(\sqrt{y_1^2 + x_1^2 y_3^2} + \ y_3\right)^2 + y_2^2}.$$

*The corresponding components $g_{ij}$ of the metric tensor are given by*

$$g_{11} = \frac{x_2^2\,(\,x_1^2\,y_3^3 + \sqrt{y_1^2 + x_1^2\,y_3^2}\,(x_1^2\,y_3^2 + y_1^2))}{(y_1^2 + x_1^2\,y_3^2)^{3/2}},$$

$$g_{22} = 1, \qquad g_{13} = \frac{x_2^2\,y_1^3}{(y_1^2 + x_1^2\,y_3^2)^{3/2}},$$

$$g_{33} = \frac{x_2^2\,(\,2x_1^4\,y_3^3 + 3x_1^2 y_1^2 y_3 + \sqrt{y_1^2 + x_1^2\,y_3^2}\,(x_1^4\,y_3^2 + x_1^2 y_1^2 + x_1^2 y_3^2 + y_1^2))}{(y_1^2 + x_1^2\,y_3^2)^{3/2}}.$$

*Also, the corresponding components $C_{ijk}$ of the Cartan tensor are calculated as follows*

$$C_{111} = -\frac{3}{2}\frac{x_1^2\,x_2^2\,y_1\,y_3^3}{(y_1^2 + x_1^2\,y_3^2)^{5/2}},\ \ C_{113} = \frac{3}{2}\frac{x_1^2\,x_2^2\,y_1^2\,y_3^2}{(y_1^2 + x_1^2\,y_3^2)^{5/2}}$$

$$C_{133} = -\frac{3}{2}\frac{x_1^2\,x_2^2\,y_1^3\,y_3}{(y_1^2 + x_1^2\,y_3^2)^{5/2}},\ \ C_{333} = \frac{3}{2}\frac{x_1^2\,x_2^2\,y_1^4}{(y_1^2 + x_1^2\,y_3^2)^{5/2}}.$$

*The components $g^{ij}$ of the inverse metric tensor are given as follows*

$$g^{11} = \frac{x_1^6y_3^4 + 2x_1^4y_1^2y_3^2 + x_1^2y_1^4 + x_1^4y_3^4 + 2x_1y_1^2y_3^2 + y_1^4 + \sqrt{y_1^2+x_1^2y_3^2}(2x_1^4y_3^3 + 3x_1^2y_1^2y_3)}{x_2^2x_1^2(\sqrt{y_1^2+x_1^2y_3^2}(3x_1^2y_3^3 + 3y_1^2y_3 + y_3^3) + x_1^4y_3^4 + 2x_1^2y_1^2y_3^2 + 3x_1^2y_3^4 + y_1^4 + 3y_1^2y_3^2)},$$

$$g^{13} = \frac{-y_1^3\sqrt{y_1^2+x_1^2y_3^2}}{x_2^2x_1^2(\sqrt{y_1^2+x_1^2y_3^2}(3x_1^2y_3^3 + 3y_1^2y_3 + y_3^3) + x_1^4y_3^4 + 2x_1^2y_1^2y_3^2 + 3x_1^2y_3^4 + y_1^4 + 3y_1^2y_3^2)},$$

$$g^{33} = \frac{x_1^4y_3^4 + 2x_1^2y_1^2y_3^2 + y_1^4 + x_1^2y_3^3\sqrt{y_1^2+x_1^2y_2^2}}{x_2^2x_1^2(\sqrt{y_1^2+x_1^2y_3^2}(3x_1^2y_3^3 + 3y_1^2y_3 + y_3^3) + x_1^4y_3^4 + 2x_1^2y_1^2y_3^2 + 3x_1^2y_3^4 + y_1^4 + 3y_1^2y_3^2)},$$

$$g^{22} = 1.$$

*Straightforward calculations by using the Finsler package [27] and to avoid complicated or big formulas, we list the required coefficients of Cartan connection, precisely:*

$$\Gamma^1_{12} = \frac{1}{x_2}, \quad \Gamma^3_{23} = \frac{1}{x_2}, \quad \Gamma^2_{22} = 0.$$

*Since we have $p^iC_{ijk} = 0$ and*

$$p^1_{|1} = \delta_1 p^1 + p^1\Gamma^1_{11} + p^2\Gamma^1_{12} + p^3\Gamma^1_{13} = 1,$$

*similarly, $p^2_{|2} = 1$, $p^3_{|3} = 1$ and all other components of $p^i_{|j}$ vanish, then we conclude that this metric provides a concurrent $\pi$-vector field given by $\overline{p} = p^i\overline{\partial_i}$, where $\overline{\partial_i}$ are the basis of fibers of $\pi^{-1}(TM)$, $p^1(x) = 0,\ p^2(x) = x_2,\ p^3(x) = 0$.*
*Now, the components of the corresponding $\pi$-form $\mathbf{B}$ are given by $\mathbf{B}^1 = 0$, $\mathbf{B}^2 = x_2$, $\mathbf{B}^3 = 0$, and hence the associated one form $\mathfrak{B}$ becomes $\mathfrak{B}(x,y) = x_2y_2$. Therefore, we have*

$$\widetilde{L}(x,y) = \frac{(L(x,y)+\mathfrak{B}(x,y))^2}{L(x,y)} = \frac{\left\{\sqrt{x_2^2\left(\sqrt{y_1^2+x_1^2y_3^2} + \ y_3\right)^2 + y_2^2} + x_2y_2\right\}^2}{\sqrt{x_2^2\left(\sqrt{y_1^2+x_1^2y_3^2} + \ y_3\right)^2 + y_2^2}},$$

*which defines a generalized Shen square metric over $M = \mathbb{R}^3$.*

## Concluding remark

It is worth mentioning that the Shen square metric has many applications, and in the near future, as a continuation of this work, we will investigate intrinsically some geometric consequences for this metric related to some applications.

## References

[1] A. Bejancu and H. Farran, *Generalized Landsberg manifolds of scalar curvature*, Bull. Korean Math. Soc. **37** (3) (2000), 543–550.

[2] S.-S. Chern and Z. Shen, *Riemann-Finsler geometry*, Nankai Tracts in Mathematics, 6, World Scientific, Hackensack, 2005.

[3] A. Frölicher and A. Nijenhuis, *Theory of vector-valued differential forms*, Ann. Proc. Kon. Ned. Akad. **59** (1956), 338–359.
https://dx.doi.org/10.1016/s1385-7258(58)50058-4

[4] J. Grifone, *Structure présque-tangente et connexions,* I, Ann. Inst. Fourier Grenoble **22** (1) (1972), 287–334.
https://dx.doi.org/10.5802/aif.407

[5] J. Grifone, *Structure presque-tangente et connexions,* II, Ann. Inst. Fourier Grenoble **22** (3) (1972), 291–338.
https://dx.doi.org/10.5802/aif.431

[6] J. Klein and A. Voutier, *Formes extérieures génératrices de sprays*, Ann. Inst. Fourier, Grenoble, **18** (1) (1968), 241–260.
https://doi.org/10.5802/aif.282

[7] M. Matsumoto, *On Finsler spaces with Randers metric and special forms of important tensors*, J. Math. Kyoto Univ., **14** (1974), 477–498.
https://doi.org/10.1215/kjm/1250523171

[8] M. Matsumoto, *On three-dimensional Finsler spaces satisfying the $T$-condition and $B^P$-condition*, Tensor N.S., **29** (1975), 13–20.

[9] M. Matsumoto and C. Shibata, *On semi $C$-reducible, $T$-tensor$=0$ and $S_4$-likeness of Finsler spaces*, J. Math. Kyoto Univ., **19** (2) (1979), 301–314.
https://doi.org/10.1215/kjm/1250522434

[10] M. Matsumoto, *The Berwald connection of Finsler with an $(\alpha, \beta)$-metric*, Tensor (NS), **50** (1991), 18–21.

[11] M. Matsumoto, *Theory of Finsler spaces with $(\alpha, \beta)$-metrics*, Rep. Math. Phys., **31** (1991), 43–47.
https://doi.org/10.1016/0034-4877(92)90005-L

[12] R. Miron and M. Anastasiei, *The geometry of Lagrange spaces: Theory and applications*, Kluwer Acad. Publ., **59** (1994).
https://doi.org/10.1007/978-94-011-0788-4_9

[13] Z. Shen and G. Civi Yildirim, *On a class of projectively flat metrics with constant flag curvature*, Can. J. Math., **60:2** (2008), 443–456. Zbl 1157.53014.
https://doi.org/10.4153/cjm-2008-021-1

[14] Z. Shen and C. Yu, *On Einstein square metrics*, Publ. Math. Debrecen, **85** (3–4) (2014), 413–424.
https://doi.org/10.5486/pmd.2014.6015

[15] A. Soleiman, *Recurrent Finsler manifolds under projective change*, Int. J. Geom. Meth. Mod. Phys., **13** (2016), 1650126 (10 pages).
https://doi.org/10.1142/s0219887816501267

[16] A. Soleiman and S. G. Elgendi, *The geometry of Finsler spaces of Hp-scalar curvature*, Differential Geometry - Dynamical Systems, **22** (2020), 254–268.

[17] S. G. Elgendi and A. Soleiman, *An intrinsic proof of Numata's theorem on Landsberg spaces*, J. Korean Math. Soc. **61** (1), (2023), 149–160.
https://doi.org/10.4134/JKMS.j230263

[18] Z.I. Szabo, *Positive definite Finsler spaces satisfying the $T$-condition are Riemannian*, Tensor N.S., **35** (1981), 247–248.

[19] J. Szilasi, R.L. Lovas and D.Cs. Kertész, *Connections, Sprays and Finsler Structures*, World Scientific, 2014.
https://doi.org/10.1142/8659

[20] S. Tachibana, *On Finsler spaces which admit concurrent vector field*, Tensor, N. S., **1** (1950), 1–5.

[21] A. A. Tamim, *Special Finsler manifolds*, J. Egypt. Math. Soc., **10(2)** (2002), 149–177.

[22] Nabil L. Youssef, S. H. Abed and A. Soleiman, *Cartan and Berwald connections in the pullback formalism*, Algebras, Groups and Geometries, **25** (2008), 363–384. arXiv: 0707.1320 [math. DG].

[23] Nabil L. Youssef, S. H. Abed and A. Soleiman, *Concurrent $\pi$-vector fields and eneregy $\beta$-change*, Int. J. Geom. Meth. Mod. Phys., **06** (2011).
https://doi.org/10.1142/S0219887809003904

[24] Nabil L. Youssef, S. H. Abed and A. Soleiman, *A global approach to the theory of special Finsler manifolds*, J. Math. Kyoto Univ., **48** (2008), 857–893. arXiv: 0704.0053 [math. DG].
https://doi.org/10.1215/kjm/1250271321

[25] Nabil L. Youssef, S. H. Abed and A. Soleiman, *A global approach to the theory of connections in Finsler geometry*, Tensor, N. S., **71** (2009), 187–208. arXiv: 0801.3220 [math.DG].

[26] Nabil L. Youssef, S. H. Abed and A. Soleiman, *Geometric objects associated with the fundamental connections in Finsler geometry*, J. Egypt. Math. Soc., **18** (2010), 67–90. arXiv: 0805.2489 [math.DG].

[27] Nabil L. Youssef and S. G. Elgendi, *New Finsler package*, Comput. Phys. Commun., **185** (3) (2014), 986–997.
https://doi.org/10.1016/j.cpc.2013.10.024

[28] Nabil L. Youssef, S. G. Elgendi and Ebtsam H. Taha, *Semi-concurrent vector fields in Finsler geometry*, Differ. Geom. Appl., **65** (2019), 1–15.
https://doi.org/10.1016/j.difgeo.2019.02.011

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