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Classical and Bayesian inferences of stress-strength reliability model based on record data

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Abstract

In reliability analysis, the probability $P(Y < X)$ is significant because it denotes availability and dependability in a stress-strength model where Y and X are the stress and strength variables, respectively. In reliability theory, the inverse Lomax distribution is a well-established lifetime model, and the literature is developing inference techniques for its reliability attributes. In this article, we are interested in estimating the stress-strength reliability $R = P(Y < X)$, where X and Y have an unknown common scale parameter and follow the inverse Lomax distribution. Using Bayesian and non-Bayesian approaches, we discuss this issue when both stress and strength are expressed in terms of lower record values. The parametric bootstrapping techniques of R are taken into consideration. The stress-strength reliability estimator is investigated using uniform and gamma priors with several loss functions. Based on the proposed loss functions, the reliability R is estimated using Bayesian analyses with Gibbs and Metropolis-Hasting samplers. Monte Carlo simulation studies and real-data-based examples are also performed to analyze the behavior of the proposed estimators. We analyze electrical insulating fluids, particularly those used in transformers, for data sets using the stress-strength model. In conclusion, as expected, the study's results showed that the mean squared error values decreased as the record number increased. In most cases, Bayesian estimates under the precautionary loss function are more suitable in terms of simulation conclusions than other specified loss functions.

Keywords: inverse Lomax distribution, lower record values, Bayesian estimation, bootstrap confidence intervals, Markov Chain Monte Carlo

1. Introduction

In mechanical reliability, Y represents the random stress employed to a system with strength X, and $R = P(Y < X)$ represents the system's reliability. In a stress-strength (SS) model, the system breaks down on the condition that the applied stress surpasses its strength. The SS model appears in several applications in various fields, such as in the medical sciences, structural engineering, and in natural occurrences such as floods, earthquakes, and other natural phenomena. In life test experiments, X and Y represent the useful life of a product manufactured by two companies with the same warranty period, and $P(Y < X)$ represents the probability that one is superior to the other. In medicine, the random

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variables X and Y measure the effects of the control treatment and the new treatment, respectively. When a new therapy is used in place of a control treatment, the measure $R = P(Y < X)$ represents how effective the new treatment is in comparison to the control treatment. In structural engineering, Y and X represent the bridge's stress (load) and strength (capacity). A bridge can only withstand tension if its strength is greater than the stress. As a result, $R = P(Y < X)$ describes the bridge's survival probability. Several authors have predicted the estimators of R. For an extensive literature review regarding the estimation and application of SS reliability, readers can refer to the works of Johnson (1998) and Kotz *et al.* (2003) for a thorough assessment of the literature on the estimate and use of SS reliability.

Record values (RVs) and the associated statistics have an interest and importance in numerous areas of real-world applications, including data relating to economics, meteorology, athletic events, sports, oil, mining surveys, and life testing. The RV for a phenomenon is the largest (smallest) observation anyone ever made. The mathematical theory and statistical analysis of RVs presented by Chandler (1952) have now spread in a variety of areas.

Let $X_1, X_2, \ldots$ represent a series of independent and identically distributed (iid) continuous random variables with a cumulative distribution function (CDF) $F(x)$ and its corresponding probability density function (PDF) $f(x)$. An observation X_j is called a lower record value (LRV) and denoted by $X_{L(j)}$ if $X_j < X_{j-1},\ \ j > 1$. Let $R_i,\ i = 1, 2, \ldots, n$ be the first n LRV resulting from any distribution with a specific PDF and CDF. As reported by Arnold *et al.* (1998), the joint PDF of R_i is given by

$$f(r_1, r_2, \ldots, r_n) = f(r_n) \prod_{i=1}^{n-1} \frac{f(r_i)}{F(r_i)}, \quad -\infty < r_n < \cdots < r_1 < \infty.$$

Several researchers have discussed the estimation of reliability parameter R based on RVs with various lifetime models over the last two decades. Baklizi (2008) introduced the likelihood and Bayesian estimation of $R = P(Y < X)$ using LRVs from the generalized exponential distribution. Based on upper RVs, Tarvirdizade and Kazemzadeh (2016) have deduced the maximum likelihood estimates (MLEs), Bayes estimates (BEs), and accurate confidence intervals (CIs) of R for bathtub-shaped distribution. Condino *et al.* (2016) studied a proportional reversed hazard family and obtained a non-Bayesian estimate and BE of the parameter R. Amin (2017) described the estimation of R via upper RVs for Kumaraswamy exponential distribution. The MLE and BE under the squared error loss function (SELF), and the linear exponential (LINEX) loss function of R based on LRVs for power function distribution were obtained by Dhanya and Jeevavand (2018). Hassan *et al.* (2018a,b) examined the estimation of R for the generalized inverse exponential using advanced RVs. Based on RVs, Raqab *et al.* (2018) considered estimation for the bathtub-shaped distribution. For more recent studies the reader can refer to Hassan *et al.* (2020), Khan and Khatoon (2020), Chaturvedi and Malhotra (2020), Pak *et al.* (2021), Hassan *et al.* (2022), Mohamed (2022), and Hassan *et al.* (2024a,b).

In the literature, there has been little research to examine the SS reliability models based on LRVs. To the best of our knowledge, no studies have been performed on the inference of the SS reliability for the inverse Lomax distribution (ILD) using LRV data. So, the primary goal in this work is to establish some inferential procedures assuming that the stress (Y) and strength (X) are two independently distributed but not identically distributed random variables from a flexible ILD with common scale parameter. This distribution has received significant consideration from several authors and widespread applications (see Section 2). This model has many failure rate forms, which is beneficial for a variety of data applications. On the other hand, the RVs are a different kind of incomplete data that are often seen in a wide range of practical applications. Thus, we provide the following summary of our study.

1. The Bayesian estimator and non-Bayesian (ML) estimator of R are derived. The BE of R is considered using non-informative prior (NIP) and informative prior (IP).
2. The BE of R is determined under symmetric (SELF). Also, the BE of R is provided under asymmetric loss functions (LFs), including weighted SELF (WSELF), precautionary LF (PLF), and minimum expected LF (MELF). The significance of the selected loss functions is mentioned in sub-section 5.1.
3. Bootstrap confidence intervals (BCIs), including percentile bootstrap and bootstrap-t for R, are obtained.
4. We adopt the Markov chain Monte Carlo (MCMC) approach due to the complicated forms of the BEs of R.
5. Finally, we analyze the SS model using electrical insulating fluid data to study the performances of several estimates.

The rest of the paper is organized as follows. In Section 2, we provide the reliability function R when X and Y have an ILD with the same scale parameter but different shape parameters. The MLEs and BCIs for R are obtained based on the LRVs in Sections 3 and 4, respectively. The BE under symmetric and asymmetric LFs is obtained in Section 5. In Section 6, the MCMC technique is discussed. A numerical simulation study and different real data sets are provided to report the performances of the different estimation methods in Section 7. The paper ends with some concluding remarks.

2. Model description

Kleiber and Kotz (2003) introduced the ILD, which has a wide application in stochastic modeling of the decreasing failure rate of life components and is one of the most frequently used distributions in economics (Kleiber and Kotz, 2003), geography data (Kleiber, 2004; McKenzie *et al.*, 2011), actuarial science (Reyad and Othman, 2018), and medical data (Hassan *et al.*, 2023). Research on statistical inference and prediction on the ILD using the Bayesian method was conducted by Rahman *et al.* (2013). Rahman and Aslam (2014) have recently examined the two-component mixture ILD for the prediction of future ordered observations in the Bayesian framework utilizing the predictive model. Singh *et al.* (2016) examined reliability estimates of the ILD under the Type-II censoring scheme. Yadav *et al.* (2019) studied the Bayesian estimation of the SS reliability parameter of the ILD. AlOmari *et al.* (2021) discussed the estimation of SS reliability for ILD based on the extreme ranked set sampling data. Sharma and Kumar (2021) have explored the Bayesian and E-Bayesian estimation of ILD parameters based on Type-II censored samples.

A random variable X is considered to have an ILD with scale parameter η and a shape parameter ρ, denoted by $X \sim \text{ILD}(\rho, \eta)$, then the PDF and CDF are given, respectively, by

$$f(x;\rho,\eta) = \frac{\rho\,\eta}{x^2}\left[1+\frac{\eta}{x}\right]^{-(1+\rho)}, \quad x \geq 0, \quad \rho\,,\,\eta > 0,$$

and,

$$F(x;\rho,\eta) = \left[1+\frac{\eta}{x}\right]^{-\rho}, \quad x \geq 0, \quad \rho\,,\,\eta > 0.$$

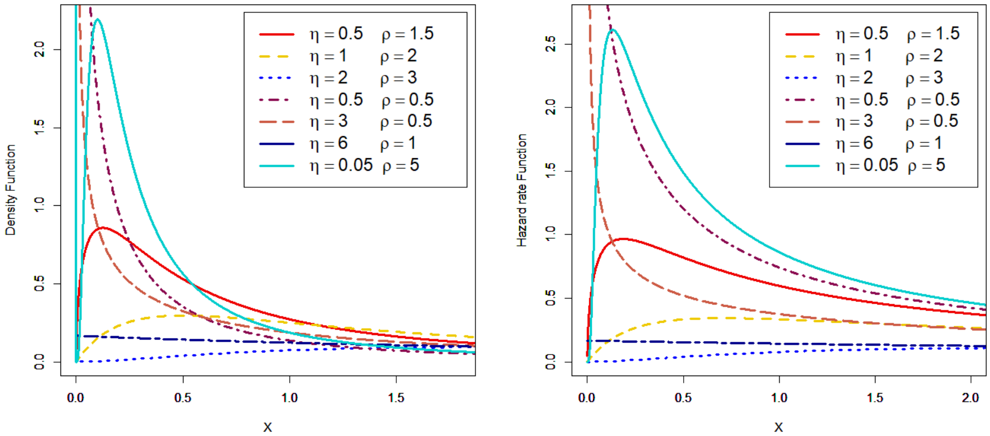


Figure 1: *Plots of PDF and HRF of the ILD.*

The survival function and hazard function (HRF) of the ILD are as follows;

$$\overline{F}(x) = 1 - \left[1+\frac{\eta}{x}\right]^{-\rho}, \quad x \geq 0, \quad \rho\,,\,\eta > 0,$$

and

$$H(x) = \frac{\rho\,\eta\left[1+\frac{\eta}{x}\right]^{-(1+\rho)}}{x^2\left[1-\left[1+\frac{\eta}{x}\right]^{-\rho}\right]}, \quad x \geq 0, \quad \rho\,,\,\eta > 0.$$

Figure 1 illustrates plots of the ILD PDF and HRF for some selected values of the parameters. The versatility of the ILD to describe data that is right-skewed, decreasing, and in the shape of an upside-down bathtub is demonstrated by these figures.

Let X and Y be two independent SS random variables with $X \sim$ ILD $(\rho,\,\eta)$ and $Y \sim$ ILD $(\delta,\,\eta)$, and the SS reliability parameter R is,

$$R = P(Y<X) = \int_{x=0}^{\infty} F_Y(x)\; f_X(x)\;\mathrm{dx} = \int_{x=0}^{\infty} \frac{\rho\,\eta}{x^2}\left[1+\frac{\eta}{x}\right]^{-(1+\rho)}\left[1+\frac{\eta}{x}\right]^{-\delta}\mathrm{dx} = \frac{\rho}{\rho+\delta}. \tag{2.1}$$

It is clear that the SS reliability R given in Equation (2.1) depends on the parameters ρ and δ.

3. Maximum likelihood estimation

This section provides the MLE of R. To obtain the MLE of R, we must first determine the MLEs of the unknown parameters ρ, δ, and η in the model. Let $X_1, X_2, \ldots$ be a succession of *iid* random variables have, an ILD $(\rho,\,\eta)$, and let $\underline{r} = (r_1, r_2, \ldots, r_n)$ be the corresponding group of first n LRVs.

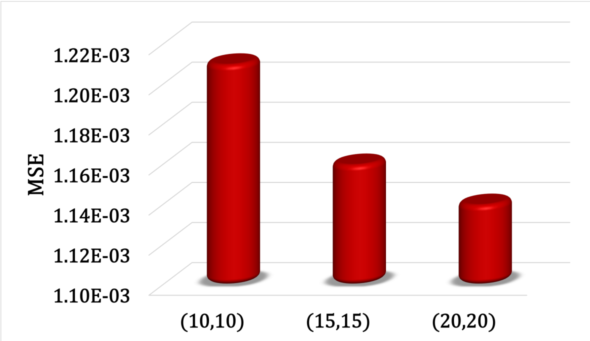


Figure 2: *The MSEs of $\hat{R}_{ML}$ for different record numbers at true value $R = 0.3$.*

Likewise, suppose $Y_1, Y_2, \ldots$ be another succession of *iid* random variables, have an ILD (δ, η), and be the corresponding group of the first m LRVs, which are $\underline{s} = (s_1, s_2, \ldots, s_m)$. Then, let's denote the parameter $\theta = (\rho, \delta, \eta)$, then the likelihood function based on these LRVs values can be written as:

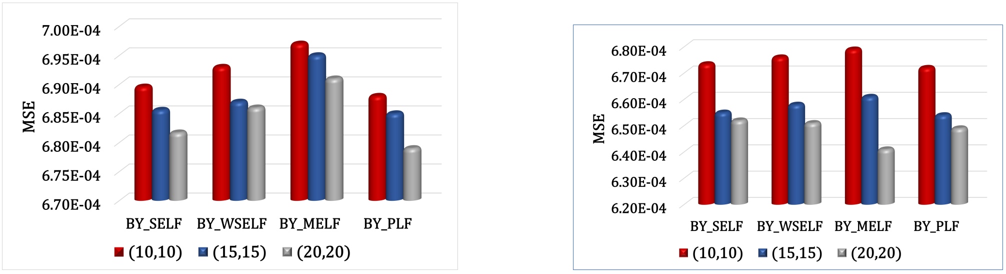
$$L(\underline{r}, \underline{s} \mid \theta) = \left[f(r_n) \prod_{i_1=1}^{n-1} \frac{f(r_{i_1})}{F(r_{i_1})} \right] \left[g(s_m) \prod_{i_2=1}^{m-1} \frac{g(s_{i_2})}{G(s_{i_2})} \right], \tag{3.1}$$
$$r_1 < r_2 < \cdots < r_n, \quad s_1 < s_2 < \cdots < s_m,$$

where $f(.;\rho,\eta)$, $F(.;\rho,\eta)$, $g(.;\delta,\eta)$, and $G(.;\delta,\eta)$ are the PDF, CDF of ILD (ρ, η) and the ILD (δ, η) respectively. By inserting the values of these PDFs and CDFs in Equation (3.1), the likelihood function takes the form

$$\begin{aligned} L(\underline{r}, \underline{s} \mid \theta) &= \rho^n \delta^m \eta^{n+m} \left[1 + \frac{\eta}{r_n}\right]^{-\rho} \left[1 + \frac{\eta}{s_m}\right]^{-\delta} \prod_{i_1=1}^{n} \frac{1}{r_{i_1}^2} \left[1 + \frac{\eta}{r_{i_1}}\right]^{-1} \prod_{i_2=1}^{m} \frac{1}{s_{i_2}^2} \left[1 + \frac{\eta}{s_{i_2}}\right]^{-1} \\ &= \rho^n \delta^m \eta^{n+m} \exp\left\{ -\rho \log\left[1 + \frac{\eta}{r_n}\right] - 2 \sum_{i_1=1}^{n} \log[r_{i_1}] - \sum_{i_1=1}^{n} \log\left[1 + \frac{\eta}{r_{i_1}}\right] \right. \\ &\left. - \delta \log\left[1 + \frac{\eta}{s_m}\right] - 2 \sum_{i_2=1}^{m} \log[s_{i_2}] - \sum_{i_2=1}^{m} \log\left[1 + \frac{\eta}{s_{i_2}}\right] \right\}. \end{aligned} \tag{3.2}$$

Consequently, the above expression's log-likelihood function say ℓ, is written as;

$$\begin{aligned} \ell &= n \log(\rho) + m \log(\delta) + (n+m) \log(\eta) - \rho \log\left[1 + \frac{\eta}{r_n}\right] - 2 \sum_{i_1=1}^{n} \log[r_{i_1}] \\ &- \sum_{i_1=1}^{n} \log\left[1 + \frac{\eta}{r_{i_1}}\right] - \delta \log\left[1 + \frac{\eta}{s_m}\right] - 2 \sum_{i_2=1}^{m} \log[s_{i_2}] - \sum_{i_2=1}^{m} \log\left[1 + \frac{\eta}{s_{i_2}}\right]. \end{aligned}$$



(a) *NIP* (b) *IP*

Figure 3: *The MSEs of different Bayes estimates for a specific number of record values at* $R = 0.3$.

The MLEs of the unknown parameter $\theta = (\rho, \delta, \eta)$, in which $\hat{\theta} = (\hat{\rho}, \hat{\delta}, \hat{\eta})$ can be obtained by differentiating ℓ according to ρ, δ, η and setting it to zero, yields

$$\frac{\partial \ell}{\partial \rho} = \frac{n}{\rho} - \log\left[1 + \frac{\eta}{r_n}\right] = 0, \tag{3.3}$$

$$\frac{\partial \ell}{\partial \delta} = \frac{m}{\delta} - \log\left[1 + \frac{\eta}{s_m}\right] = 0, \tag{3.4}$$

$$\frac{\partial \ell}{\partial \eta} = \frac{m+n}{\eta} - \frac{\delta}{s_m + \eta} - \frac{\rho}{r_n + \eta} - \sum_{i_1=1}^{n} \frac{1}{r_{i_1} + \eta} - \sum_{i_2=1}^{m} \frac{1}{s_{i_2} + \eta} = 0. \tag{3.5}$$

From Equations (3.3) and (3.4), we obtain

$$\hat{\rho}(\eta) = n\left(\log\left[1 + \frac{\eta}{r_n}\right]\right)^{-1}, \quad \hat{\delta}(\eta) = m\left(\log\left[1 + \frac{\eta}{s_m}\right]\right)^{-1}. \tag{3.6}$$

By putting the values of $\hat{\rho}(\eta)$ and $\hat{\delta}(\eta)$ into Equation (3.5), $\hat{\eta}$ can then be calculated as a fixed-point solution of the following equation

$$h(\eta) = \eta, \tag{3.7}$$

where

$$h(\eta) = \frac{n+m}{\frac{\hat{\delta}(\eta)}{s_m + \eta} + \frac{\hat{\rho}(\eta)}{r_n + \eta} + \sum_{i_1=1}^{n} \frac{1}{r_{i_1} + \eta} + \sum_{i_2=1}^{m} \frac{1}{s_{i_2} + \eta}}.$$

A clear and simple iterative technique $h(\eta^{(k)}) = \eta^{(k+1)}$, can be applied to determine the solution of Equation (3.7), in which $\eta^{(k)}$ is the k^{th} iterate. As soon as we have generated $\hat{\eta}_{ML}$, we will also deduce the MLE of ρ and δ from Equation (3.6) as $\hat{\rho}_{ML} = \hat{\rho}(\hat{\eta}_{ML})$ and $\hat{\delta}_{ML} = \hat{\delta}(\hat{\eta}_{ML})$. Based on the invariance property, the MLE of R is obtained by inserting $\hat{\rho}_{ML}$ and $\hat{\delta}_{ML}$ into Equation (2.1) as follows:

$$\hat{R}_{ML} = \frac{\hat{\rho}_{ML}}{\hat{\rho}_{ML} + \hat{\delta}_{ML}}.$$

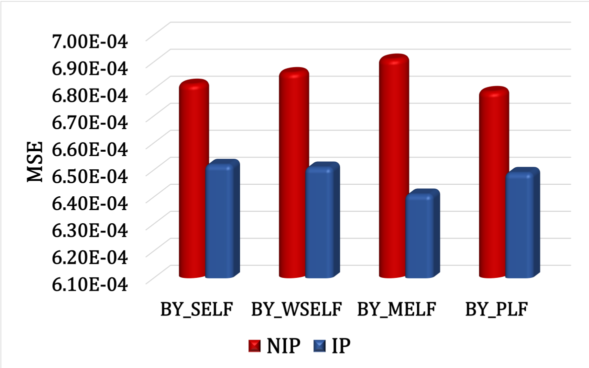


Figure 4: *The MSE of different Bayes estimates of R for different priors at true value $R = 0.3$.*

4. Bootstrap procedure

For the purpose of estimating R, this section suggests two parametric BCI approaches. They are the parametric percentile bootstrap (PPB; see Efron, 1987; Efron and Tibshirani, 1994) and the parametric bootstrap-t (PB-t; see Shao and Tu, 1995). The following is a summary of the PPB and PB-t CIs structures for R:

4.1. PPB method

Step 1 Based on LRVs, the two original samples $\underline{r} = (r_1, \ldots, r_n)$ and $\underline{s} = (s_1, \ldots, s_m)$ were generated from the ILD (ρ, η) and ILD (δ, η), respectively. Then, $\hat{\theta}$ is obtained by using the ML method of θ in (Section 2).

Step 2 Generate a first bootstrap lower record sample $\underline{r}^* = (r_1^*, r_2^*, \ldots, r_n^*)$ from the ILD $(\hat{\rho}, \hat{\eta})$, and generate a second bootstrap lower record sample $\underline{s}^* = (s_1^*, s_2^*, \ldots, s_m^*)$ from the ILD $(\hat{\delta}, \hat{\eta})$. Using these data, we calculate the bootstrap estimates $\hat{\theta}^* = (\hat{\rho}^*, \hat{\delta}^*, \hat{\eta}^*)$, and $\hat{R}^*$.

Step 3 To obtain a bootstrap sample set of $\hat{\mathfrak{R}}^* = (\hat{\mathfrak{R}}^*_{(1)}, \hat{\mathfrak{R}}^*_{(2)}, \ldots, \hat{\mathfrak{R}}^*_{(N)})$, repeat step 2 N times.

Step 4 Let's say that $h_1(x) = P(\hat{R}^* \leq x)$ be the CDF of $\hat{R}^*$. We can specify $\hat{R}_{boot}(x) = h_1^{-1}(x)$ for any given x.
The $100(1-\varepsilon)\%$ bootstrap percentile interval for R is then described as

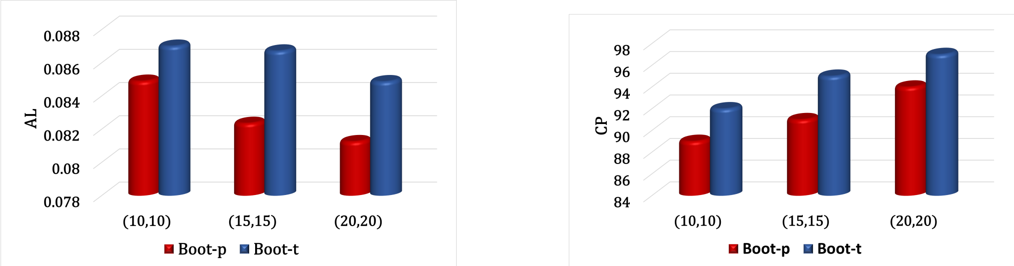
$$\left[\hat{R}_{Boot}\left(\frac{\varepsilon}{2}\right), \hat{R}_{Boot}\left(1-\frac{\varepsilon}{2}\right)\right].$$

Namely, simply use the $\varepsilon/2$ and $(1-\varepsilon/2)$ quantiles of the bootstrap sample $\hat{R}_1^*, \ldots, \hat{R}_n^*$.

4.2. PB-t method

Step 1 The same as in the previous algorithm.

Step 2 As mentioned in the PPB method, first we generate bootstrap lower record samples $\underline{r}^* = (r_1^*, r_2^*, \ldots, r_n^*)$ and $\underline{s}^* = (s_1^*, s_2^*, \ldots, s_m^*)$. By using these data, we compute the bootstrap es-



(a) *AL* (b) *CP*

Figure 5: *The AL and CP of different BCIs at* $R = 0.3$.

timates $\hat{\theta}^* = (\hat{\rho}^*, \hat{\delta}^*, \hat{\eta}^*)$, $\hat{R}^*$, and the following statistic:

$$T_b^* = \frac{\sqrt{n}\,(\hat{R}_b^* - \hat{R})}{\sqrt{\mathrm{var}(\hat{R}^*)}}, \quad b = 1, 2, \ldots, B,$$

and its $\mathrm{var}(\hat{R}^*)$ as described in the Appendix.

Step 3 To obtain a bootstrap sample set of $\hat{\mathfrak{R}}^* = (\hat{\mathfrak{R}}^*_{(1)}, \hat{\mathfrak{R}}^*_{(2)}, \ldots, \hat{\mathfrak{R}}^*_{(N)})$, repeat step 2 N times.

Step 4 Let $h_2(x) = P(T^* \leq x)$ be the CDF of T^*. We can Specify $\hat{R}_{boot}(x) = \hat{R} + h_2^{-1}(x)\sqrt{\frac{var(\hat{R})}{n}}$ for any given x. The $100(1-\varepsilon)\%$ BCI for R is then described as

$$\left[\hat{R}_{Boot}\left(\frac{\varepsilon}{2}\right), \hat{R}_{Boot}\left(1 - \frac{\varepsilon}{2}\right)\right].$$

Namely, simply use the $\varepsilon/2$ and $(1-\varepsilon/2)$ quantiles of the bootstrap sample $\hat{R}_1^*, \ldots, \hat{R}_n^*$.

5. Bayesian estimation for R

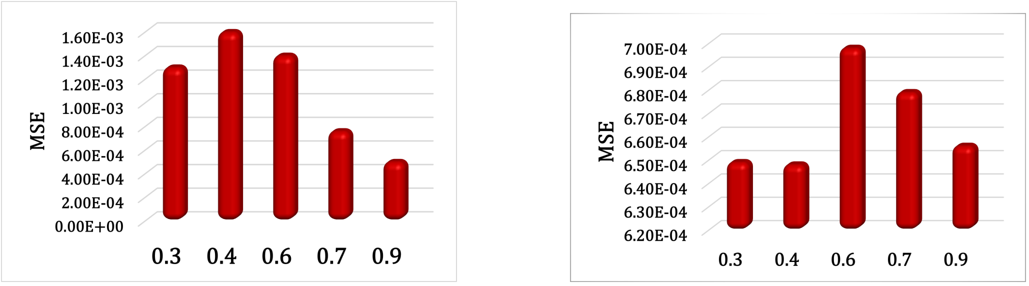
This section presents the Bayesian estimators of the ILD parameters and the reliability of the SS model using LRVs. Assume that ρ, δ, and η are independent gamma priors with the following PDFs:

$$\begin{aligned}
\pi_1(\rho) &\propto e^{-(\beta_1 \rho)} \rho^{\alpha_1 - 1}, && 0 < \rho < \infty, && \alpha_1, \beta_1 > 0, \\
\pi_2(\delta) &\propto e^{-(\beta_2 \delta)} \delta^{\alpha_2 - 1}, && 0 < \delta < \infty, && \alpha_2, \beta_2 > 0, \\
\pi_3(\eta) &\propto e^{-(\beta_3 \eta)} \eta^{\alpha_3 - 1}, && 0 < \eta < \infty, && \alpha_3, \beta_3 > 0.
\end{aligned}$$

Then, the joint IP for ρ, δ, and η is provided as:

$$\begin{aligned}
\pi(\theta) \propto\; & e^{-(\beta_1 \rho + \beta_2 \delta + \beta_3 \eta)}\, \rho^{\alpha_1 - 1}\, \delta^{\alpha_2 - 1}\, \eta^{\alpha_3 - 1}, \\
& \theta > 0,\; \alpha_i, \beta_i > 0,\; i = 1, 2, 3.
\end{aligned} \tag{5.1}$$

The selection of hyper-parameters $\alpha_i, \beta_i > 0$, $i = 1, 2, 3$ in the present study is based on the previous information of the unknown parameters. Sub-section (5.2) presents a discussion on the process of



(a) *MLE* (b) *IP*

Figure 6: *MSEs of (a) MLE (b) BE under PLF, at* $(n,m) = (20,20)$ *and all true values of* R*.*

choosing the hyper-parameter values. The joint posterior distribution of ρ, δ, and η is determined by combining the likelihood in Equation (3.2) with the joint prior densities in Equation (5.1):

$$\pi^*(\theta \mid \underline{r},\, \underline{s}) = \frac{\pi(\theta)\ L(\underline{r},\, \underline{s} \mid \theta)}{\int_0^\infty \int_0^\infty \int_0^\infty \pi(\theta)\ L(\underline{r},\, \underline{s} \mid \theta)\, d\rho\, d\eta\, d\delta}$$

$$\propto \rho^{\,n+\alpha_1-1}\, \delta^{\,m+\alpha_2-1}\, \eta^{\,n+m+\alpha_3-1} \exp\left\{ -\beta_3\eta - \sum_{i_1=1}^{n} \log\left[1+\frac{\eta}{r_{i_1}}\right] - \rho\left(\beta_1 + \log\left[1+\frac{\eta}{r_n}\right]\right) - \delta\left(\beta_2 + \log\left[1+\frac{\eta}{s_m}\right]\right) - \sum_{i_2=1}^{m} \log\left[1+\frac{\eta}{s_{i_2}}\right] \right\},$$

$$\theta > 0,\ \ \alpha_i, \beta_i > 0,\ \ i = 1,2,3.$$

The marginal posterior distributions of ρ, δ and η take the following forms:

$$\pi_1^*(\delta,\eta|\underline{r},\, \underline{s}) \propto \delta^{\,m+\alpha_2-1}\, \eta^{\,n+m+\alpha_3-1}\ \exp\left\{ -\sum_{i_1=1}^{n} \log\left[1+\frac{\eta}{r_{i_1}}\right] - \beta_3\,\eta - \sum_{i_2=1}^{m} \log\left[1+\frac{\eta}{s_{i_2}}\right] - \delta\left(\beta_2 + \log\left[1+\frac{\eta}{s_m}\right]\right) \right\} \int_0^\infty \rho^{\,n+\alpha_1-1} \exp\left\{ -\rho\left(\beta_1 + \log\left[1+\frac{\eta}{r_n}\right]\right) \right\} d\rho, \tag{5.2}$$

$$\pi_2^*(\rho,\eta \mid \underline{r},\, \underline{s}) \propto \rho^{\,n+\alpha_1-1}\, \eta^{\,n+m+\alpha_3-1}\ \exp\left\{ -\sum_{i_1=1}^{n} \log\left[1+\frac{\eta}{r_{i_1}}\right] - \beta_3\,\eta - \sum_{i_2=1}^{m} \log\left[1+\frac{\eta}{s_{i_2}}\right] - \rho\left(\beta_1 + \log\left[1+\frac{\eta}{r_n}\right]\right) \right\} \int_0^\infty \delta^{\,m+\alpha_2-1} \exp\left\{ -\delta\left(\beta_2 + \log\left[1+\frac{\eta}{s_m}\right]\right) \right\} d\delta, \tag{5.3}$$

Table 1: Values of MLEs, BEs & ALs (first row) and values of MSEs & CPs (second row) at $R = 0.3$

(n,m)	MLE	NIP				IP				Bootstrap	
		BY_{SELF}	BY_{WSELF}	BY_{MELF}	BY_{PLF}	BY_{SELF}	BY_{WSELF}	BY_{MELF}	BY_{PLF}	PPB	$PB-t$
(10,10)	0.3013	0.3004	0.2998	0.2991	0.3007	0.3008	0.30022	0.2995	0.3012	0.0849	0.08701
	1.70E-03	6.90E-04	6.93E-04	6.97E-04	6.88E-04	6.73E-04	6.76E-04	6.79E-04	6.72E-04	89.000	92.000
(10,15)	0.2265	0.2998	0.2992	0.2985	0.3001	0.2269	0.226	0.2251	0.2273	0.08923	0.09712
	6.15E-03	6.87E-04	6.91E-04	6.95E-04	6.85E-04	6.90E-04	7.00E-04	7.11E-04	6.85E-04	92.000	96.000
(15,10)	0.3925	0.2994	0.2988	0.2981	0.2997	0.3927	0.3922	0.3917	0.3929	0.08835	0.08921
	1.00E-02	6.47E-04	6.51E-04	6.50E-04	6.46E-04	6.96E-04	6.97E-04	6.99E-04	6.95E-04	93.000	94.000
(15,15)	0.3032	0.2993	0.29744	0.2955	0.3002	0.30309	0.30247	0.30185	0.3034	0.08236	0.08672
	1.53E-03	6.86E-04	6.87E-04	6.95E-04	6.85E-04	6.55E-04	6.58E-04	6.61E-04	6.54E-04	91.000	95.000
(15,20)	0.2488	0.30107	0.30043	0.2997	0.3013	0.2488	0.2472	0.2456	0.24959	0.08654	0.08725
	3.41E-03	6.93E-04	6.94E-04	6.96E-04	6.92E-04	1.27E-03	1.29E-03	1.32E-03	1.26E-03	92.000	98.000
(20,15)	0.3662	0.3005	0.2999	0.2992	0.3008	0.36486	0.3643	0.3637	0.3651	0.0846	0.0889
	5.73E-03	6.57E-04	6.61E-04	6.65E-04	6.56E-04	6.71E-04	6.75E-04	6.79E-04	6.69E-04	93.000	98.000
(20,20)	0.3074	0.3001	0.2994	0.2988	0.3004	0.2999	0.2987	0.2975	0.3005	0.08126	0.08487
	1.30E-03	6.82E-04	6.86E-04	6.91E-04	6.79E-04	6.45E-04	6.48E-04	6.41E-04	6.44E-04	94.000	97.000

$$\begin{aligned}\pi_3^*(\rho,\delta \mid \underline{r},\underline{s}) \propto \rho^{n+\alpha_1-1}\,\delta^{m+\alpha_2-1}\int_0^\infty \eta^{n+m+\alpha_3-1}\ \exp\Bigg\{&-\beta_3\,\eta-\sum_{i_1=1}^{n}\log\left[1+\frac{\eta}{r_{i_1}}\right] \\ &-\sum_{i_2=1}^{m}\log\left[1+\frac{\eta}{s_{i_2}}\right]-\rho\left(\beta_1+\log\left[1+\frac{\eta}{r_n}\right]\right)-\delta\left(\beta_2+\log\left[1+\frac{\eta}{s_m}\right]\right)\Bigg\}\,d\eta.\end{aligned} \tag{5.4}$$

It is clear from Equations (5.2 – 5.4) that the full posterior conditional distributions for ρ, δ and η are given, respectively, by:

$$\pi_1(\rho \mid \eta, \text{data}) \propto \rho^{\,n+\alpha_1-1}\ \exp\left\{-\rho\left(\beta_1+\log\left[1+\frac{\eta}{r_n}\right]\right)\right\},$$

$$\pi_2(\delta \mid \eta, \text{data}) \propto \delta^{\,m+\alpha_2-1}\ \exp\left\{-\delta\left(\beta_2+\log\left[1+\frac{\eta}{s_m}\right]\right)\right\},$$

$$\begin{aligned}\pi_3(\eta \mid \delta,\rho,\text{data}) \propto \eta^{\,n+m+\alpha_3-1}\ \exp\Bigg\{&-\beta_3\,\eta-\rho\left[\beta_1+\log\left[1+\frac{\eta}{r_n}\right]\right] \\ &-\delta\left(\beta_2+\log\left[1+\frac{\eta}{s_m}\right]\right]-\sum_{i_1=1}^{n}\log\left[1+\frac{\eta}{r_{i_1}}\right]-\sum_{i_2=1}^{m}\log\left[1+\frac{\eta}{s_{i_2}}\right]\Bigg\}.\end{aligned}$$

As a result, it becomes evident that $\rho \mid \eta,\text{data} \sim \text{Gamma}(n+\alpha_1, \beta_1+\log[1+(\eta/r_n)])$ and $\delta \mid \eta,\text{data} \sim \text{Gamma}(m+\alpha_2,\ \beta_2+\log[1+(\eta/s_m)])$. Accordingly, any gamma-generating technique can readily produce samples of ρ and δ. Nevertheless, the posterior conditional distribution of η cannot be computed in a closed form in our problem, so we solve it numerically. If the values of the hyper-parameters are set to value approach zero, the BEs of the parameters can be computed using the same approach as before; in this case, the prior is a NIP.

In the following sub-section, we derive the Bayesian estimators for the SS reliability function under symmetric and asymmetric LFs.

5.1. Loss function

An appropriate LF must be specified in order to select the best decision in decision theory. In this sub-section, we look at the symmetric and asymmetric LFs.

Table 2: Values of MLEs, BEs & ALs (first row) and values of MSEs & CPs (second row) at $R = 0.4$

(n,m)	MLE	NIP				IP				Bootstrap	
		BY_{SELF}	BY_{WSELF}	BY_{MELF}	BY_{PLF}	BY_{SELF}	BY_{WSELF}	BY_{MELF}	BY_{PLF}	PPB	$PB-t$
(10,10)	0.40271	0.3998	0.3994	0.3989	0.4001	0.40258	0.4021	0.40162	0.40281	0.10294	0.10447
	1.87E-03	6.83E-04	6.87E-04	6.92E-04	6.81E-04	6.83E-04	6.85E-04	6.88E-04	6.82E-04	88.000	92.000
(10,15)	0.30908	0.4005	0.40004	0.3995	0.4007	0.30989	0.3092	0.3086	0.3102	0.0887	0.08817
	9.65E-03	6.20E-04	6.22E-04	6.24E-04	6.19E-04	6.91E-04	6.94E-04	6.98E-04	6.89E-04	93.000	91.000
(15,10)	0.49937	0.40045	0.3999	0.39947	0.4007	0.5007	0.5003	0.4999	0.5009	0.10104	0.1023
	1.18E-02	6.87E-04	6.88E-04	6.90E-04	6.86E-04	6.26E-04	6.26E-04	6.27E-04	6.25E-04	89.000	91.000
(15,15)	0.40148	0.40005	0.39957	0.39909	0.40029	0.4008	0.4003	0.3998	0.40109	0.09754	0.09808
	1.76E-03	6.65E-04	6.68E-04	6.71E-04	6.64E-04	6.63E-04	6.67E-04	6.70E-04	6.62E-04	93.000	93.000
(15,20)	0.33678	0.4005	0.40003	0.3995	0.4007	0.3374	0.3369	0.3363	0.3377	0.0818	0.0827
	5.31E-03	6.87E-04	6.91E-04	6.94E-04	6.86E-04	6.49E-04	6.51E-04	6.54E-04	6.48E-04	84.000	90.000
(20,15)	0.4708	0.3998	0.3993	0.39883	0.40008	0.4714	0.4709	0.4705	0.4716	0.0931	0.09369
	6.75E-03	6.82E-04	6.84E-04	6.87E-04	6.81E-04	6.52E-04	6.53E-04	6.55E-04	6.51E-04	86.000	89.000
(20,20)	0.4014	0.39938	0.3989	0.39842	0.39961	0.40205	0.4015	0.40107	0.4023	0.08578	0.086759
	1.60E-03	6.53E-04	6.56E-04	6.60E-04	6.52E-04	6.49E-04	6.50E-04	6.53E-04	6.48E-04	94.000	95.000

(i) **Bayesian estimator under SELF**
A quadratic or SELF, is one of the useful symmetric LFs in nature; i.e. it gives equal importance to both over- and under-estimation. It is probably the loss functions that are most frequently applied to regression problems. It is defined by:

$$L_{SELF}(R,\hat{R}) = (\hat{R}-R)^2.$$

The BE under SELF is the posterior mean. It minimizes the average squared difference between the estimate and the true parameter value. This is a common choice due to its simplicity and focus on balancing errors. Therefore, the BE of R under the SELF is given by

$$\hat{R}_{SELF} = E(R) = \int_0^\infty \int_0^\infty \int_0^\infty R\ \pi^*(\theta \mid \underline{r},\ \underline{s}))\, d\rho\, d\eta\, d\delta.$$

(ii) **Bayesian estimator under WSELF**
A weighted version of the SELF known as WSELF was introduced by Berger (1985) as an asymmetric LF. The WSELF allows for assigning different weights to errors based on specific values of the parameter. It is given by:

$$L_{WSELF}(R,\hat{R}) = \frac{(\hat{R}-R)^2}{R}.$$

The BE will consider these weights while minimizing the expected squared loss. This is useful when the cost of errors varies across the parameter space. The Bayesian estimator of R under the WSELF is given by

$$\hat{R}_{WSELF} = \left[E(R^{-1})\right]^{-1} = \left[\int_0^\infty \int_0^\infty \int_0^\infty R^{-1}\ \pi^*(\theta \mid \underline{r},\ \underline{s}))\, d\rho\, d\eta\, d\delta\right]^{-1}.$$

(iii) **Bayesian estimator under MELF**
Tummala and Sathe (1978) developed the MELF as an asymmetric loss that is defined by:

$$L_{MELF}(R,\hat{R}) = \frac{(\hat{R}-R)^2}{R^2}.$$

Table 3: Values of MLEs, BEs & ALs (first row) and values of MSEs & CPs (second row) at $R = 0.6$

(n,m)	MLE	NIP				IP				Bootstrap	
		BY_{SELF}	BY_{WSELF}	BY_{MELF}	BY_{PLF}	BY_{SELF}	BY_{WSELF}	BY_{MELF}	BY_{PLF}	PPB	$PB-t$
(10,10)	0.59961	0.5999	0.5996	0.5993	0.60009	0.5999	0.5996	0.5992	0.60009	0.09754	0.09868
	1.46E-03	6.71E-04	6.71E-04	6.72E-04	6.70E-04	7.12E-04	7.13E-04	7.13E-04	7.11E-04	91.000	89.000
(10,15)	0.5012	0.60034	0.60003	0.5997	0.6004	0.50167	0.5009	0.5001	0.50206	0.08937	0.09004
	1.13E-02	6.60E-04	6.61E-04	6.63E-04	6.59E-04	7.05E-04	7.06E-04	7.89E-04	7.53E-04	85.000	84.000
(15,10)	0.68904	0.6003	0.60003	0.59971	0.60052	0.68891	0.6886	0.6883	0.68906	0.07677	0.0773
	9.11E-03	6.71E-04	6.73E-04	6.75E-04	6.71E-04	7.02E-04	7.03E-04	7.03E-04	7.01E-04	90.000	86.000
(15,15)	0.59802	0.6002	0.5999	0.5996	0.60043	0.5976	0.5972	0.5969	0.5977	0.08454	0.0854
	1.43E-03	6.58E-04	6.59E-04	6.60E-04	6.57E-04	7.06E-04	7.07E-04	7.09E-04	7.05E-04	92.500	90.000
(15,20)	0.5288	0.5998	0.5995	0.5992	0.600008	0.5296	0.5288	0.5281	0.5299	0.07749	0.0778
	6.70E-03	6.87E-04	6.88E-04	6.90E-04	6.87E-04	6.98E-04	6.98E-04	6.96E-04	6.95E-04	91.000	90.000
(20,15)	0.6649	0.60063	0.6003	0.5999	0.6008	0.6655	0.6652	0.6649	0.6657	0.0748	0.0755
	5.40E-03	6.86E-04	6.87E-04	6.88E-04	6.85E-04	6.91E-04	6.92E-04	6.93E-04	6.91E-04	88.000	87.000
(20,20)	0.5981	0.60001	0.5997	0.5993	0.6001	0.59768	0.5973	0.59703	0.5978	0.0796	0.0802
	1.41E-03	6.44E-04	6.45E-04	6.46E-04	6.44E-04	6.99E-04	7.01E-04	7.02E-04	6.98E-04	95.000	92.000

The BE, by definition, minimizes the expected loss regardless of the chosen function. It essentially finds the estimate that leads to the lowest average loss across all possible parameter values. The Bayesian estimator of R under the MELF is given by:

$$\hat{R}_{MELF} = \frac{E(R^{-1})}{E(R^{-2})} = \left[\frac{\int_0^\infty \int_0^\infty \int_0^\infty R^{-1} \; \pi^*(\theta \mid \underline{r}, \underline{s}))\, d\rho\, d\eta\, d\delta}{\int_0^\infty \int_0^\infty \int_0^\infty R^{-2} \; \pi^*(\theta \mid \underline{r}, \underline{s}))\, d\rho\, d\eta\, d\delta} \right].$$

(iv) **Bayesian estimator under PLF**

Norstrom (1996) introduced an alternative asymmetric precautionary loss function.

$$L_{PLF}(R, \hat{R}) = \frac{(\hat{R} - R)^2}{\hat{R}}.$$

This function prioritizes avoiding underestimation. The resulting BE will tend to favor values that are less likely to underestimate the true parameter, even if it means sacrificing some accuracy on the overestimation side. The Bayesian estimator of R under the PLF is given by

$$\hat{R}_{PLF} = \sqrt{E(R^2)} = \left[\int_0^\infty \int_0^\infty \int_0^\infty R^2 \; \pi^*(\theta \mid \underline{r}, \underline{s}))\, d\rho\, d\eta\, d\delta \right]^{\frac{1}{2}}.$$

All of the equations listed above cannot be solved analytically. As a result, one of the simulation techniques, such as MCMC, is used to generate samples and calculate BEs with symmetric and asymmetric LFs for these types of equations.

5.2. Hyper-parameter elicitation

The main concern in the Bayesian approach is the elicitation technique used for identifying the hyper-parameter value when an informative prior of the parameter is taken into consideration. This issue has been explained in the literature by Dey *et al.* (2016), and Singh and Tripathi (2018). Furthermore, the value of hyper-parameters for the unknown parameter of interest is determined by assuming two independent pieces of information: the prior mean and variance of the model parameters under

Table 4: Values of MLEs, BEs & ALs (first row) and values of MSEs & CPs (second row) at $R = 0.7$

(n,m)	MLE	NIP				IP				Bootstrap	
		BY_{SELF}	BY_{WSELF}	BY_{MELF}	BY_{PLF}	BY_{SELF}	BY_{WSELF}	BY_{MELF}	BY_{PLF}	PPB	$PB-t$
(10,10)	0.69903	0.6997	0.6994	0.6992	0.6998	0.69906	0.69879	0.6985	0.6992	0.0927	0.09328
	1.12E-03	6.83E-04	6.84E-04	6.85E-04	6.83E-04	7.15E-04	7.15E-04	7.16E-04	7.15E-04	92.000	88.000
(10,15)	0.6082	0.69929	0.6990	0.6987	0.6994	0.60731	0.6069	0.60668	0.60747	0.0982	0.0989
	9.79E-03	6.72E-04	6.72E-04	6.73E-04	6.71E-04	6.97E-04	6.98E-04	7.00E-04	6.96E-04	92.000	88.000
(15,10)	0.7734	0.6994	0.69918	0.6989	0.6996	0.77435	0.7741	0.7738	0.7744	0.0975	0.0983
	6.09E-03	6.89E-04	6.89E-04	6.89E-04	6.88E-04	6.58E-04	6.58E-04	6.58E-04	6.58E-04	92.000	88.000
(15,15)	0.6949	0.69853	0.69825	0.6979	0.6986	0.69471	0.6944	0.6941	0.6948	0.08983	0.186777
	8.88E-04	6.73E-04	6.73E-04	6.74E-04	6.72E-04	7.02E-04	7.03E-04	7.04E-04	7.01E-04	94.000	96.000
(15,20)	0.6326	0.69957	0.6993	0.69903	0.69971	0.6998	0.6995	0.6993	0.700007	0.09646	0.16599
	5.50E-03	6.29E-04	6.29E-04	6.30E-04	6.28E-04	7.02E-04	7.03E-04	7.04E-04	7.01E-04	90.000	96.000
(20,15)	0.752	0.70046	0.70018	0.69991	0.70059	0.7003	0.70007	0.6998	0.7004	0.07999	0.14419
	3.33E-03	6.92E-04	6.92E-04	6.92E-04	6.92E-04	6.69E-04	6.69E-04	6.70E-04	6.88E-04	90.000	98.000
(20,20)	0.6965	0.70036	0.70009	0.6998	0.7005	0.7001	0.6998	0.6995	0.7002	0.08756	0.08857
	7.60E-04	6.71E-04	6.72E-04	6.72E-04	6.71E-04	6.79E-04	6.80E-04	6.80E-04	6.79E-04	95.0000	98.0000

consideration. In this regard, the gamma prior, $\pi(\theta) \propto \theta^{h_1-1}e^{-(h_2\,\theta)}$, with $\text{Mean}(\theta) = h_1/h_2$ and $\text{Variance}(\theta) = h_1/h_2^2$, has been taken into consideration in this study. Notice that here θ represents a parameter of interest of the ILD (ρ, η) and ILD (δ, η) distributions, and therefore

- for $\theta = \rho$ we have $h_1 = \alpha_1$, $h_2 = \beta_1$.
- for $\theta = \delta$ we have $h_1 = \alpha_2$, $h_2 = \beta_2$.
- for $\theta = \eta$ we have $h_1 = \alpha_3$, $h_2 = \beta_3$.

In this respect, we propose the following steps to select the values of hyper-parameters

1. Set the initial parameter value of (ρ, δ, η).
2. Set $j = 1$.
3. Based on LRVs, the two original samples $\underline{r} = (r_1, \ldots, r_n)$ and $\underline{s} = (s_1, \ldots, s_m)$ were generated from the *ILD* (ρ, η) and *ILD* (δ, η), respectively.
4. Obtain $\hat{\theta}$ by using the ML method of θ in (Section 2).
5. Repeat steps 2-4 k times to get $(\hat{\rho}^j, \hat{\delta}^j, \hat{\eta}^j)$, $j = 1, 2, \ldots, k$.
6. Equating the mean and variance of gamma priors with the mean and variance of $\hat{\theta}^j, j = 1, 2, \ldots, k$ is
$$\frac{1}{k}\sum_{j=1}^{k}\hat{\theta}^j = \frac{h_1}{h_2}, \quad \frac{1}{k-1}\sum_{j=1}^{k}\left[\hat{\theta}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\theta}^j\right]^2 = \frac{h_1}{h_2^2}.$$
7. Now, on solving the above equations, the estimated hyper-parameters turn out as,
$$\hat{h_1} = \frac{\left[\frac{1}{k}\sum_{j=1}^{k}\hat{\theta}^j\right]^2}{\frac{1}{k-1}\sum_{j=1}^{k}\left[\hat{\theta}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\theta}^j\right]^2}, \quad \hat{h_2} = \frac{\left[\frac{1}{k}\sum_{j=1}^{k}\hat{\theta}^j\right]}{\frac{1}{k-1}\sum_{j=1}^{k}\left[\hat{\theta}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\theta}^j\right]^2}.$$

Table 5: Values of MLEs, BEs & ALs (first row) and values of MSEs & CPs (second row) at $R = 0.9$

(n,m)	MLE	NIP				IP				Bootstrap	
		BY_{SELF}	BY_{WSELF}	BY_{MELF}	BY_{PLF}	BY_{SELF}	BY_{WSELF}	BY_{MELF}	BY_{PLF}	PPB	$PB-t$
(10,10)	0.8979	0.9002	0.90005	0.8998	0.9003	0.8974	0.8972	0.897	0.8975	0.04618	0.05087
	6.60E-03	6.59E-04	6.59E-04	6.60E-04	6.59E-04	6.96E-04	6.97E-04	6.98E-04	6.95E-04	89.000	88.000
(10,15)	0.8539	0.90004	0.89982	0.899606	0.90015	0.85471	0.85448	0.85425	0.85483	0.06248	0.06644
	2.71E-03	6.96E-04	6.97E-04	6.97E-04	6.96E-04	7.08E-04	7.09E-04	7.09E-04	7.08E-04	95.000	88.000
(15,10)	0.9095	0.9008	0.9006	0.9003	0.9009	0.90978	0.9095	0.90934	0.9098	0.03223	0.03322
	2.58E-03	6.29E-04	6.30E-04	6.31E-04	6.29E-04	6.93E-04	6.93E-04	6.94E-04	6.92E-04	92.000	88.000
(15,15)	0.8879	0.8981	0.8979	0.8977	0.8982	0.88824	0.88803	0.8878	0.8883	0.04329	0.0458
	3.10E-03	6.42E-04	6.44E-04	6.46E-04	6.41E-04	6.59E-04	6.60E-04	6.60E-04	6.59E-04	92.000	91.000
(15,20)	0.8597	0.8998	0.8995	0.8993	0.8999	0.8599	0.8597	0.8595	0.86008	0.06634	0.06224
	2.00E-03	6.94E-04	6.95E-04	6.97E-04	6.93E-04	6.81E-04	6.81E-04	6.81E-04	6.80E-04	92.000	91.000
(20,15)	0.8995	0.9016	0.9013	0.9011	0.9017	0.8989	0.8987	0.8985	0.899	0.05566	0.09411
	1.90E-03	6.98E-04	6.98E-04	6.99E-04	6.97E-04	6.61E-04	6.63E-04	6.62E-04	6.61E-04	91.000	96.000
(20,20)	0.8857	0.8995	0.8992	0.899	0.8996	0.88501	0.88481	0.8846	0.8851	0.04116	0.03991
	1.30E-03	6.38E-04	6.38E-04	6.39E-04	6.37E-04	6.56E-04	6.57E-04	6.57E-04	6.56E-04	94.000	93.000

6. Markov Chain Monte Carlo

The MCMC simulation method is employed to assess the performances of various estimators derived from Bayes computation. In the Bayesian paradigm, we obtain various posterior samples of different sample size values using different MCMC sampling algorithms. Gibbs sampling and the more generic Metropolis-Hasting (M-H) within-Gibbs samplers with normal proposal distribution are significant subclasses of MCMC algorithms. The hybrid M-H and Gibbs sampler is summarized below:

1. Start with the initial value $(\rho^{(0)} = \hat{\rho}, \delta^{(0)} = \hat{\delta}, \eta^{(0)} = \hat{\eta})$, $M = \text{Nburn}$.

2. Place $k = 1$.

3. From Gamma $\left[n+\alpha_1, \beta_1 + \ln\left(1+\frac{\eta^{(k-1)}}{r_n}\right)\right]$, generate $\rho_1^{(k)}$.

4. From Gamma $\left[m+\alpha_2, \beta_2 + \ln\left(1+\frac{\eta^{(k-1)}}{s_m}\right)\right]$, generate $\delta_1^{(k)}$.

5. To Generate $\eta^{(k)}$

   (a) Generate η_1^* from the normal proposal distribution $N(\eta^{(k-1)}, V_\eta)$.

   (b) Evaluate the acceptance probability

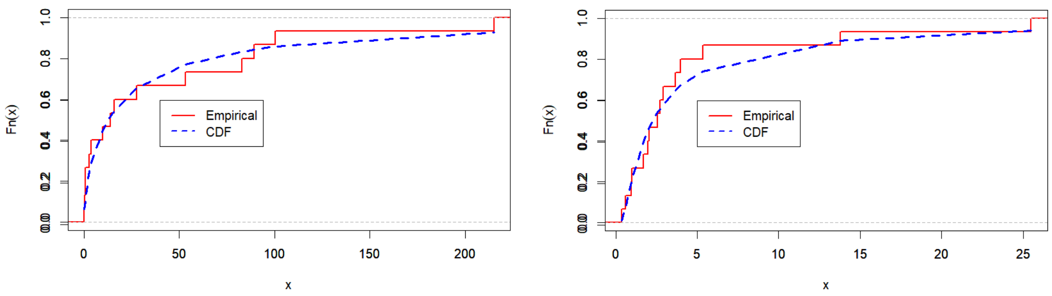
   $$\Omega_\eta\left(\eta^{(k-1)},\ \eta_1^*\right) = \min\left[1, \frac{\pi_3^*(\eta_1^* \mid \delta^{(k)}, \rho^{(k)}, \text{data})}{\pi_3^*(\eta^{(k-1)} \mid \delta^{(k)}, \rho^{(k)}, \text{data})}\right].$$

   (c) Achieve U from $U(0,1)$

   (d) Confirm the proposal and place $\eta_1^{(k)} = \eta^*$, If $U \leq \Omega_\eta(\eta^{(k-1)},\ \eta_1^*)$. Elsewhere, deny the proposal and place $\eta_1^{(k)} = \eta_1^{(k-1)}$.

6. Obtain $\rho^{(k)}$, $\delta^{(k)}$, $\eta^{(k)}$ and $R^{(k)}$.

7. Place $k = k+1$



(a) *32 kv* (b) *36 kv*

Figure 7: *The empirical and theoretical CDFs of data sets.*

8. Repeat N times of steps 3–7.

Now, based on SELF, the Bayes estimate of R is given by:

$$\hat{R}_{SELF} = E(R \mid \text{data}) = \frac{1}{N-M} \sum_{k=M+1}^{N} R^k,$$

where M is the burn-in period.
Based on the WSELF,

$$\hat{R}_{WSELF} = \left[E\left(R^{-1}\right)\right]^{-1} = \left[\frac{1}{N-M} \sum_{k=M+1}^{N} \left(R^k\right)^{-1}\right]^{-1}.$$

Based on the MELF,

$$\hat{R}_{MELF} = \frac{E(R^{-1})}{E(R^{-2})} = \left[\frac{\frac{1}{N-M} \sum_{k=M+1}^{N} (R^k)^{-1}}{\frac{1}{N-M} \sum_{k=M+1}^{N} (R^k)^{-2}}\right].$$

Based on the PLF,

$$\hat{R}_{PLF} = \sqrt{E(R^2)} = \left[\frac{1}{N-M} \sum_{k=M+1}^{N} (R^k)^2\right]^{\frac{1}{2}}.$$

7. Numerical illustration & real data analysis

This section conducts a Monte Carlo simulation analysis to evaluate the performance of MLEs, BCIs, and BEs for R and examines how the various techniques behave under a various number of records. Furthermore, using ILD fitting distribution, electrical insulating fluids used in transformers for genuine data sets are investigated. R 4.2.2 software does all the computations.

7.1. Simulation study

The MCMC simulations are utilized here to evaluate the performance of different methods. The outcomes of the MLEs and BEs for the various LFs are compared in terms of mean square errors (MSEs). Also, we compare different BCIs with respect to average lengths (ALs) and coverage probability (CP). We consider seven sample sizes such as: $(n,m) = (10,10), (15,10), (10,15), (15,15), (15,20), (20,15)$, and $(20,20)$ for five sets of true parameters and the corresponding real value of R, in which the common parameter is $\eta = 2$. These sets are as shown below:

- Set 1: $\rho = 0.12$, $\delta = 0.28$, and $R = 0.3$
- Set 2: $\rho = 0.1$, $\delta = 0.15$, and $R = 0.4$
- Set 3: $\rho = 0.225$, $\delta = 0.15$, and $R = 0.6$
- Set 4: $\rho = 0.21$, $\delta = 0.09$, and $R = 0.7$
- Set 5: $\rho = 0.45$, $\delta = 0.05$, and $R = 0.9$

We first calculate the MLEs of R in each set and then generate the 95% BCIs of R using boot-p and boot-t methods. The MCMC technique uses a hybrid M-H sampler to generate the Bayes estimates. Consequently, the MCMC approach is used to generate the Bayes estimates of R under SELF, WSELF, PLF, and MELF. The hyper-parameter values are determined by applying the technique described in Section 5.2 when the prior is IP, and are set to value approach zero when the prior is NIP. The first 20% of observations are discarded after 10000 MCMC sample iterations. Also, we took 1000 replications for boot-p and boot-t. The simulation results are presented in Tables (1–5), and the chosen values are shown in Figures (2–6). These tables and figures suggest the following conclusions:

1. The performances of the Bayes estimates of R, obtained from different LFs, are better than that of MLEs.

2. Regarding, the number of records (n and m) for the stress and strength variables, it is observed that for small values of n and m, the MSE of MLEs and BEs increases. While the MSE decreases as the record numbers increase, as seen in Figures 2 and 3.

3. Figure 3 showed that the BEs of R under PLF are superior to the others in terms of producing fewer MSEs for several values of n and m. However, compared to other BEs, the BEs under MELF are the worst, and result in a large MSE.

4. The BEs in the case of IP are preferred over the others in the case of NIP as seen in Figure 4.

5. We observed that the ALs for PPB are less than those for PB-t. When the record numbers increase, the length of BCIs decreases as seen in Figure 5(a).

6. The CPs for the BCIs using PPB are lower than the nominal level 0.95, however, the CPs for BCIs using the PB-t are more significant than the nominal level 0.95, as shown in Figure 5(b).

7. The MSE of the MLEs is smaller for the high value of R, while for the BEs, the minimum MSE is performed when R is close to 0.5 as seen in Figure 6.

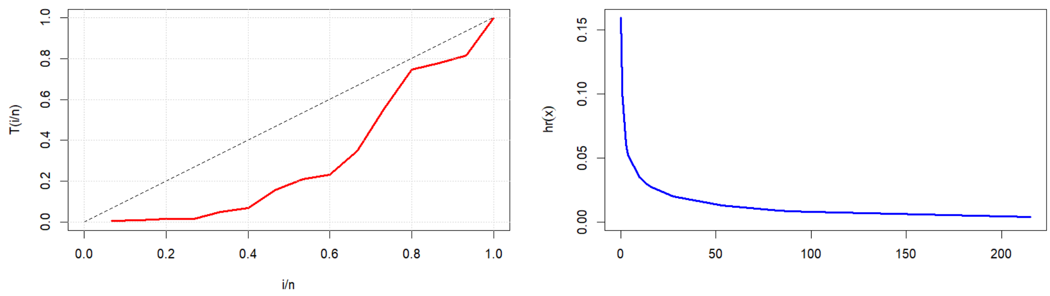


Figure 8: *Scale TTT-transform and HRF plots of ILD for data set I.*

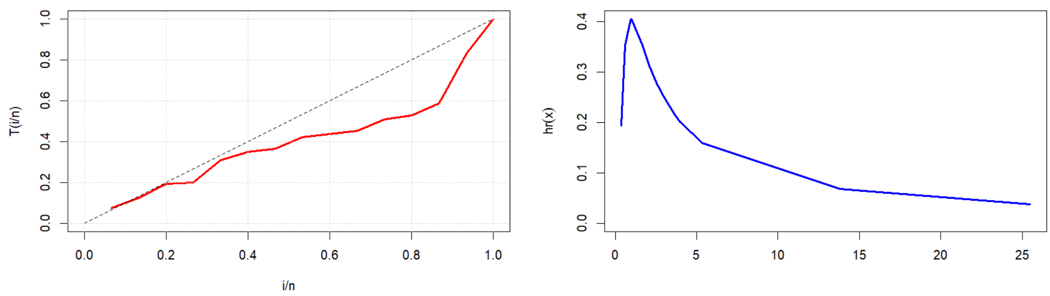


Figure 9: *Scale TTT-transform and HRF plots of ILD for data set II.*

7.2. Data analysis

In a life test experiment, specimens of an electrical insulating fluid, designed specifically for use in transformers were exposed to a constant voltage stress. Each specimen's failure or breakdown was timed in minutes. The observations of seven groups of specimens tested at voltages ranging from 26 to 38 kilovolts (kV) were provided by Nelson (1972) for lifetime modeling using the inverse power-law model. This data sets were also given in Lawless (2002). To be self-contained, the failure times in 180 minutes for groups of specimens subjected to 32 kV and 36 kV are indicated in Tables 6 and 7 for illustration.

We wish to compute the SS model's reliability $P(Y < X)$. Primarily, Table 8 displays the MLEs, and Kolmogorov-Smirnov (K-S) distances besides the p-values (PVs) for the ILD based on each sample of the data set.

Figure 7 also confirms this, as the empirical and theoretical CDF nearly overlap for both groups of specimens subjected to 32 kV and 36 kV, respectively.

An important graphical method to check if the data may be applied to a certain distribution or not is the total time test (TTT) plot, which was first described by Aarset (1987). Figures 8 and 9 show how each sample fits into the ILD, which indicates a declining failure rate for the 32 kV dataset I and a uni-modal failure rate for the 36 kV dataset II.

It is first assumed that $X \sim \text{ILD}(\rho, \eta)$ and $Y \sim \text{ILD}(\delta, \eta)$, the MLEs are $\hat{\rho} = 1.44224$, $\hat{\delta} = 1.04537$, $\hat{\eta} = 3.18457$, and $L_0 = -107.764$ for the conforming log-likelihood value. Second, when

Table 6: The failure rate subjected to 32 kV : Data Set I

0.4	82.85	9.88	89.29	215.10	2.75	0.79	15.93
3.91	0.27	0.69	100.58	27.80	13.95	53.24	

Table 7: The failure rate subjected to 36 kV : Data Set II

1.97	0.59	2.58	1.69	2.71	25.50	0.35	0.99
3.99	3.67	2.07	0.96	5.35	2.90	13.77	

Table 8: Distribution parameters with K-S statistics based on a real-data set

	MLEs	K-S	PV
Data Set-I (Y)	$(0.541755, 32.4963)$	0.13489	0.9142
Data Set-II (X)	$(13.5373, 0.118294)$	0.1335	0.9198

Table 9: Mean estimates of R with their standard error of MLE

	MLE	Standard error
	0.343549	0.23622

Table 10: Mean estimates of R with their PSDs under different prior and LFs

		Estimate	PSD
NIP	BY_{SELF}	0.580008	0.057838
	BY_{WSELF}	0.578330	0.058078
	BY_{MELF}	0.576656	0.058340
	BY_{PLF}	0.580844	0.057728

considering $X \sim \text{ILD}(\rho, \eta_1)$ and $Y \sim \text{ILD}(\delta, \eta_2)$, the MLEs are $\hat{\rho} = 0.541755$, $\hat{\delta} = 13.5373$, $\hat{\eta_1} = 32.4963$, $\hat{\eta_2} = 0.118294$, and $L_1 = -102.633$ for the conforming log-likelihood value. At this point, we have completed the following hypotheses tests:

$$H_0 : \eta_1 = \eta_2; \qquad H_1 : \eta_1 \neq \eta_2$$

In this case, the likelihood ratio test value is $-2(L_0 - L_1) = -10.262$, with degrees of freedom as 1 (the number of parameters differs between the two models). Furthermore, the Chi-squared test has a PV of one; and because this PV is more than 0.05, the null hypothesis cannot be rejected. As a result, H_0 is a reasonable assumption in this case.

The LRVs from data set I are 0.40 and 0.27, and the LRVs from data set II are 1.97, 0.59, and 0.35. Using the information from the LRVs for both, we must now estimate the probability $P(Y < X)$ that the failure times in 180 minutes for the second group of specimens subjected to 36 kV are lower than the failure times for the first group of specimens subjected to 32 kV. As a result, Tables 9 and 10 show the MLE and BE for the entire sample. Furthermore, the 95% corresponding BCI in case of PPB is $(0.1305644, 0.7925648)$, and the BCI in case of PB-t is $(0.009701, 0.856001)$.

The BEs have a lower posterior standard deviation (PSD) than the other MLEs. As a result, the BE of the ILD's parameters is the most accurate. The $P(Y < X)$ reliability of the Bayesian technique is greater than that of the ML method, which proves the displayed conclusion.

Figure 10 displays a trace plot for different SS reliability estimates with a number of repetitions (10000) for the symmetric and asymmetric LFs. The SS reliability indicates that all of the generated posteriors fit fairly well with the theoretical posterior density functions, and it is obvious that a large

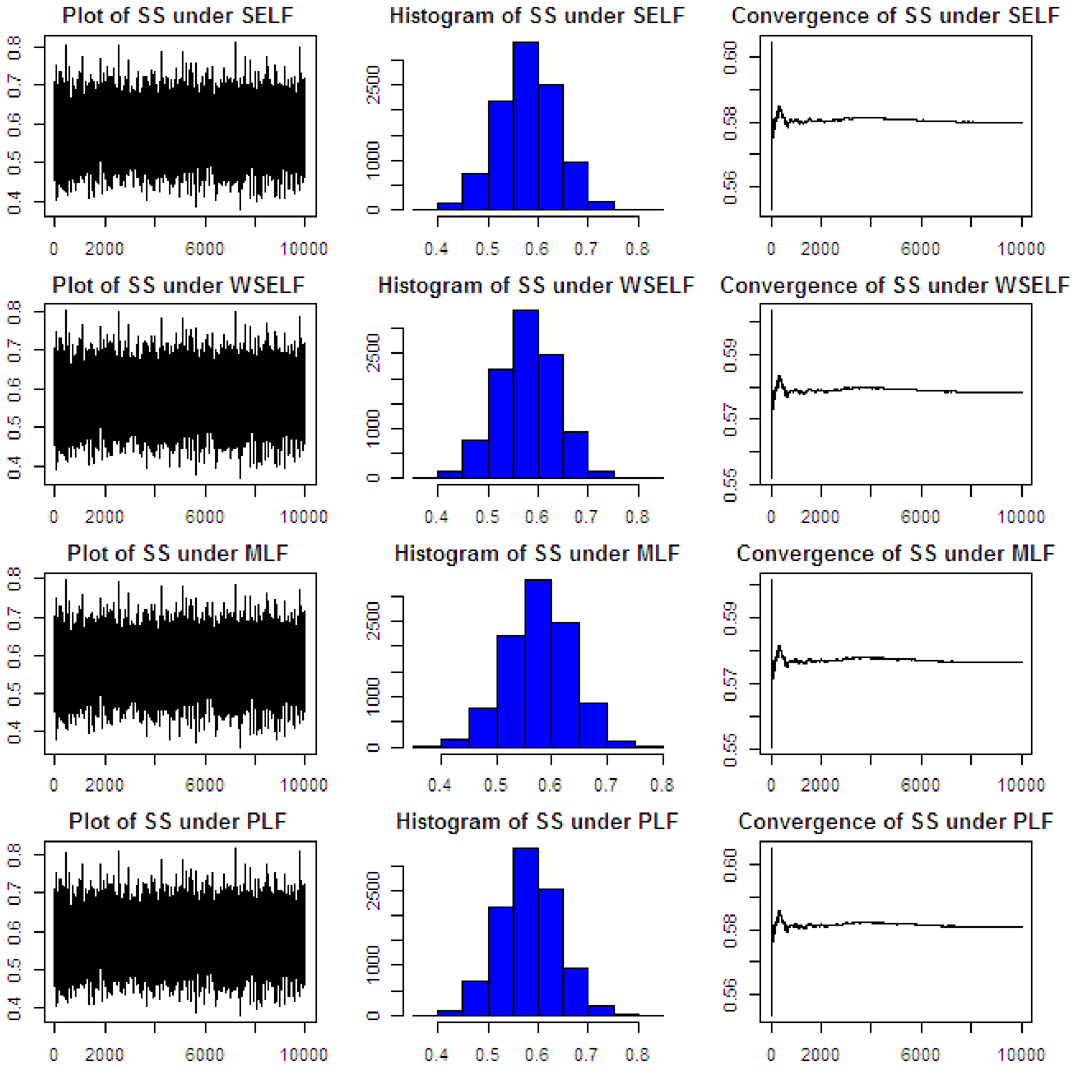


Figure 10: *The MCMC plots for data for the SS reliability with NIP.*

loop of MCMC would present the same and more efficient results. These plots resemble a horizontal band with no long upward or downward trends, indicating convergence.

8. Conclusion

In this paper, the ML and Bayesian estimators are obtained for reliability $R = P(Y < X)$ based on lower record values, where X and Y are independent random variables, from ILD with common scale parameters but different shape parameters. The Bayesian procedure, given either NIP or IP distribution, under symmetric and asymmetric LFs, is adopted to develop inferences of R. The BCIs using the parametric PPB and PP-t methods are given. The Bayesian estimates are computed using the MCMC method based on the Metropolis algorithm and Gibbs sampling in terms of their mean squared error. Application to electrical insulating fluids real data and simulation issues are provided.

From the simulation study, it is observed that the BE performs better than the MLE of the reliability parameter in terms of MSE. Further, comparing the MSE of the different Bayes estimates, we found that the performance of the R estimate obtained from PLF is superior to the other BEs for different values of n and m. Overall, the ALs and CPs for the PPB intervals are less than the PB-t intervals. Moreover, it was checked that the MSE of MLEs and BEs of the SS decreases when the number of records (n and m) increases. We observe that the MSE of MLEs is smaller for high R values, whereas the MSE of Bayesian estimates is accomplished when R is close to 0.5.

Appendix:

Here we obtain the variance of the MLE, $\hat{R}$, using the delta method. Let us denote the Fisher information matrix of $\theta = (\rho, \delta, \eta)$ as $J(\theta) = E(I(\theta))$ where $I(\theta) = (I_{i_1 i_2}(\theta))$ for $i_1, i_2 = 1, 2, 3$. Therefore,

$$I(\theta) = -\begin{pmatrix} \frac{\partial^2 \ell}{\partial \rho^2} & \frac{\partial^2 \ell}{\partial \rho \partial \delta} & \frac{\partial^2 \ell}{\partial \rho \partial \eta} \\ \frac{\partial^2 \ell}{\partial \delta \partial \rho} & \frac{\partial^2 \ell}{\partial \delta^2} & \frac{\partial^2 \ell}{\partial \delta \partial \eta} \\ \frac{\partial^2 \ell}{\partial \eta \partial \rho} & \frac{\partial^2 \ell}{\partial \eta \partial \delta} & \frac{\partial^2 \ell}{\partial \eta^2} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}$$

It is easy to see that

$$I_{11} = \frac{-n}{\rho^2} \qquad I_{12} = I_{21} = 0 \qquad I_{13} = I_{31} = \frac{-1}{r_n + \eta}$$

$$I_{22} = \frac{-m}{\delta^2} \qquad I_{23} = I_{32} = \frac{-1}{s_m + \eta}$$

$$I_{33} = \frac{-(n+m)}{\eta^2} + \frac{\rho}{(r_n+\eta)^2} + \frac{\delta}{(s_m+\eta)^2} + \sum_{i_1=1}^{n} \frac{1}{(r_{i_1}+\eta)^2} + \sum_{i_2}^{m} \frac{1}{(s_{i_2}+\eta)^2}$$

Because of the complexity of the expectations, the observed information matrix is used as a consistent estimator of the information matrix I. An approximation of the variance-covariance matrix of (ρ, δ, η) is $I^{-1} \mid_{\hat{\rho},\hat{\delta},\hat{\eta}}$. We use the delta method to get approximate estimates of the variance of $\hat{R}$ (see Krishnamoorthy and Lin (2010)). Let $B' = (\frac{\partial R}{\partial \rho}, \frac{\partial R}{\partial \delta}, \frac{\partial R}{\partial \eta})$ where

$$\frac{\partial R}{\partial \rho} = \frac{\delta}{(\rho+\delta)^2}, \quad \frac{\partial R}{\partial \delta} = \frac{-\rho}{(\rho+\delta)^2}, \quad \text{and} \quad \frac{\partial R}{\partial \eta} = 0.$$

Then, the approximate estimate of $\mathrm{Var}(\hat{R})$ is $\hat{\mathrm{Var}}(\hat{R}) \simeq [B' I^{-1} B]_{\hat{\rho},\hat{\delta},\hat{\eta}}$. It should be noted that the variance of $\hat{R}$ must be estimated. We recommend estimating $\hat{R}$ using the MLE of ρ, δ, and η , which is very convenient.

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