East Asian Math. J.
Vol. 40 (2024), No. 5, pp. 515–526
http://dx.doi.org/10.7858/eamj.2024.036

THE SCHRÖDINGER EQUATION FOR AN EULER OPERATOR ON FOCK SPACES*

Hong Rae Cho

Abstract. We consider the initial value problem of the Schrödinger equation for an Euler operator $\mathcal{R}$ on $\mathbb{C}^n$ that is an analogue of the harmonic oscillator in $\mathbb{R}^n$. We get some regularity results of the Schrödinger equation on Fock spaces.

1. Introduction

Let H be the most basic Schrödinger operator in $\mathbb{R}^n, n \geq 1$, the Hermite operator (or the harmonic oscillator):

$$H = -\Delta + |x|^2.$$

Then the Schrödinger equation for H can be written by

$$(i\partial_t - H)u = 0.$$

This is an important model in quantum mechanics (see for example [4] and [6]). In [6], Nandakumarana and Ratnakumar considered the regularity of the following initial value problem for the Schrödinger equation for H:

$$\begin{cases} (i\partial_t - H)u &= 0 \quad \text{on} \quad \mathbb{R}^n \times (0,\infty) \\ u(\cdot,0) &= f \quad \text{on} \quad \mathbb{R}^n. \end{cases} \tag{1}$$

Let $\mathbb{C}^n$ be the complex n-space. If $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ are points in $\mathbb{C}^n$, we write

$$z \cdot w = \sum_{j=1}^{n} z_j w_j, \qquad |z| = (z \cdot \overline{z})^{1/2}.$$

Received September 13, 2024; Accepted September 19, 2024.
2020 *Mathematics Subject Classification.* Primary 30H20; Secondary 35Q41; 32W50.
Key words and phrases. Fock space; Schrödinger equation; Euler operator; Hermite operator; Bargmann transform; Euler semigroup.
*This work was financially supported by a 2-Year Research Grant of Pusan National University.

(pISSN 1226-6973, eISSN 2287-2833)

There is an interesting operator $\mathcal{R}$ on $\mathbb{C}^n$, given by

$$\mathcal{R} = 2\sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} + n.$$

This $\mathcal{R}$ is an Euler operator.

The Bargmann transform $\mathcal{B}$ is defined by

$$\mathcal{B}f(z) = \frac{1}{\pi^{n/4}} e^{\frac{1}{4}z^2} \int_{\mathbb{R}^n} f(x) e^{-\frac{1}{2}(z-x)^2}\, dx,$$

where dx is the volume measure on $\mathbb{R}^n$, $x^2 = x \cdot x$, and $z^2 = z \cdot z$. We know that

$$\mathcal{B}H = \mathcal{R}\mathcal{B} \quad \text{on} \quad L^2(\mathbb{R}^n).$$

By this relation, the Bargmann transform $\mathcal{B}$ maps the initial value problem (1) to the equivalent form:

$$\begin{cases} (i\partial_t - \mathcal{R})u & = 0 \quad \text{on} \quad \mathbb{C}^n \times (0,\infty) \\ u(\cdot, 0) & = f \quad \text{on} \quad \mathbb{C}^n. \end{cases} \tag{2}$$

Let dV be the ordinary volume measure on $\mathbb{C}^n$. For any $0 < p \le \infty$ we let $L^p_G(\mathbb{C}^n)$ denote the space of Lebesgue measurable functions f on $\mathbb{C}^n$ such that the function $f(z)e^{-\frac{1}{4}|z|^2}$ is in $L^p(\mathbb{C}^n, dV)$. When $0 < p < \infty$, it is clear that

$$L^p_G(\mathbb{C}^n) = L^p\left(\mathbb{C}^n, e^{-\frac{p}{4}|z|^2}\, dV(z)\right).$$

We define

$$\|f\|_{L^p_G} = \left[\left(\frac{p}{4\pi}\right)^n \int_{\mathbb{C}^n} \left|f(z)e^{-\frac{1}{4}|z|^2}\right|^p dV(z)\right]^{\frac{1}{p}}.$$

For $p = \infty$ the norm in $L^\infty_G(\mathbb{C}^n)$ is defined by

$$\|f\|_{L^\infty_G} = \operatorname{esssup}\left\{|f(z)|e^{-\frac{1}{4}|z|^2} : z \in \mathbb{C}^n\right\}.$$

Let $F^p(\mathbb{C}^n)$ denote the space of entire functions in $L^p_G(\mathbb{C}^n)$. If $0 < p < q$, then $F^p \subset F^q$, and the inclusion is proper and continuous (see [9]). Note that F^2 is a closed subspace of the Hilbert space L^2_G with inner product

$$\langle f, g\rangle_{F^2} = \frac{1}{(2\pi)^n} \int_{\mathbb{C}^n} f(z)\overline{g(z)} e^{-\frac{1}{2}|z|^2}\, dV(z).$$

In this paper, we consider the regularity of the regularized problem

$$\begin{cases} (i\partial_t - \mathcal{R})u & = 0 \quad \text{on} \quad \mathbb{C}^n \times (0,\infty) \\ u(\cdot, 0) & = e^{-r\mathcal{R}}f \quad \text{on} \quad \mathbb{C}^n. \end{cases} \tag{3}$$

Theorem 1.1. *Let $r \ge 0$. Then $u_r(z,t) = e^{-(r+it)\mathcal{R}}f(z)$ is the solution of the regularized problem (3) satisfying the inequality*

$$\sup_{t\in\mathbb{R}} \|u_r(\cdot,t)\|_{F^p} \le \|f\|_{F^{p'}},$$

where $1 \le p' \le 2$, $2 \le p \le \infty$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$.

2. Hermite operator and Euler operator

2.1. Hermite operator

The Hermite operator

$$H = -\Delta + |x|^2$$

is self-adjoint on the set of infinitely differentiable functions with compact support $C_c^\infty(\mathbb{R}^n)$, and it can be factorized as

$$H = \frac{1}{2}\sum_{j=1}^{n}\left(a_j a_j^\dagger + a_j^\dagger a_j\right),$$

where

$$a_j = \frac{\partial}{\partial x_j} + x_j \quad \text{and} \quad a_j^\dagger = -\frac{\partial}{\partial x_j} + x_j, \quad 1 \le j \le n.$$

In one dimension, the Hermite polynomials H_k are defined by

$$H_k(x) = (-1)^k e^{x^2}\frac{d^k}{dx^k}\left(e^{-x^2}\right), \quad x \in \mathbb{R},$$

and by normalization we obtain the Hermite functions,

$$h_k(x) = \frac{1}{\pi^{1/4}}\frac{1}{\sqrt{2^k k!}}e^{-\frac{1}{2}x^2}H_k(x), \quad x \in \mathbb{R}.$$

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be the set of nonnegative integer. In higher dimensions, for each multi-index $I = (I_1, \cdots, I_n) \in \mathbb{N}_0^n$, the Hermite polynomials H_I are defined by

$$H_I(x) = \prod_{j=1}^{n} H_{I_j}(x_j), \quad x = (x_1, \cdots, x_n) \in \mathbb{R}^n$$

and the Hermite functions h_I are defined by

$$\begin{aligned} h_I(x) &= \prod_{j=1}^{n} h_{I_j}(x_j) \\ &= \frac{1}{\pi^{n/4}}\frac{1}{\sqrt{2^{|I|}I!}}e^{-\frac{1}{2}x^2}H_I(x), \quad x = (x_1, \cdots, x_n) \in \mathbb{R}^n. \end{aligned}$$

Then $\{h_I : I \in \mathbb{N}_0^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$.

Lemma 2.1 ([9]).

$$Hh_I = (2|I| + n)h_I.$$

Let $\mathcal{H}$ be the space of finite linear combinations of Hermite functions,

$$f = \sum_{|I| \le N} \langle f, h_I \rangle_{L^2(\mathbb{R}^n)} h_I,$$

where

$$\langle f, h_I \rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f(x) h_I(x) \, dx.$$

The space $\mathcal{H}$ is dense in $L^2(\mathbb{R}^n)$, and so, by the orthonormality of the Hermite functions,

$$\|f\|_{L^2(\mathbb{R}^n)} = \left(\sum_{I \in \mathbb{N}_0^n} |\langle f, h_I \rangle_{L^2(\mathbb{R}^n)}|^2 \right)^{1/2}.$$

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of rapidly decreasing $C^\infty(\mathbb{R}^n)$ functions. For $f \in \mathcal{S}(\mathbb{R}^n)$, the Hermite series expansion

$$\sum_{I \in \mathbb{N}_0^n} \langle f, h_I \rangle_{L^2(\mathbb{R}^n)} h_I$$

converges to f uniformly in $\mathbb{R}^n$ (and also in $L^2(\mathbb{R}^n)$), since $\|h_I\|_{L^\infty(\mathbb{R}^n)} \le C$, for all $I \in \mathbb{N}_0^n$, and for each $m \in \mathbb{N}$, we have (see [8])

$$|\langle f, h_I \rangle_{L^2(\mathbb{R}^n)}| \le \|H^m f\|_{L^2(\mathbb{R}^n)} (2|I| + n)^{-m}.$$

The spectral decomposition of H on $\mathbb{R}^n$ is given by

$$Hf = \sum_{I \in \mathbb{N}_0^n} (2|I| + n) \langle f, h_I \rangle_{L^2(\mathbb{R}^n)} h_I.$$

2.2. Euler operator

The Euler operator $\mathcal{R}$ can be written by

$$\mathcal{R} = \frac{1}{2} \sum_{j=1}^{n} \left(A_j A_j^* + A_j^* A_j \right),$$

where

$$A_j = 2 \frac{\partial}{\partial z_j}, \qquad A_j^* = z_j, \quad 1 \le j \le n.$$

Both A_j and A_j^*, as defined above, are densely defined linear operators on F^p (unbounded though).

Remark 1. Let

$$f(z) = \sum_{k=0}^{\infty} \frac{z_1^k}{\sqrt{2}^k (k+1) \sqrt{k!}}.$$

Then $f \in F^2$, but $\mathcal{R} f \notin F^2$.

The remark above tells us that $\mathrm{Dom}(\mathcal{R}) \subsetneq F^2$. Thus $\mathcal{R}$ is an unbounded operator on F^2. Moreover, we know that $\mathcal{R}$ is a positive, self-adjoint operator on $\mathrm{Dom}(\mathcal{R})$.

For $f \in F^2$ let

$$f(z) = \sum_{I \in \mathbb{N}_0^n} c_I e_I(z)$$

be the orthonormal decomposition of f. Since $\mathcal{R}$ has the discrete spectrum $\sigma(\mathcal{R}) = \{2|I| + n : I \in \mathbb{N}_0^n\}$, $\mathcal{R}f$ is given by

$$\mathcal{R}f(z) = \sum_{I \in \mathbb{N}_0^n} (2|I| + n) c_I e_I(z), \quad f \in \mathrm{Dom}(\mathcal{R}).$$

2.3. Bargmann transform

It is well-known that the Bargmann transform $\mathcal{B}$ is a unitary isomorphism between $L^2(\mathbb{R}^n)$ and $F^2(\mathbb{C}^n)$ ([1], [9]).

Lemma 2.2 ([9]). *For each $j = 1, \ldots, n$, we have*

$$\mathcal{B}(a_j f) = A_j \mathcal{B}_(f)$$
$$\mathcal{B}(a_j^\dagger f) = A_j^* \mathcal{B}(f).$$

Lemma 2.3 ([9]). *Let*

$$e_I(z) = \frac{z^I}{\sqrt{2^{|I|} I!}}.$$

Then $\{e_I : I \in \mathbb{N}_0^n\}$ is an orthonormal basis for F^2 and $\mathcal{B}(h_I) = e_I$.

Corollary 2.4. *We have*

$$\mathcal{B}H = \mathcal{R}\mathcal{B}.$$

Proof. For $f \in \mathcal{S}(\mathbb{R}^n)$ we have

$$Hf = \sum_{I \in \mathbb{N}_0^n} (2|I| + n) \langle f, h_I \rangle_{L^2(\mathbb{R}^n)} h_I$$

and so

$$\mathcal{B}(Hf) = \sum_{I \in \mathbb{N}_0^n} (2|I| + n) \langle f, h_I \rangle_{L^2(\mathbb{R}^n)} e_I.$$

Since $\mathcal{B}$ is a unitary isomorphism, we have $\langle f, h_I \rangle_{L^2(\mathbb{R}^n)} = \langle \mathcal{B}(f), e_I \rangle_{F^2}$, hence

$$\mathcal{B}(Hf) = \sum_{I \in \mathbb{N}_0^n} (2|I| + n) \langle \mathcal{B}(f), e_I \rangle_{F^2} \, e_I = \mathcal{R}\mathcal{B}(f).$$

Thus we get the result. □

3. Regularized Schrödinger equation

3.1. Euler semigroup

We know that $\{e_I : I \in \mathbb{N}_0^n\}$ is an orthonormal basis for F^2. For $f \in F^2$ let

$$f(z) = \sum_{I \in \mathbb{N}_0^n} c_I e_I(z)$$

be the orthonormal decomposition of f. Associated with the operator $\mathcal{R}$ is a semigroup $\{B_t\}_{t\geq 0}$ defined by the expansion

$$B_t f(z) = \sum_{I \in \mathbb{N}_0^n} e^{-i(2|I|+n)t} c_I e_I(z).$$

It is easy to see that $B_t f(z) \to f(z)$ in F^2 as $t \to 0^+$ by the dominated convergence theorem since $|e^{-i(2|\alpha|+n)t} - 1| \leq 2$. We know that $\{B_t\}_{t\geq 0}$ is a strongly continuous semigroup. Moreover, $-i\mathcal{R}$ is the infinitesimal generator of $\{B_t\}_{t\geq 0}$.

Proposition 3.1. *$-i\mathcal{R}$ is the infinitesimal generator of $\{B_t\}_{t\geq 0}$. That is,*

$$\lim_{t\to 0^+} \frac{B_t f - f}{t} = -i\mathcal{R} f$$

for $f \in \mathrm{Dom}(\mathcal{R})$.

Proof. Let $f \in \mathrm{Dom}(\mathcal{R})$. Then we have

$$\frac{B_t f(z) - f(z)}{t} - (-i\mathcal{R} f(z)) = \sum_{I \in \mathbb{N}_0^n} \left(\frac{e^{-i(2|I|+n)t} - 1}{t} + i(2|I| + n) \right) c_I e_I(z).$$

We note that

$$\left| \frac{e^{-i(2|I|+n)t} - 1}{t} + i(2|I| + n) \right| |c_I||e_I(z)| \leq 2(2|I| + n)|c_I||e_I(z)|$$

for small $t > 0$. Since

$$2 \sum_{I \in \mathbb{N}_0^n} (2|I| + n)|c_I||e_I(z)| < \infty,$$

by the dominated convergence theorem, we have

$$\begin{aligned} &\lim_{t\to 0^+} \sum_{I \in \mathbb{N}_0^n} \left(\frac{e^{-i(2|I|+n)t} - 1}{t} + i(2|I| + n) \right) c_I e_I(z) \\ &\quad = \sum_{I \in \mathbb{N}_0^n} \lim_{t\to 0^+} \left(\frac{e^{-i(2|I|+n)t} - 1}{t} + i(2|I| + n) \right) c_I e_I(z) = 0. \end{aligned}$$

Hence

$$\lim_{t\to 0^+} \frac{B_t f(z) - f(z)}{t} = -i\mathcal{R} f(z).$$

Since $B_t f$ and $\mathcal{R}f$ belong to F^2, by the dominated convergence theorem again, we have

$$\lim_{t\to 0^+} \left\| \frac{B_t f - f}{t} - (-i\mathcal{R}f) \right\|_{F^2}^2 = 0.$$

Thus we get the result. □

Thus, we have (see [3])

$$B_t = e^{-it\mathcal{R}}$$

and so $u(z,t) = e^{-it\mathcal{R}}$ is the solution of the initial value problem:

$$\begin{cases} (i\partial_t - \mathcal{R})u &= 0 \quad \text{on} \quad \mathbb{C}^n \times (0,\infty) \\ u(\cdot,0) &= f \quad \text{on} \quad \mathbb{C}^n. \end{cases}$$

Proposition 3.2. *The operator* $e^{-it\mathcal{R}}$ *is unitary in* F^2*. Hence* $\mathrm{Dom}(e^{-it\mathcal{R}}) = F^2$ *and* $(e^{-it\mathcal{R}})^{-1} = e^{it\mathcal{R}}$*.*

Proof. For $f \in F^2$, we have a holomorphic expansion of $f(z) = \sum c_\alpha e_\alpha(z)$. Then

$$\begin{aligned} u(z,t) &= e^{-it\mathcal{R}} f(z) \\ &= e^{-int} \sum_\alpha e^{-2it|\alpha|} c_\alpha e_\alpha(z). \end{aligned}$$

So we have

$$\begin{aligned} \|u(\cdot,t)\|_{F^2}^2 &= \langle u(\cdot,t), u(\cdot,t)\rangle \\ &= \left\langle e^{-int} \sum_\alpha e^{-2it|\alpha|} c_\alpha e_\alpha, e^{-int} \sum_\beta e^{-2it|\beta|} c_\beta e_\beta \right\rangle_{F^2} \\ &= \sum_{\alpha,\beta} c_\alpha \overline{c_\beta} e^{-2it(|\alpha|-|\beta|)} \langle e_\alpha, e_\beta\rangle_{F^2} \\ &= \sum_\alpha |c_\alpha|^2 = \|f\|_{F^2}^2. \end{aligned}$$

□

3.2. The kernel associated to the Euler semigroup

It is well-known ([1], [9]) that for $f \in F^2$ we have the reproducing formula such that

$$f(z) = \frac{1}{(2\pi)^n} \int_{\mathbb{C}^n} f(w) K(z,w) e^{-\frac{1}{2}|z|^2} \, dV(w),$$

where $K(z,w)$ is the reproducing kernel defined by

$$K(z,w) = \sum_I e_I(z) \overline{e_I(w)}.$$

In fact, we know that

$$K(z,w) = e^{\frac{1}{2}z\cdot\bar{w}}.$$

By the spectral theory,

$$\begin{aligned}
u(z,t) &= e^{-it\mathcal{R}} f(z) \\
&= e^{-it\mathcal{R}} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{C}^n} f(w) \sum_I e_I(z)\overline{e_I(w)} e^{-\frac{1}{2}|w|^2} dV(w) \right) \\
&= e^{-it\mathcal{R}} \left(\sum_I e_I(z) \right) \frac{1}{(2\pi)^n} \int_{\mathbb{C}^n} f(w)\overline{e_I(w)} e^{-\frac{1}{2}|w|^2} dV(w) \\
&= \sum_I e^{-it(2|I|+n)} e_I(z) \frac{1}{(2\pi)^n} \int_{\mathbb{C}^n} f(w)\overline{e_I(w)} e^{-\frac{1}{2}|w|^2} dV(w) \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{C}^n} f(w) \sum_I e^{-it(2|I|+n)} e_I(z)\overline{e_I(w)} e^{-\frac{1}{2}|w|^2} dV(w) \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{C}^n} f(w) K_t(z,w) e^{-|w|^2} dV(w).
\end{aligned}$$

Interchanging the order of summation and integration is justified by the dominated convergence theorem since

$$\sum_I |e_I(z)| \int_{\mathbb{C}^n} |f(w)||e_I(w)| e^{-\frac{1}{2}|w|^2} dV(w) \le \sum_I \frac{|z^I|}{\sqrt{2^{|I|} I!}} \|f\|_{F^2}$$

and the power series on the right side of the inequality above is convergent for every $z \in \mathbb{C}^n$.

Note that

$$\begin{aligned}
K_t(z,w) &= \sum_I e^{-it(2|I|+n)} e_I(z)\overline{e_I(w)} \\
&= e^{-int} \sum_I e^{-2it|I|} \frac{z^I \bar{w}^I}{2^{|I|} I!} \\
&= e^{-int} \exp\left[\frac{1}{2} e^{-2it} z \cdot \bar{w} \right].
\end{aligned}$$

Hence $K_{t+2\pi}(z,w) = K_t(z,w)$ and

$$\begin{aligned}
|K_t(z,w)| &= \exp\left[\operatorname{Re}\left(\frac{1}{2} e^{-2it} z \cdot \bar{w} \right) \right] \\
&\le \exp\left(\frac{1}{2} |z \cdot \bar{w}| \right).
\end{aligned}$$

3.3. Regularity of the regularized Schrödinger equation

By using Gross's logarithmic Sobolev inequality [5], Carlen proved the hypercontractivity inequality:

Lemma 3.3 ([2]). *Let $f \in H(\mathbb{C}^n)$. Let $r > 0$ and $0 < p \le q < \infty$. Then*

$$\left\| |f|^r \right\|_{L^q_G} \le \left\| |f|^r \right\|_{L^p_G}$$

and the estimate is sharp.

Proposition 3.4. *Let $0 < p < \infty$ and $r > 0$. Then $e^{-r\mathcal{R}}$ is contraction on F^p.*

Proof. Let $f \in F^p$. Then

$$\begin{aligned}
\|e^{-r\mathcal{R}} f\|_{F^p}^p &= \left(\frac{p}{4\pi}\right)^n \int_{\mathbb{C}^n} \left| e^{-r\mathcal{R}} f(z) e^{-\frac{1}{4}|z|^2} \right|^p dV(z) \\
&= \left(\frac{p}{4\pi}\right)^n e^{-rnp} \int_{\mathbb{C}^n} \left| f(e^{-2r} z) e^{-\frac{1}{4}|z|^2} \right|^p dV(z) \\
&\le \left(\frac{p}{4\pi}\right)^n \int_{\mathbb{C}^n} |f(w)|^p e^{-\frac{p}{4} e^{4r} |w|^2} e^{4nr} \, dV(w) \\
&\le \left(\frac{pe^{4r}}{4\pi}\right)^n \int_{\mathbb{C}^n} \left| |f(w)|^s e^{-\frac{1}{4}|w|^2} \right|^{pe^{4r}} dV(w) \\
&= \left\| |f|^s \right\|_{L_G^{pe^{4r}}}^{pe^{4r}},
\end{aligned}$$

where $s = e^{-4r}$. By Lemma 3.3, we have

$$\left\| |f|^s \right\|_{L_G^{pe^{4r}}}^{pe^{4r}} \le \left\| |f|^s \right\|_{L_G^p}^{pe^{4r}}.$$

Hence

$$\|e^{-r\mathcal{R}} f\|_{F^p}^p \le \left\| |f|^s \right\|_{L_G^p}^{pe^{4r}}$$

By Jensen's inequality, we have

$$\begin{aligned}
\left\| |f|^s \right\|_{L_G^p}^{pe^{4r}} &= \left[\left(\frac{p}{4\pi}\right)^n \int_{\mathbb{C}^n} |f(z)|^{\frac{p}{e^{4r}}} e^{-\frac{p}{4}|z|^2} dV(z) \right]^{e^{4r}} \\
&\le \left(\frac{p}{4\pi}\right)^n \int_{\mathbb{C}^n} |f(z)|^p e^{-\frac{p}{4}|z|^2} dV(z).
\end{aligned}$$

Therefore

$$\|e^{-r\mathcal{R}} f\|_{F^p} \le \|f\|_{F^p}.$$

□

Now, we consider the regularity of the regularized problem

$$\begin{cases} (i\partial_t - \mathcal{R})u &= 0 \quad \text{on} \quad \mathbb{C}^n \times (0, \infty) \\ u(\cdot, 0) &= e^{-r\mathcal{R}} f \quad \text{on} \quad \mathbb{C}^n. \end{cases}$$

Let

$$f(z) = \sum_{k=0}^{\infty} f_k(z),$$

where

$$f_k(z) = \sum_{|I|=k} c_I e_I(z).$$

Then the solution in this case is given by

$$u_r(z,t) = e^{-it\mathcal{R}} e^{-r\mathcal{R}} f(z) = \sum_{k=0}^{\infty} e^{-(r+it)(2k+n)} f_k(z).$$

Let $\zeta = r + it$, $r > 0, t \in \mathbb{R}$. Then

$$\begin{aligned} u_r(z,t) &= e^{-it\mathcal{R}} e^{-r\mathcal{R}} f(z) \\ &= e^{-\zeta\mathcal{R}} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{C}^n} f(w) \sum_I e_I(z)\overline{e_I(w)} e^{-\frac{1}{2}|w|^2} dV(w) \right) \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{C}^n} f(w) \sum_I e^{-\zeta(2|I|+n)} e_I(z)\overline{e_I(w)} e^{-\frac{1}{2}|w|^2} dV(w) \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{C}^n} f(w) K_\zeta(z,w) e^{-|w|^2} dV(w), \end{aligned}$$

where

$$K_\zeta(z,w) = \sum_{k=0}^{\infty} e^{-\zeta(2k+n)} \sum_{|I|=k} e_I(z)\overline{e_I(w)} \tag{4}$$

which is the kernel associated to the semigroup $e^{-\zeta\mathcal{R}}$. Clearly, the semigroup $e^{-\zeta\mathcal{R}}$ is also periodic in t with period 2π.

Lemma 3.5. *Let $\zeta = r + it$, $r > 0, 0 < |t| \le \pi$. Then*

$$|K_\zeta(z,w)| \le e^{-nr} \exp\left[\frac{1}{2} e^{-2r} |z \cdot \bar{w}|\right].$$

Proof. The above series can be re-written as

$$K_\zeta(z,w) = e^{-n(r+it)} \exp\left[\frac{1}{2} e^{-2\zeta} z \cdot \bar{w}\right].$$

Hence

$$\begin{aligned} |K_\zeta(z,w)| &= e^{-nr} \exp\left[\frac{1}{2} \mathrm{Re}(e^{-2\zeta} z \cdot \bar{w})\right] \\ &= e^{-nr} \exp\left[\frac{1}{2} e^{-2r} \mathrm{Re}(e^{-2it} z \cdot \bar{w})\right]. \end{aligned}$$

□

Theorem 3.6. *Let $r \ge 0$. Then $u_r(z,t) = e^{-(r+it)\mathcal{R}} f(z)$ is the solution of the regularized problem (3) satisfying the inequality*

$$\sup_{t\in\mathbb{R}} \|u_r(\cdot,t)\|_{F^p} \le \|f\|_{F^{p'}},$$

where $1 \le p' \le 2$, $2 \le p \le \infty$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Let $\zeta = r + it$. We note that

$$\begin{aligned} |K_\zeta(z,w)|e^{-\frac{1}{2}|w|^2-\frac{1}{4}|z|^2} &\le e^{-nr}\exp\left[\frac{1}{2}e^{-2r}\mathrm{Re}(e^{-2it}z\cdot\bar{w})\right]e^{-\frac{1}{2}|w|^2-\frac{1}{4}|z|^2} \\ &\le e^{-nr}\exp\left[\frac{1}{2}e^{-2r}|z\cdot\bar{w}|\right]e^{-\frac{1}{2}|w|^2-\frac{1}{4}|z|^2} \\ &= e^{-nr}e^{\frac{1}{2}e^{-2r}|z\cdot\bar{w}|-\frac{1}{2}|w|^2-\frac{1}{4}|z|^2} \end{aligned}$$

and

$$-\frac{1}{2}|w|^2 - \frac{1}{4}|z|^2 + \frac{1}{2}|z\cdot\bar{w}| \le -\frac{1}{2}|w|^2 - \frac{1}{4}|z|^2 + \frac{1}{2}|z||w| \le -\frac{1}{4}|w|^2.$$

Hence

$$\begin{aligned} \|u_r(\cdot,t)\|_{F^\infty} &= \sup_{z\in\mathbb{C}^n}|u_r(z,t)|e^{-\frac{1}{4}|z|^2} \\ &\le \frac{1}{(2\pi)^n}\sup_{z\in\mathbb{C}^n}\left[\int_{\mathbb{C}^n}|f(w)||K_\zeta(z,w)|e^{-\frac{1}{2}|w|^2-\frac{1}{4}|z|^2}dV(w)\right] \\ &\le e^{-nr}\frac{1}{(2\pi)^n}\left[\int_{\mathbb{C}^n}|f(w)|e^{-\frac{1}{4}|w|^2}dV(w)\right] \le \|f\|_{F^1}. \end{aligned}$$

On the other hand, for $f \in F^2$, we have

$$u_r(z,t) = e^{-it\mathcal{R}}\big(e^{-r\mathcal{R}}f\big)(z).$$

By Proposition 3.2 and Proposition 3.4, we have

$$\begin{aligned} \|u_r(\cdot,t)\|_{F^2}^2 &= \|e^{-it\mathcal{R}}\big(e^{-r\mathcal{R}}f\big)\|_{F^2}^2 \\ &= \|e^{-r\mathcal{R}}f\|_{F^2}^2 \\ &\le \|f\|_{F^2}^2. \end{aligned}$$

Hence by Riesz-Thorin interpolation theorem [7], for $p \in [1,2]$ we have

$$\sup_{t\in\mathbb{R}}\|u_r(\cdot,t)\|_{F^p} \le \|f\|_{F^{p'}},$$

where $1 \le p' \le 2$, $2 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. □

References

[1] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform I*, Comm. Pure Appl. Math. 14 (1961), 187-214.

[2] E. A. Carlen, *Some integral identities and inequalities for entire functions and their application to the coherent state transform*, J. Funct. Anal. 97 (1991), 231-249.

[3] K.-J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics 194, Springer-Verlag, Berlin, New York, 2000.

[4] R. P. Feynman and A. R. Hibbs, *Quantum mechanics and path integrals*, McGraw-Hill, Maidenhead, 1965.

[5] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. 97 (1975), 1061-1083.

[6] A. K. Nandakumaran and P. K. Ratnakumar, *Schrödinger equation and the oscillatory semigroup for the Hermite operator*, J. Funct. Anal. 224 (2005), 371-385.
[7] E. Stein and G. Weiss, *Interpolation of operators with change of measures*, Trans. Amer. Math. Soc. 87 (1958), 159-172.
[8] P. R. Stinga and J. L. Torrea, *Regularity theory for the fractional harmonic oscillator*, J. Funct. Anal. 260 (2011), no. 10, 3097-3131.
[9] K. Zhu, *Analysis on Fock spaces*, Graduate Texts in Mathematics, 263. Springer, New York, 2012.

Hong Rae Cho
Department of Mathematics, Pusan National University, Busan 46241, KOREA
Email address: chohr@pusan.ac.kr