Honam Mathematical J. **46** (2024), No. 3, pp. 407–427
https://doi.org/10.5831/HMJ.2024.46.3.407

HORADAM 3-PARAMETER GENERALIZED QUATERNIONS

Zehra İşbilir and Nurten Gürses*

Abstract. The purpose of this article is to bring together the Horadam numbers and 3-parameter generalized quaternions, which are a general form of the quaternion algebra according to 3-parameters. With this purpose, we introduce and examine a new type of quite big special numbers system, which is called Horadam 3-parameter generalized quaternions (shortly, Horadam 3PGQs), and special cases of them. Besides, we compute both some new equations and classical well-known equations such as; Binet formulas, generating function, exponential generating function, Poisson generating function, sum formulas, Cassini identity, polar representation, and matrix equation. Furthermore, this article concludes by presenting the determinant, characteristic polynomial, characteristic equation, eigenvalues, and eigenvectors in relation to the matrix representation of Horadam 3PGQ.

1. Introduction

Throughout history, number systems have been an attractive concept for lots of researchers in several disciplines because they have been used in lots of areas and there are several applications. One of the most popular number systems is the quaternions, which were investigated by William Rowan Hamilton in order to extend the complex numbers in 1843 [17–19]. The quaternion algebra is a non-commutative, associative, and 4-dimensional Clifford algebra. Several applications and usage areas can be listed in many disciplines, such as; mathematics (especially in graph theory, computer sciences, and differential geometry), physics, and others. The set of the quaternions (real or Hamilton's quaternions) is represented by

$$\mathbb{H} := \{q | q = q_0 + q_1e_1 + q_2e_2 + q_3e_3,\ q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where e_1, e_2, e_3 are quaternionic units, which satisfy the following rules, $e_1^2 = e_2^2 = e_3^2 = -1$, $e_1e_2 = -e_2e_1 = e_3$, $e_2e_3 = -e_3e_2 = e_1$, $e_3e_1 = -e_1e_3 = e_2$ [17–19]. After investigating the real quaternions, the split quaternions were studied by James Cockle [11]. The split quaternions hold the

Received December 29, 2023. Accepted March 12, 2024.
2020 Mathematics Subject Classification. 11B37, 11K31, 11R52, 11Y55.
Key words and phrases. Horadam numbers, 3-parameter generalized quaternions, special recurrence sequences, fundamental matrix.

*Corresponding author

following rules $e_1^2=-1$, $e_2^2=e_3^2=1, e_1e_2=-e_2e_1=e_3$, $e_2e_3=-e_3e_2=-e_1$, $e_1e_2e_3=1$ [11]. Further away, the generalized quaternions (or 2-parameter generalized quaternions, shortly 2PGQs) have been examined in lots of studies (see [12, 14, 31–33, 38, 43–45, 58]). The set of 2PGQs is denoted as $\mathbb{H}_{\lambda_1\lambda_2}$ and identified as

$$\mathbb{H}_{\lambda_1\lambda_2} := \{q|q=q_0+q_1e_1+q_2e_2+q_3e_3,\ q_0,q_1,q_2,q_3,\lambda_1,\lambda_2\in\mathbb{R}\},$$

where the quaternionic units e_1,e_2,e_3 satisfy the following rules:

$$e_1^2=-\lambda_1,\ e_2^2=-\lambda_2,\ e_3^2=-\lambda_1\lambda_2,$$
$$e_1e_2=-e_2e_1=e_3,\ e_2e_3=-\,e_3e_2=\lambda_2e_1,\ e_3e_1=-e_1e_3=\lambda_1e_2.$$

For $\lambda_1=\lambda_2=1$, q is a real quaternion; for $\lambda_1=1,\lambda_2=-1$, q is a split quaternion; for $\lambda_1=1,\lambda_2=0$, q is a semi-quaternion; for $\lambda_1=-1$, $\lambda_2=0$, q is a split semi-quaternion, and for $\lambda_1=\lambda_2=0$, q is a 1/4-quaternion [11, 12, 14, 19, 31–33, 38, 39, 43–45, 48, 49, 58].

On the other hand, T. D. Şentürk and Z. Ünal have introduced a new type of quaternion family, which is called the 3-parameter generalized quaternion (shortly, 3PGQ) in [48, 49]. In order to achieve a generalization of real, split, and 2PGQ, the authors derive a comprehensive understanding of the quaternion algebra based on the 3-parameters. The set of 3PGQs is denoted by $\mathbb{K}$ and defined as

$$\mathbb{K} := \{q=q_0+q_1e_1+q_2e_2+q_3e_3,\ q_0,q_1,q_2,q_3,\lambda_1,\lambda_2,\lambda_3\in\mathbb{R}\},$$

where the quaternionic units e_1,e_2,e_3 satisfy the rules given in Table 1.

Table 1. Multiplication rules of quaternionic units for 3PGQ [48, 49].

$\cdot$	$\mathbf{1}$	$\boldsymbol{e_1}$	$\boldsymbol{e_2}$	$\boldsymbol{e_3}$
$\mathbf{1}$	1	e_1	e_2	e_3
$\boldsymbol{e_1}$	e_1	$-\lambda_1\lambda_2$	λ_1e_3	$-\lambda_2e_2$
$\boldsymbol{e_2}$	e_2	$-\lambda_1e_3$	$-\lambda_1\lambda_3$	λ_3e_1
$\boldsymbol{e_3}$	e_3	λ_2e_2	$-\lambda_3e_1$	$-\lambda_2\lambda_3$

According to the values $\lambda_{i\in\{1,2,3\}}$, we get some special cases. The following Table 2 includes some special cases of 3PGQs. Also, the other types of special cases can be studied for $\lambda_{i\in\{1,2,3\}}$ [48, 49].

Table 2. Classification of 3PGQs [48, 49].

For	**Some Types of 3PGQs**
$\lambda_1=1,\lambda_2,\lambda_3\in\mathbb{R}$	2PGQs [12, 14, 31–33, 38, 43–45, 58]
$\lambda_1=1,\lambda_2=1,\lambda_3=-1$	Split quaternion [11]
$\lambda_1=1,\lambda_2=1,\lambda_3=1$	Hamilton quaternions [17–19]
$\lambda_1=1,\lambda_2=1,\lambda_3=0$	Semi-quaternions [39, 44]
$\lambda_1=1,\lambda_2=-1,\lambda_3=0$	Split semi-quaternions [44]
$\lambda_1=1,\lambda_2=0,\lambda_3=0$	1/4-quaternions [19, 44]

In addition, special sequences (or numbers) are interesting and quite popular work-frames for several researchers. Accordingly, a great number of studies have been done and are ongoing, and they are linked to them in different ways when the existing literature is examined. In this paper, we deal with the generalization of the second-order recurrence sequences, called Horadam sequence (or numbers) [29]. Some of the most popular special cases of Horadam numbers are the Fibonacci and Lucas numbers (see [36, 50]). For all $n \geq 2$, the Horadam numbers ($\{Q_n(Q_0, Q_1; r, s)\}_{n\geq 0}$ or $\{Q_n\}_{n\geq 0}$) satisfy the following recurrence relation

$$Q_n = rQ_{n-1} + sQ_{n-2}, \tag{1}$$

where the initial values $Q_0 = a, Q_1 = b$ are arbitrary integers and r, s are real numbers [51]. For more detailed information about Horadam numbers and special cases of them, the studies [20, 22–24, 26–29, 37, 51–53] can be examined.

One can observe that bringing together the various types of quaternions and special recurrence sequence components is quite an attractive concept for several researchers in the literature. Horadam [21, 25] and Iyer [30] studied the Fibonacci quaternions and quaternion recurrence, and Swamy obtained the generalized Fibonacci quaternions in [54]. Polatlı et al. introduced the split k-Fibonacci and k-Lucas quaternions in [40]. Additionally, Halıcı investigated the Fibonacci quaternions and a new generalization of them in [15,16], respectively. Tan gave a new generalization of Fibonacci quaternions in [56]. Also, in [55] Jacobsthal quaternions, in [10] Pell and Pell-Lucas quaternions were examined. Catarino introduced the $h(x)$-Fibonacci quaternions in [7] and modified Pell and modified k-Pell quaternions in [8]. Tokeşer et al. studied the split Pell and Pell-Lucas quaternions in [57]. Then, Flaut and Savin determined the generalized Fibonacci-Lucas quaternions [13], as well. The generalization of Fibonacci/Lucas quaternions was studied in [41, 42]. Akyiğit et al. examined the split Fibonacci/Lucas quaternions [1] and generalized (2-parameter) Fibonacci/Lucas quaternions [2]. In [46], Şentürk et al. scrutinized the unrestricted Horadam generalized quaternions and Horadam hybrid numbers [47]. k-Fibonacci and k-Lucas generalized quaternions were obtained by Bilgici et al. [4]. Yüce and Torunbalcı Aydın examined the dual Fibonacci quaternions [61] and generalizations of them [60]. Moreover, Bród introduced the split Horadam quaternions in [6]. Then, Bilgici investigated the Fibonacci and Lucas 3PGQs [3], and Jacobsthal and Jacobsthal-Lucas 3PGQ in [5]. Also, [59] determined the dual Fibonacci and Lucas 3PGQs. Recently, Chaker and Boua examined the generalized quaternions algebra with generalized Fibonacci quaternions in [9]. Also, Kızılateş and Kibar determined the 3PGQs with higher order generalized Fibonacci numbers components in [35].

In this study, we intend to combine the Horadam numbers and 3PGQ; namely, we investigate 3PGQ with Horadam numbers components. Also, we examine the recurrence relation, Binet formula, generating function, exponential generating function, Poisson generating function, summing formulas, matrix formulas, Cassini identity, and some special equations as well. The subsequent analysis involves the examination of the determinant, characteristic polynomial, characteristic equation, eigenvalues, and eigenvectors in relation to the matrix representation of Horadam 3PGQ.

2. Preliminaries

In this section, we recall some terminology that is used throughout this paper with respect to both 3PGQs and Horadam numbers.

For $q = q_0 + q_1e_1 + q_2e_2 + q_3e_3,\ p = p_0 + p_1e_1 + p_2e_2 + p_3e_3 \in \mathbb{K}$, taking into account the rules in Table 1, some basic algebraic properties are listed below [48, 49]:

- ✳ *Equality:* $q = p \Leftrightarrow q_0 = p_0, \quad q_1 = p_1, \quad q_2 = p_2, \quad q_3 = p_3.$
- ✳ *Addition and subtraction:*
$$q \pm p = q_0 \pm p_0 + (q_1 \pm p_1)e_1 + (q_2 \pm p_2)e_2 + (q_3 \pm p_3)e_3.$$
- ✳ *Multiplication by a scalar:* $cq = cq_0 + cq_1e_1 + cq_2e_2 + cq_3e_3$, $c \in \mathbb{R}$.
- ✳ *Scalar and vector part:* q occurs from scalar and vector part such as; $q = S_q + V_q$, where $S_q = q_0$ is scalar part and $V_q = q_1e_1 + q_2e_2 + q_3e_3$ is vector part.
- ✳ *Multiplication:* $qp = S_qS_p - f(V_q, V_p) + S_qV_q + S_pV_p + V_q \wedge V_p$, where
$$f(V_q, V_p) = \lambda_1\lambda_2q_1p_1 + \lambda_1\lambda_3q_2p_2 + \lambda_2\lambda_3q_3p_3$$
and
$$V_q \wedge V_p = \begin{vmatrix} \lambda_3e_1 & \lambda_2e_2 & \lambda_1e_3 \\ q_1 & q_2 & q_3 \\ p_1 & p_2 & p_3 \end{vmatrix}.$$
Here $V_q \wedge V_p = \lambda_3(q_2p_3 - q_3p_2)e_1 + \lambda_2(q_3p_1 - q_1p_3)e_2 + \lambda_1(q_1p_2 - q_2p_1)e_3$.
- ✳ *Conjugation:* $\overline{q} = q_0 - q_1e_1 - q_2e_2 - q_3e_3$.
- ✳ *Inverse:* $q^{-1} = \dfrac{\overline{q}}{N_q} = \dfrac{q_0 - q_1e_1 - q_2e_2 - q_3e_3}{q_0^2 + \lambda_1\lambda_2q_1^2 + \lambda_1\lambda_3q_2^2 + \lambda_2\lambda_3q_3^2}$, where $N_q \neq 0$.
- ✳ *Inner product:* $\langle q, p\rangle = q_0p_0 + \lambda_1\lambda_2q_1p_1 + \lambda_1\lambda_3q_2p_2 + \lambda_2\lambda_3q_3p_3$.
- ✳ *Norm:* $N_q = q\overline{q} = \overline{q}q = q_0^2 + \lambda_1\lambda_2q_1^2 + \lambda_1\lambda_3q_2^2 + \lambda_2\lambda_3q_3^2$.

They can be seen easily that $S_{q\pm p} = q_0 \pm p_0 = S_q \pm S_p$, $V_{q\pm p} = V_q \pm V_p$ and $\overline{q} = S_q - V_q$. If $N_q = 1$, then q is a 3-parameter generalized unit quaternion. For more detailed terminology for 3PGQs, we refer to the studies [48, 49].

Also, for $N_q > 0$ and $\lambda_1\lambda_2q_1^2 + \lambda_1\lambda_3q_2^2 + \lambda_2\lambda_3q_3^2 \neq 0$, q can be written in a polar form as follows:
$$q = \sqrt{N_q}\left(\cos\theta + \hat{q}\sin\theta\right),$$

where

$$\hat{q}=\frac{1}{\sqrt{\lambda_1\lambda_2 q_1^2+\lambda_1\lambda_3 q_2^2+\lambda_2\lambda_3 q_3^2}}\left(q_1,q_2,q_3\right)\cdot$$

Here

$$\cos\theta=\frac{q_0}{\sqrt{N_q}},\quad \sin\theta=\sqrt{\frac{\lambda_1\lambda_2 q_1^2+\lambda_1\lambda_3 q_2^2+\lambda_2\lambda_3 q_3^2}{N_q}},$$

and $\hat{q}$ is called 3-parameter generalized unit vector. Moreover for q, the following fundamental matrix M_q is constructed:

$$M_q=\begin{pmatrix} q_0 & -\lambda_1\lambda_2 q_1 & -\lambda_1\lambda_3 q_2 & -\lambda_2\lambda_3 q_3\\ q_1 & q_0 & -\lambda_3 q_3 & \lambda_3 q_2\\ q_2 & \lambda_2 q_3 & q_0 & -\lambda_2 q_1\\ q_3 & -\lambda_1 q_2 & \lambda_1 q_1 & q_0\end{pmatrix}.$$

According to the values of $\lambda_{i\in\{1,2,3\}}$, we can classify the matrix M_q. For $\lambda_1=1$, λ_2, $\lambda_3\in\mathbb{R}$, the fundamental matrix for 2PGQ is constructed. For $\lambda_1=1$, $\lambda_2=1,\lambda_3=-1$, then the fundamental matrix for split quaternions is given. Also, for $\lambda_1=\lambda_2=\lambda_3=1$, then the fundamental matrix for Hamilton quaternions is written.

Additionally, one can give some algebraic calculations for M_q: the determinant of M_q is $\det\left(M_q\right)=N_q^2$. The characteristic polynomial of M_q is:

$$P_{M_q}(t)=\left(t^2-2tq_0+q_0^2+\lambda_1\lambda_2 q_1^2+\lambda_1\lambda_3 q_2^2+\lambda_2\lambda_3 q_3^2\right)^2.$$

Hence, the characteristic equation of M_q is:

$$\det\left(M_q-tI_4\right)=0\Leftrightarrow P_{M_q}(t)=\left(t^2-2tq_0+q_0^2+\lambda_1\lambda_2 q_1^2+\lambda_1\lambda_3 q_2^2+\lambda_2\lambda_3 q_3^2\right)^2=0.$$

It enables to compute the eigenvalues as follows:

$$t_{1,2}=q_0+\sqrt{-\lambda_1\lambda_2 q_1^2-\lambda_1\lambda_3 q_2^2-\lambda_2\lambda_3 q_3^2},$$
$$t_{3,4}=q_0-\sqrt{-\lambda_1\lambda_2 q_1^2-\lambda_1\lambda_3 q_2^2-\lambda_2\lambda_3 q_3^2}.$$

This gives the relation:

$$t_{1,2}t_{3,4}=q_0^2+\lambda_1\lambda_2 q_1^2+\lambda_1\lambda_3 q_2^2+\lambda_2\lambda_3 q_3^2=N_q.$$

The eigenvectors corresponding to the eigenvalue $t_{1,2}$ are computed as:

$$\begin{pmatrix}\frac{\lambda_1 q_2\sqrt{-\lambda_1\lambda_2 q_1^2-\lambda_1\lambda_3 q_2^2-\lambda_2\lambda_3 q_3^2}-\lambda_1\lambda_2 q_1 q_3}{\lambda_1 q_2^2+\lambda_2 q_3^2} & \frac{q_3\sqrt{-\lambda_1\lambda_2 q_1^2-\lambda_1\lambda_3 q_2^2-\lambda_2\lambda_3 q_3^2}+\lambda_1 q_1 q_2}{\lambda_1 q_2^2+\lambda_2 q_3^2} & 1 & 0\end{pmatrix}^T$$

and

$$\begin{pmatrix}\frac{\lambda_2 q_3\sqrt{-\lambda_1\lambda_2 q_1^2-\lambda_1\lambda_3 q_2^2-\lambda_2\lambda_3 q_3^2}+\lambda_1\lambda_2 q_1 q_2}{\lambda_1 q_2^2+\lambda_2 q_3^2} & -\frac{q_2\sqrt{-\lambda_1\lambda_2 q_1^2-\lambda_1\lambda_3 q_2^2-\lambda_2\lambda_3 q_3^2}-\lambda_2 q_1 q_3}{\lambda_1 q_2^2+\lambda_2 q_3^2} & 0 & 1\end{pmatrix}^T.$$

The eigenvectors corresponding to the eigenvalue $t_{3,4}$ are

$$\begin{pmatrix}\frac{\lambda_1 q_2\sqrt{-\lambda_1\lambda_2 q_1^2-\lambda_1\lambda_3 q_2^2-\lambda_2\lambda_3 q_3^2}+\lambda_1\lambda_2 q_1 q_3}{\lambda_1 q_2^2+\lambda_2 q_3^2} & -\frac{q_3\sqrt{-\lambda_1\lambda_2 q_1^2-\lambda_1\lambda_3 q_2^2-\lambda_2\lambda_3 q_3^2}-\lambda_1 q_1 q_2}{\lambda_1 q_2^2+\lambda_2 q_3^2} & 1 & 0\end{pmatrix}^T$$

and

$$\begin{pmatrix} -\frac{\lambda_2 q_3\sqrt{-\lambda_1\lambda_2 q_1^2-\lambda_1\lambda_3 q_2^2-\lambda_2\lambda_3 q_3^2}-\lambda_1\lambda_2 q_1 q_2}{\lambda_1 q_2^2+\lambda_2 q_3^2} & \frac{q_2\sqrt{-\lambda_1\lambda_2 q_1^2-\lambda_1\lambda_3 q_2^2-\lambda_2\lambda_3 q_3^2}+\lambda_2 q_1 q_3}{\lambda_1 q_2^2\lambda_2 q_3^2} & 0 & 1 \end{pmatrix}^T.$$

On the other hand, the characteristic equation of Horadam numbers (see equation (1)) is $x^2 - rx - s = 0$, and its roots are as follows:

$$x_1 = \frac{r+\sqrt{r^2+4s}}{2} \quad \text{and} \quad x_2 = \frac{r-\sqrt{r^2+4s}}{2},$$

where $x_1 + x_2 = r$ and $x_1x_2 = -s$ [24, 51]. The Binet formula for Horadam numbers is [24, 51]:

$$Q_n = \frac{Ax_1^n - Bx_2^n}{x_1 - x_2}, \tag{2}$$

where

$$A = Q_1 - Q_0x_2 \quad \text{and} \quad B = Q_1 - Q_0x_1. \tag{3}$$

The Horadam sequence can be classified with respect to the initial conditions and r, s values. In the following Table 3, some of its members are given [24, 51]:

TABLE 3. Some special cases of Horadam numbers.

Name	$\{Q_n\} = \{Q_n(Q_0, Q_1; r, s)\}$	**Recurrence Relation**
Fibonacci	$\{F_n\} = \{Q_n(0, 1; 1, 1)\}$	$F_n = F_{n-1} + F_{n-2}$
Lucas	$\{L_n\} = \{Q_n(2, 1; 1, 1)\}$	$L_n = L_{n-2} + L_{n-3}$
Pell	$\{P_n\} = \{Q_n(0, 1; 2, 1)\}$	$P_n = 2P_{n-1} + P_{n-2}$
Pell-Lucas	$\{B_n\} = \{Q_n(2, 2; 2, 1)\}$	$B_n = 2B_{n-1} + B_{n-2}$
Jacobsthal	$\{J_n\} = \{Q_n(0, 1; 1, 2)\}$	$J_n = J_{n-1} + 2J_{n-2}$
Jacobsthal-Lucas	$\{C_n\} = \{Q_n(2, 1; 1, 2)\}$	$C_n = C_{n-1} + 2C_{n-2}$

Moreover, the following matrix equation for Horadam numbers can be written depending on the r and s values such that ([34, 51]):

$$\begin{pmatrix} Q_{n+1} \\ Q_n \end{pmatrix} = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} Q_1 \\ Q_0 \end{pmatrix}.$$

3. Horadam 3-Parameter Generalized Quaternions

This section presents Horadam 3PGQs as a novel family of special number systems. Also, we examine some algebraic properties and obtain the Binet formula, generating function, exponential generating function, Poisson generating function, some summation formulas, matrix equations, Cassini identity, polar representation, and some new interesting equations. Also, via the matrix representation of Horadam 3PGQ, we get some properties such as; determinant, characteristic polynomial and equation, eigenvalues, and eigenvectors.

Definition 3.1. *Let $\mathcal{Q}_n$ be nth Horadam 3PGQ. Then, it is defined as*

$$\mathcal{Q}_n = Q_n + Q_{n+1}e_1 + Q_{n+2}e_2 + Q_{n+3}e_3 \quad \text{for every} \quad n \geq 0, \tag{4}$$

where Q_n is the nth Horadam number and e_1, e_2 and e_3 satisfy the rules given in Table 1. Also, initial values are written as

$$\begin{cases} \mathcal{Q}_0 = a + be_1 + (sa + rb)\, e_2 + \left[rsa + (r^2 + s)b\right] e_3, \\ \mathcal{Q}_1 = b + (sa + rb)\, e_1 + \left[rsa + (r^2 + s)b\right] e_2 + \left[\left(r^2 s + s^2\right) a + \left(r^3 + 2rs\right) b\right] e_3. \end{cases}$$

In the following Table 4, we classify Horadam 3PGQs.

TABLE 4. Classification of Horadam 3PGQs.

For	**Types**
$\lambda_1 = 1, \quad \lambda_2, \lambda_3 \in \mathbb{R}$	Horadam 2PGQ [2]
$\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = -1$	Horadam split quaternions [6]
$\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 1$	Horadam Hamilton quaternions [15, 54]
$\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 0$	Horadam semi-quaternions
$\lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = 0$	Horadam split semi-quaternions
$\lambda_1 = 1, \quad \lambda_2 = 0, \quad \lambda_3 = 0$	Horadam $\frac{1}{4}$-quaternions

Also, one can see Table 5 for some special cases of Horadam 3PGQs.

TABLE 5. Some special cases of Horadam 3PGQs.

Name	**Definition**	**Recurrence**
Fibonacci 3PGQ [3, 35]	$\mathcal{F}_n = F_n + F_{n+1}e_1 + F_{n+2}e_2 + F_{n+3}e_3$	$\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2}$
Lucas 3PGQ [3]	$\mathcal{L}_n = L_n + L_{n+1}e_1 + L_{n+2}e_2 + L_{n+3}e_3$	$\mathcal{L}_n = \mathcal{L}_{n-1} + \mathcal{L}_{n-2}$
Pell 3PGQ	$\mathcal{P}_n = P_n + P_{n+1}e_1 + P_{n+2}e_2 + P_{n+3}e_3$	$\mathcal{P}_n = 2\mathcal{P}_{n-1} + \mathcal{P}_{n-2}$
Pell-Lucas 3PGQ	$\mathcal{B}_n = B_n + B_{n+1}e_1 + B_{n+2}e_2 + B_{n+3}e_3$	$\mathcal{B}_n = 2\mathcal{B}_{n-1} + \mathcal{B}_{n-2}$
Jacobsthal 3PGQ [5]	$\mathcal{J}_n = J_n + J_{n+1}e_1 + J_{n+2}e_2 + J_{n+3}e_3$	$\mathcal{J}_n = \mathcal{J}_{n-1} + 2\mathcal{J}_{n-2}$
Jacobsthal-Lucas 3PGQ [5]	$\mathcal{C}_n = C_n + C_{n+1}e_1 + C_{n+2}e_2 + C_{n+3}e_3$	$\mathcal{C}_n = \mathcal{C}_{n-1} + 2\mathcal{C}_{n-2}$

*Chaker and Boua ([9]) examined some results on generalized quaternions algebra with generalized Fibonacci quaternions.

Now, let us examine some algebraic properties such as equality, addition/ subtraction, multiplication by scalar, scalar and vector parts, multiplication, conjugation, norm, inverse, and inner product of Horadam 3PGQs.

For every $n, m \geq 0$, let $\mathcal{Q}_n = Q_n + Q_{n+1}e_1 + Q_{n+2}e_2 + Q_{n+3}e_3$ and $\mathcal{Q}_m = Q_m + Q_{m+1}e_1 + Q_{m+2}e_2 + Q_{m+3}e_3$ be the *nth* and *mth* Horadam 3PGQ, respectively. The following properties can be obtained:

✳ *Equality:*

$$\mathcal{Q}_n = \mathcal{Q}_m \Leftrightarrow Q_n = Q_m, \quad Q_{n+1} = Q_{m+1}, \quad Q_{n+2} = Q_{m+2}, \quad Q_{n+3} = Q_{m+3}.$$

✳ *Addition/Subtraction:*

$$\begin{aligned} \mathcal{Q}_n \pm \mathcal{Q}_m = & Q_n \pm Q_m + \left(Q_{n+1} \pm Q_{m+1}\right) e_1 + \left(Q_{n+2} \pm Q_{m+2}\right) e_2 \\ & + \left(Q_{n+3} \pm Q_{m+3}\right) e_3. \end{aligned}$$

✳ *Multiplication by a scalar:*

$$c\mathcal{Q}_n = cQ_n + cQ_{n+1}e_1 + cQ_{n+2}e_2 + cQ_{n+3}e_3, \quad c \in \mathbb{R}.$$

✳ *Scalar and Vector Parts:* The scalar part of $\mathcal{Q}_n$ is denoted by $S_{\mathcal{Q}_n}$ and $S_{\mathcal{Q}_n} = Q_n$. Also, the vector part of $\mathcal{Q}_n$ is denoted by $V_{\mathcal{Q}_n}$ and $V_{\mathcal{Q}_n} = Q_{n+1}e_1 + Q_{n+2}e_2 + Q_{n+3}e_3$. This implies that $S_{\mathcal{Q}_n \pm \mathcal{Q}_m} = Q_n \pm Q_m = S_{\mathcal{Q}_n} \pm S_{\mathcal{Q}_m}$ and $V_{\mathcal{Q}_n \pm \mathcal{Q}_m} = V_{\mathcal{Q}_n} \pm V_{\mathcal{Q}_m}$.

✳ *Multiplication:*

$$\mathcal{Q}_n\mathcal{Q}_m = S_{\mathcal{Q}_n}S_{\mathcal{Q}_m} - f(V_{\mathcal{Q}_n}, V_{\mathcal{Q}_m}) + S_{\mathcal{Q}_n}V_{\mathcal{Q}_n} + S_{\mathcal{Q}_m}V_{\mathcal{Q}_m} + V_{\mathcal{Q}_n} \wedge V_{\mathcal{Q}_m},$$

where

$$f(V_{\mathcal{Q}_n}, V_{\mathcal{Q}_m}) = \lambda_1\lambda_2 Q_{n+1}Q_{m+1} + \lambda_1\lambda_3 Q_{n+2}Q_{n+2} + \lambda_2\lambda_3 Q_{n+3}Q_{n+3}$$

and

$$\begin{aligned} V_{\mathcal{Q}_n} \wedge V_{\mathcal{Q}_m} = & \begin{vmatrix} \lambda_3 e_1 & \lambda_2 e_2 & \lambda_1 e_3 \\ Q_{n+1} & Q_{n+2} & Q_{n+3} \\ Q_{m+1} & Q_{m+2} & Q_{m+3} \end{vmatrix} \\ = & \lambda_3(Q_{n+2}Q_{m+3} - Q_{n+3}Q_{m+2})e_1 \\ & + \lambda_2(Q_{n+3}Q_{m+1} - Q_{n+1}Q_{m+3})e_2 \\ & + \lambda_1(Q_{n+1}Q_{m+2} - Q_{n+2}Q_{m+1})e_3. \end{aligned}$$

We can also give the following form of multiplication as

$$\begin{aligned} \mathcal{Q}_n\mathcal{Q}_m = & Q_nQ_m - \lambda_1\lambda_2 Q_{n+1}Q_{m+1} - \lambda_1\lambda_3 Q_{n+2}Q_{m+2} - \lambda_2\lambda_3 Q_{n+3}Q_{m+3} \\ & + \left(Q_nQ_{m+1} + Q_mQ_{n+1} + \lambda_3\left(Q_{n+2}Q_{m+3} - Q_{n+3}Q_{m+2}\right)\right)e_1 \\ & + \left(Q_nQ_{m+2} + Q_mQ_{n+2} + \lambda_2\left(Q_{n+3}Q_{m+1} - Q_{n+1}Q_{m+3}\right)\right)e_2 \\ & + \left(Q_nQ_{m+3} + Q_mQ_{n+3} + \lambda_1\left(Q_{n+1}Q_{m+2} - Q_{n+2}Q_{m+1}\right)\right)e_3. \end{aligned} \tag{5}$$

✳ *Conjugation:* $\overline{\mathcal{Q}}_n = Q_n - Q_{n+1}e_1 - Q_{n+2}e_2 - Q_{n+3}e_3$.

✳ *Inverse:*

$$\mathcal{Q}_n^{-1} = \frac{\overline{\mathcal{Q}}_n}{N_{\mathcal{Q}_n}} = \frac{Q_n - Q_{n+1}e_1 - Q_{n+2}e_2 - Q_{n+3}e_3}{Q_n^2 + \lambda_1\lambda_2 Q_{n+1}^2 + \lambda_1\lambda_3 Q_{n+2}^2 + \lambda_2\lambda_3 Q_{n+3}^2}, \ N_{\mathcal{Q}_n} \neq 0.$$

✳ *Inner product:*

$$\begin{aligned} \langle \mathcal{Q}_n, \mathcal{Q}_m \rangle = & Q_nQ_m + \lambda_1\lambda_2 Q_{n+1}Q_{m+1} + \lambda_1\lambda_3 Q_{n+2}Q_{m+2} \\ & + \lambda_2\lambda_3 Q_{n+3}Q_{m+3}. \end{aligned}$$

✳ *Norm:*

$$N_{\mathcal{Q}_n} = \mathcal{Q}_n\overline{\mathcal{Q}}_n = \overline{\mathcal{Q}}_n\mathcal{Q}_n = Q_n^2 + \lambda_1\lambda_2 Q_{n+1}^2 + \lambda_1\lambda_3 Q_{n+2}^2 + \lambda_2\lambda_3 Q_{n+3}^2.$$

Theorem 3.2 (Recurrence Relation)**.** *Let $\mathcal{Q}_n$ be the nth Horadam 3PGQ. Then the following recurrence relation holds:*

$$\mathcal{Q}_n = r\mathcal{Q}_{n-1} + s\mathcal{Q}_{n-2}, \quad \forall n \geq 2. \tag{6}$$

Proof. By using equations (1) and (4), we obtain:

$$\begin{aligned} r\mathcal{Q}_{n-1} + s\mathcal{Q}_{n-2} =& r\left(Q_{n-1} + Q_n e_1 + Q_{n+1}e_2 + Q_{n+2}e_3\right) \\ & + s\left(Q_{n-2} + Q_{n-1}e_1 + Q_n e_2 + Q_{n+1}e_3\right) \\ =& rQ_{n-1} + sQ_{n-2} + \left(rQ_n + sQ_{n-1}\right)e_1 \\ & + \left(rQ_{n+1} + sQ_n\right)e_2 + \left(rQ_{n+2} + sQ_{n+1}\right)e_3 \\ =& Q_n + Q_{n+1}e_1 + Q_{n+2}e_2 + Q_{n+3}e_3 \\ =& \mathcal{Q}_n. \end{aligned}$$

This finishes the proof. □

Theorem 3.3 (Binet Formula)**.** *Let $\mathcal{Q}_n$ be the nth Horadam 3PGQ. For every $n \geq 0$, the following Binet formula is satisfied:*

$$\mathcal{Q}_n = \frac{Ax_1^n\,\widetilde{x}_1 - Bx_2^n\,\widetilde{x}_2}{x_1 - x_2}\,, \tag{7}$$

where A and B are given in equation (3) *and*

$$\widetilde{x}_1 = 1 + x_1e_1 + x_1^2e_2 + x_1^3e_3,\ \ \widetilde{x}_2 = 1 + x_2e_1 + x_2^2e_2 + x_2^3e_3.$$

Proof. Via equations (2) and (4), the proof can be completed as follows:

$$\begin{aligned} \mathcal{Q}_n =& Q_n + Q_{n+1}e_1 + Q_{n+2}e_2 + Q_{n+3}e_3 \\ =& \frac{Ax_1^n - Bx_2^n}{x_1 - x_2} + \left(\frac{Ax_1^{n+1} - Bx_2^{n+1}}{x_1 - x_2}\right)e_1 + \left(\frac{Ax_1^{n+2} - Bx_2^{n+2}}{x_1 - x_2}\right)e_2 + \left(\frac{Ax_1^{n+3} - Bx_2^{n+3}}{x_1 - x_2}\right)e_3 \\ =& \frac{Ax_1^n\,\widetilde{x}_1 - Bx_2^n\,\widetilde{x}_2}{x_1 - x_2}, \end{aligned}$$

where $\widetilde{x}_1 = 1 + x_1e_1 + x_1^2e_2 + x_1^3e_3$ and $\widetilde{x}_2 = 1 + x_2e_1 + x_2^2e_2 + x_2^3e_3$. □

In following Table 6, Binet formulas of some special cases of Horadam 3PGQ are given. In this table, (also in following Table 8 and Table 9), we use:

- ➢ $\omega_1 = (1+\sqrt{5})/2$, $\omega_2 = (1-\sqrt{5})/2$ are the roots of equation $x^2 - x - 1 = 0$ and $\widetilde{\omega}_1 = 1 + \omega_1e_1 + \omega_1^2e_2 + \omega_1^3e_3$, $\widetilde{\omega}_2 = 1 + \omega_2e_1 + \omega_2^2e_2 + \omega_2^3e_3$,
- ➢ $\nu_1 = 1 + \sqrt{2}$, $\nu_2 = 1 - \sqrt{2}$ are the roots of equation $x^2 - 2x - 1 = 0$ and $\widetilde{\nu}_1 = 1 + \nu_1e_1 + \nu_1^2e_2 + \nu_1^3e_3$, $\widetilde{\nu}_2 = 1 + \nu_2e_1 + \nu_2^2e_2 + \nu_2^3e_3$,
- ➢ $\mu_1 = 2$, $\mu_2 = -1$ are the roots of equation $x^2 - x - 2 = 0$ (see [51]) and $\widetilde{\mu}_1 = 1 + \mu_1e_1 + \mu_1^2e_2 + \mu_1^3e_3$, $\widetilde{\mu}_2 = 1 + \mu_2e_1 + \mu_2^2e_2 + \mu_2^3e_3$.

TABLE 6. Binet formulas for special cases of Horadam 3PGQs.

Name	**Binet Formula**
Fibonacci 3PGQ [3, 35]	$\mathcal{F}_n = (\omega_1^n\,\widetilde{\omega}_1 - \omega_2^n\,\widetilde{\omega}_2)/\sqrt{5}$
Lucas 3PGQ [3]	$\mathcal{L}_n = \omega_1^n\,\widetilde{\omega}_1 + \omega_2^n\,\widetilde{\omega}_2$
Pell 3PGQ	$\mathcal{P}_n = (\nu_1^n\,\widetilde{\nu}_1 - \nu_2\,\widetilde{\nu}_2)/2\sqrt{2}$
Pell-Lucas 3PGQ	$\mathcal{B}_n = \nu_1^n\,\widetilde{\nu}_1 + \nu_2^n\,\widetilde{\nu}_2$
Jacobsthal 3PGQ [5]	$\mathcal{J}_n = (\mu_1^n\,\widetilde{\mu}_1 - \mu_2^n\,\widetilde{\mu}_2)/3$
Jacobsthal-Lucas 3PGQ [5]	$\mathcal{C}_n = \mu_1^n\,\widetilde{\mu}_1 + \mu_2^n\,\widetilde{\mu}_2$

Theorem 3.4 (Generating Function)**.** *Let $\mathcal{Q}_n$ be the nth Horadam 3PGQ. Then the following generating function is written:*

$$\sum_{n=0}^{\infty} \mathcal{Q}_n x^n = \frac{\mathcal{Q}_0 + (\mathcal{Q}_1 - r\mathcal{Q}_0)x}{1 - rx - sx^2}.$$

Proof. Suppose that $\sum\limits_{n=0}^{\infty} \mathcal{Q}_n x^n = \mathcal{Q}_0 + \mathcal{Q}_1 x + \mathcal{Q}_2 x^2 + \ldots + \mathcal{Q}_n x^n + \ldots$ be generating function of Horadam 3PGQ. Let us multiply both sides of the equation by rx and sx^2:

$$rx \sum_{n=0}^{\infty} \mathcal{Q}_n x^n = r\mathcal{Q}_0 x + r\mathcal{Q}_1 x^2 + r\mathcal{Q}_2 x^3 + \ldots + r\mathcal{Q}_n x^{n+1} + \ldots$$
$$sx^2 \sum_{n=0}^{\infty} \mathcal{Q}_n x^n = s\mathcal{Q}_0 x^2 + s\mathcal{Q}_1 x^3 + s\mathcal{Q}_2 x^4 + \ldots + s\mathcal{Q}_n x^{n+2} + \ldots$$

Then, by adding these two equations and considering equation (6), we get:

$$(1 - rx - sx^2) \sum_{n=0}^{\infty} \mathcal{Q}_n x^n = \mathcal{Q}_0 + (\mathcal{Q}_1 - r\mathcal{Q}_0)x.$$

Hence, the proof is completed. □

Also, the following Table 7 includes the generating functions of special cases of Horadam 3PGQ:

TABLE 7. Generating functions for special cases of Horadam 3PGQs.

Name	**Generating Function**
Fibonacci 3PGQ [3,35]	$\sum\limits_{n=0}^{\infty} \mathcal{F}_n x^n = \dfrac{e_1 + e_2 + 2e_3 + (1 + e_2 + e_3)\, x}{1 - x - x^2}$
Lucas 3PGQ [3]	$\sum\limits_{n=0}^{\infty} \mathcal{L}_n x^n = \dfrac{2 + e_1 + 3e_2 + 4e_3 + (-1 + 2e_1 + e_2 + 3e_3)x}{1 - x - x^2}$
Pell 3PGQ	$\sum\limits_{n=0}^{\infty} \mathcal{P}_n x^n = \dfrac{e_1 + 2e_2 + 5e_3 + (1 + e_2 + 2e_3)}{1 - 2x - x^2}$
Pell-Lucas 3PGQ	$\sum\limits_{n=0}^{\infty} \mathcal{B}_n x^n = \dfrac{2 + 2e_1 + 6e_2 + 14e_3 + (-2 + 2e_1 + 2e_2 + 6e_3)\, x}{1 - 2x - x^2}$
Jacobsthal 3PGQ [5]	$\sum\limits_{n=0}^{\infty} \mathcal{J}_n x^n = \dfrac{e_1 + e_2 + 3e_3 + (1 + 2e_2 + 2e_3)\, x}{1 - x - 2x^2}$
Jacobsthal-Lucas 3PGQ [5]	$\sum\limits_{n=0}^{\infty} \mathcal{C}_n x^n = \dfrac{2 + e_1 + 5e_2 + 7e_3 + (-1 + 4e_1 + 2e_2 + 10e_3)\, x}{1 - x - 2x^2}$

Theorem 3.5 (Exponential Generating Function)**.** *Let $\mathcal{Q}_n$ be the nth Horadam 3PGQ. Then the exponential generating function is written as:*

$$\sum_{n=0}^{\infty} \mathcal{Q}_n \frac{y^n}{n!} = \frac{Ae^{x_1 y}\, \widetilde{x}_1 - Be^{x_2 y}\, \widetilde{x}_2}{x_1 - x_2}. \tag{8}$$

Proof. By using equation (7), we have:

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{Q}_n\frac{y^n}{n!}&=\sum_{n=0}^{\infty}\frac{Ax_1^n\,\widetilde{x}_1}{x_1-x_2}\frac{y^n}{n!}-\sum_{n=0}^{\infty}\frac{Bx_2^n\,\widetilde{x}_2}{x_1-x_2}\frac{y^n}{n!}\\&=\frac{A\,\widetilde{x}_1}{x_1-x_2}\sum_{n=0}^{\infty}\frac{x_1^ny^n}{n!}-\frac{B\,\widetilde{x}_2}{x_1-x_2}\sum_{n=0}^{\infty}\frac{x_2^ny^n}{n!}\\&=\frac{Ae^{x_1y}\,\widetilde{x}_1-Be^{x_2y}\,\widetilde{x}_2}{x_1-x_2}\cdot\end{aligned}$$

Therefore, we get the desired result. □

In addition to these, exponential functions for special cases of Horadam 3PGQs are written in Table 8:

TABLE 8. Exponential functions for special cases of Horadam 3PGQs.

Name	**Exponential Generating Function**
Fibonacci 3PGQ	$\sum_{n=0}^{\infty}\mathcal{F}_n\frac{y^n}{n!}=\frac{\widetilde{\omega}_1e^{\omega_1y}-\widetilde{\omega}_2e^{\omega_2y}}{\sqrt{5}}$
Lucas 3PGQ	$\sum_{n=0}^{\infty}\mathcal{L}_n\frac{y^n}{n!}=\widetilde{\omega}_1e^{\omega_1y}+\widetilde{\omega}_2e^{\omega_1y}$
Pell 3PGQ	$\sum_{n=0}^{\infty}\mathcal{P}_n\frac{y^n}{n!}=\frac{\widetilde{\nu}_1e^{\nu_1y}-\widetilde{\nu}_2e^{\nu_2y}}{2\sqrt{2}}$
Pell-Lucas 3PGQ	$\sum_{n=0}^{\infty}\mathcal{B}_n\frac{y^n}{n!}=\widetilde{\nu}_1e^{\nu_1y}+\widetilde{\nu}_2e^{\nu_2y}$
Jacobsthal 3PGQ	$\sum_{n=0}^{\infty}\mathcal{J}_n\frac{y^n}{n!}=\frac{\widetilde{\mu}_1e^{\mu_1y}-\widetilde{\mu}_2e^{\mu_2y}}{3}$
Jacobsthal-Lucas 3PGQ	$\sum_{n=0}^{\infty}\mathcal{C}_n\frac{y^n}{n!}=\widetilde{\mu}_1e^{\mu_1y}+\widetilde{\mu}_2e^{\mu_2y}$

Theorem 3.6 (Poisson Generating Function)**.** *Let $\mathcal{Q}_n$ be the nth Horadam 3PGQ. The Poisson generating function is written as:*

$$e^{-y}\sum_{n=0}^{\infty}\mathcal{Q}_n\frac{y^n}{n!}=\frac{Ae^{x_1y}\,\widetilde{x}_1-Be^{x_2y}\,\widetilde{x}_2}{e^y\,(x_1-x_2)}\cdot$$

Proof. With the help of equation (8), we get the desired result since the Poisson generating function is given as multiplying the exponential generating function by e^{-y} (see also [47]). □

In the following Table 9, we get the Poisson generating function for special cases of Horadam 3PGQs:

Table 9. Poisson generating function for special cases of Horadam 3PGQs.

Name	**Poisson Generating Function**
Fibonacci 3PGQ	$e^{-y}\sum\limits_{n=0}^{\infty}\mathcal{F}_n\dfrac{y^n}{n!}=\dfrac{\widetilde{\omega}_1 e^{\omega_1 y}-\widetilde{\omega}_2 e^{\omega_2 y}}{e^y\sqrt{5}}$
Lucas 3PGQ	$e^{-y}\sum\limits_{n=0}^{\infty}\mathcal{L}_n\dfrac{y^n}{n!}=e^{-y}\left(\widetilde{\omega}_1 e^{\omega_1 y}+\widetilde{\omega}_2 e^{\omega_1 y}\right)$
Pell 3PGQ	$e^{-y}\sum\limits_{n=0}^{\infty}\mathcal{P}_n\dfrac{y^n}{n!}=\dfrac{\widetilde{\nu}_1 e^{\nu_1 y}-\widetilde{\nu}_2 e^{\nu_2 y}}{2\sqrt{2}e^y}$
Pell-Lucas 3PGQ	$e^{-y}\sum\limits_{n=0}^{\infty}\mathcal{B}_n\dfrac{y^n}{n!}=e^{-y}\left(\widetilde{\nu}_1 e^{\nu_1 y}+\widetilde{\nu}_2 e^{\nu_2 y}\right)$
Jacobsthal 3PGQ	$e^{-y}\sum\limits_{n=0}^{\infty}\mathcal{J}_n\dfrac{y^n}{n!}=\dfrac{\widetilde{\mu}_1 e^{\mu_1 y}-\widetilde{\mu}_2 e^{\mu_2 y}}{3e^y}$
Jacobsthal-Lucas 3PGQ	$e^{-y}\sum\limits_{n=0}^{\infty}\mathcal{C}_n\dfrac{y^n}{n!}=e^{-y}\left(\widetilde{\mu}_1 e^{\mu_1 y}+\widetilde{\mu}_2 e^{\mu_2 y}\right)$

Thanks to the study [52, 53], we can obtain the following sum formulas for Horadam 3PGQ without proofs for the sake of brevity.

Theorem 3.7. *Let $\mathcal{Q}_n$ be the nth Horadam 3PGQ. For every $n, m \in \mathbb{N}$, the following summation formulas are satisfied:*

(a) $\sum\limits_{n=0}^{m}\mathcal{Q}_n=\dfrac{\mathcal{Q}_{m+2}+(1-r)\mathcal{Q}_{m+1}-\mathcal{Q}_1+(r-1)\mathcal{Q}_0}{r+s-1}$,

(b) $\sum\limits_{n=0}^{m}\mathcal{Q}_{2n}=\dfrac{(1-s)\mathcal{Q}_{2m+2}+rs\mathcal{Q}_{2m+1}+(s-1)\mathcal{Q}_2-rs\mathcal{Q}_1+(r^2-s^2+2s-1)\mathcal{Q}_0}{(r+s-1)(r-s+1)}$,

(c) $\sum\limits_{n=0}^{m}\mathcal{Q}_{2n+1}=\dfrac{r\mathcal{Q}_{2m+2}+(s-s^2)\mathcal{Q}_{2m+1}-r\mathcal{Q}_2+(-1+s+r^2)\mathcal{Q}_1}{(r-s+1)(r+s-1)}$,

where $r+s-1\neq 0$ and $(r-s+1)(r+s-1)\neq 0$.

Now, we shall give some special equations for Horadam 3PGQ in the following Theorem 3.8 and Theorem 3.9:

Theorem 3.8. *Let $\mathcal{Q}_n$ be the nth Horadam 3PGQ. For all $n \geq 0$, the following properties hold:*

(a) $\mathcal{Q}_n-\mathcal{Q}_{n+1}e_1-\mathcal{Q}_{n+2}e_2-\mathcal{Q}_{n+3}e_3$
$=Q_n+\lambda_1\lambda_2 Q_{n+2}+\lambda_1\lambda_3 Q_{n+4}+\lambda_2\lambda_3 Q_{n+6}$,

(b) $\mathcal{Q}_n+\mathcal{Q}_{n+1}e_1+\mathcal{Q}_{n+2}e_2+\mathcal{Q}_{n+3}e_3$
$=2\mathcal{Q}_n-(Q_n+\lambda_1\lambda_2 Q_{n+2}+\lambda_1\lambda_3 Q_{n+4}+\lambda_2\lambda_3 Q_{n+6})$,

(c) $\mathcal{Q}_n+\overline{\mathcal{Q}}_n=2Q_n$,

(d) $\mathcal{Q}_n-\overline{\mathcal{Q}}_n=2\mathcal{Q}_n-2Q_n$,

(e) $\mathcal{Q}_n^2=2Q_n\mathcal{Q}_n-(Q_n^2+\lambda_1\lambda_2 Q_{n+1}^2+\lambda_1\lambda_3 Q_{n+2}^2+\lambda_2\lambda_3 Q_{n+3}^2)$,

(f) $\overline{\mathcal{Q}}_n^{\,2}=-2Q_n\mathcal{Q}_n-(-3Q_n^2-\lambda_1\lambda_2 Q_{n+1}^2-\lambda_1\lambda_3 Q_{n+2}^2-\lambda_2\lambda_3 Q_{n+3}^2)$.

Proof. (a) By using Table 1 and equation (4), we get:

$$\begin{aligned}\mathcal{Q}_n - \mathcal{Q}_{n+1}e_1 - \mathcal{Q}_{n+2}e_2 - \mathcal{Q}_{n+3}e_3 =& Q_n + Q_{n+1}e_1 + Q_{n+2}e_2 + Q_{n+3}e_3 \\ &- (Q_{n+1} + Q_{n+2}e_1 + Q_{n+3}e_2 + Q_{n+4}e_3)e_1 \\ &- (Q_{n+2} + Q_{n+3}e_1 + Q_{n+4}e_2 + Q_{n+5}e_3)e_2 \\ &- (Q_{n+3} + Q_{n+4}e_1 + Q_{n+5}e_2 + Q_{n+6}e_3)e_3 \\ =& Q_n + \lambda_1\lambda_2 Q_{n+2} + \lambda_1\lambda_3 Q_{n+4} + \lambda_2\lambda_3 Q_{n+6}.\end{aligned}$$

(b) With the help of Table 1 and equation (4), we achieve:

$$\begin{aligned}\mathcal{Q}_n + \mathcal{Q}_{n+1}e_1 + \mathcal{Q}_{n+2}e_2 + \mathcal{Q}_{n+3}e_3 =& Q_n + Q_{n+1}e_1 + Q_{n+2}e_2 + Q_{n+3}e_3 \\ &+ (Q_{n+1} + Q_{n+2}e_1 + Q_{n+3}e_2 + Q_{n+4}e_3)e_1 \\ &+ (Q_{n+2} + Q_{n+3}e_1 + Q_{n+4}e_2 + Q_{n+5}e_3)e_2 \\ &+ (Q_{n+3} + Q_{n+4}e_1 + Q_{n+5}e_2 + Q_{n+6}e_3)e_3 \\ =& 2\mathcal{Q}_n - (Q_n + \lambda_1\lambda_2 Q_{n+2} + \lambda_1\lambda_3 Q_{n+4} + \lambda_2\lambda_3 Q_{n+6}).\end{aligned}$$

(c) By means of equation (4) and conjugation of $\mathcal{Q}_n$, we get:

$$\begin{aligned}\mathcal{Q}_n + \overline{\mathcal{Q}}_n =& (Q_n + Q_{n+1}e_1 + Q_{n+2}e_2 + Q_{n+3}e_3) + (Q_n - Q_{n+1}e_1 - Q_{n+2}e_2 - Q_{n+3}e_3) \\ =& 2Q_n.\end{aligned}$$

(d) With the help of equation (4) and conjugation of $\mathcal{Q}_n$, we have:

$$\begin{aligned}\mathcal{Q}_n - \overline{\mathcal{Q}}_n =& (Q_n + Q_{n+1}e_1 + Q_{n+2}e_2 + Q_{n+3}e_3) - (Q_n - Q_{n+1}e_1 - Q_{n+2}e_2 - Q_{n+3}e_3) \\ =& 2\mathcal{Q}_n - 2Q_n.\end{aligned}$$

(e) By using equations (4) and (5), we have:

$$\begin{aligned}\mathcal{Q}_n^2 =& Q_n^2 - \lambda_1\lambda_2 Q_{n+1}^2 - \lambda_1\lambda_3 Q_{n+2}^2 - \lambda_2\lambda_3 Q_{n+3}^2 \\ &+ \left(2Q_nQ_{n+1} + \lambda_3\left(Q_{n+2}Q_{n+3} - Q_{n+3}Q_{n+2}\right)\right)e_1 \\ &+ \left(2Q_nQ_{n+2} + \lambda_2\left(Q_{n+3}Q_{n+1} - Q_{n+1}Q_{n+3}\right)\right)e_2 \\ &+ \left(2Q_nQ_{n+3} + \lambda_1\left(Q_{n+1}Q_{n+2} - Q_{n+2}Q_{n+1}\right)\right)e_3.\end{aligned}$$

Then, we obtain

$$\mathcal{Q}_n^2 = 2Q_n\mathcal{Q}_n - (Q_n^2 + \lambda_1\lambda_2 Q_{n+1}^2 + \lambda_1\lambda_3 Q_{n+2}^2 + \lambda_2\lambda_3 Q_{n+3}^2).$$

(f) By using equations (4), (5) and conjugate of $\mathcal{Q}_n$ we attain:

$$\begin{aligned}\overline{\mathcal{Q}}_n^2 =& Q_n^2 - \lambda_1\lambda_2 Q_{n+1}^2 - \lambda_1\lambda_3 Q_{n+2}^2 - \lambda_2\lambda_3 Q_{n+3}^2 \\ &- 2\left(Q_nQ_{n+1}e_1 + Q_nQ_{n+2}e_2 + Q_nQ_{n+3}e_3\right).\end{aligned}$$

Then, we obtain

$$\overline{\mathcal{Q}}_n^{\ 2} = -2Q_n\mathcal{Q}_n - (-3Q_n^2 + \lambda_1\lambda_2 Q_{n+1}^2 + \lambda_1\lambda_3 Q_{n+2}^2 + \lambda_2\lambda_3 Q_{n+3}^2).$$

We finished the proof. □

Theorem 3.9. *Let $\mathcal{Q}_n$ and $\mathcal{Q}_m$ be the nth and mth Horadam 3PGQ. For all $n, m \geq 0$, the following properties can be written:*

(a) $\mathcal{Q}_n\mathcal{Q}_m - \overline{\mathcal{Q}}_n\overline{\mathcal{Q}}_m = 2[(Q_n\mathcal{Q}_{m+1} + Q_{n+1}\mathcal{Q}_m)\, e_1 + (Q_n\mathcal{Q}_{m+2} + Q_{n+2}\mathcal{Q}_m)\, e_2 + (Q_n\mathcal{Q}_{m+3} + Q_{n+3}\mathcal{Q}_m)\, e_3]$,

(b) $\mathcal{Q}_n\mathcal{Q}_m + \overline{\mathcal{Q}}_n\overline{\mathcal{Q}}_m = 2[Q_nQ_m - \lambda_1\lambda_2Q_{n+1}Q_{m+1} - \lambda_1\lambda_3Q_{n+2}Q_{m+2} - \lambda_2\lambda_3Q_{n+3}Q_{m+3} + \lambda_3(Q_{n+2}Q_{m+3} - Q_{n+3}Q_{m+2})e_1 + \lambda_2(Q_{n+3}Q_{m+1} - Q_{n+1}Q_{m+3})e_2 + \lambda_1(Q_{n+1}Q_{m+2} - Q_{n+2}Q_{m+1})e_3]$,

(c) $\mathcal{Q}_n\overline{\mathcal{Q}}_m - \overline{\mathcal{Q}}_n\mathcal{Q}_m = 2[(Q_{n+1}Q_m - Q_nQ_{m+1})e_1 + (Q_{n+2}Q_m - Q_nQ_{m+2})e_2 + (Q_{n+3}Q_m - Q_nQ_{m+3})e_3]$,

(d) $\mathcal{Q}_n\overline{\mathcal{Q}}_m + \overline{\mathcal{Q}}_n\mathcal{Q}_m = 2[Q_nQ_m + \lambda_1\lambda_2Q_{n+1}Q_{m+1} + \lambda_1\lambda_3Q_{n+2}Q_{m+2} + \lambda_2\lambda_3Q_{n+3}Q_{m+3} + \lambda_3(Q_{n+3}Q_{m+2} - Q_{n+2}Q_{m+3})e_1 + \lambda_2(Q_{n+1}Q_{m+3} - Q_{n+3}Q_{m+1})e_2 + \lambda_1(Q_{n+2}Q_{m+1} - Q_{n+1}Q_{m+2})e_3]$.

Proof. (a) Via Table 1, equation (4), conjugation and multiplication properties, we get:

$$\begin{aligned}
&\mathcal{Q}_n\mathcal{Q}_m - \overline{\mathcal{Q}}_n\overline{\mathcal{Q}}_m \\
&= (Q_n + Q_{n+1}e_1 + Q_{n+2}e_2 + Q_{n+3}e_3)(Q_m + Q_{m+1}e_1 + Q_{m+2}e_2 + Q_{m+3}e_3) \\
&\quad - (Q_n - Q_{n+1}e_1 - Q_{n+2}e_2 - Q_{n+3}e_3)(Q_m - Q_{m+1}e_1 - Q_{m+2}e_2 - Q_{m+3}e_3) \\
&= 2\left[(Q_n\mathcal{Q}_{m+1} + Q_{n+1}\mathcal{Q}_m)e_1 + (Q_n\mathcal{Q}_{m+2} + Q_{n+2}\mathcal{Q}_m)e_2 + (Q_n\mathcal{Q}_{m+3} + Q_{n+3}\mathcal{Q}_m)e_3\right]
\end{aligned}$$

(b) With the help of Table 1, equation (4), conjugation and multiplication properties, we obtain:

$$\begin{aligned}
&\mathcal{Q}_n\mathcal{Q}_m + \overline{\mathcal{Q}}_n\overline{\mathcal{Q}}_m \\
&= (Q_n + Q_{n+1}e_1 + Q_{n+2}e_2 + Q_{n+3}e_3)(Q_m + Q_{m+1}e_1 + Q_{m+2}e_2 + Q_{m+3}e_3) \\
&\quad + (Q_n - Q_{n+1}e_1 - Q_{n+2}e_2 - Q_{n+3}e_3)(Q_m - Q_{m+1}e_1 - Q_{m+2}e_2 - Q_{m+3}e_3) \\
&= 2[Q_nQ_m - \lambda_1\lambda_2Q_{n+1}Q_{m+1} - \lambda_1\lambda_3Q_{n+2}Q_{m+2} - \lambda_2\lambda_3Q_{n+3}Q_{m+3} \\
&\quad + \lambda_3(Q_{n+2}Q_{m+3} - Q_{n+3}Q_{m+2})e_1 + \lambda_2(Q_{n+3}Q_{m+1} - Q_{n+1}Q_{m+3})e_2 \\
&\quad + \lambda_1(Q_{n+1}Q_{m+2} - Q_{n+2}Q_{m+1})e_3].
\end{aligned}$$

(c) By means of Table 1, equation (4), conjugation and multiplication properties, we get:

$$\begin{aligned}
&\mathcal{Q}_n\overline{\mathcal{Q}}_m - \overline{\mathcal{Q}}_n\mathcal{Q}_m \\
&= (Q_n + Q_{n+1}e_1 + Q_{n+2}e_2 + Q_{n+3}e_3)(Q_m - Q_{m+1}e_1 - Q_{m+2}e_2 - Q_{m+3}e_3) \\
&\quad - (Q_n - Q_{n+1}e_1 - Q_{n+2}e_2 - Q_{n+3}e_3)(Q_m + Q_{m+1}e_1 + Q_{m+2}e_2 + Q_{m+3}e_3) \\
&= 2\left[(Q_{n+1}Q_m - Q_nQ_{m+1})e_1 + (Q_{n+2}Q_m - Q_nQ_{m+2})e_2 \right.\\
&\quad \left. + (Q_{n+3}Q_m - Q_nQ_{m+3})e_3\right].
\end{aligned}$$

(d) By utilizing Table 1, equation (4), conjugation and multiplication properties, we have:

$$\begin{aligned}
&\mathcal{Q}_n\overline{\mathcal{Q}}_m + \overline{\mathcal{Q}}_n\mathcal{Q}_m \\
&= (Q_n + Q_{n+1}e_1 + Q_{n+2}e_2 + Q_{n+3}e_3)(Q_m - Q_{m+1}e_1 - Q_{m+2}e_2 - Q_{m+3}e_3) \\
&\quad + (Q_n - Q_{n+1}e_1 - Q_{n+2}e_2 - Q_{n+3}e_3)(Q_m + Q_{m+1}e_1 + Q_{m+2}e_2 + Q_{m+3}e_3) \\
&= 2\,[Q_nQ_m + \lambda_1\lambda_2Q_{n+1}Q_{m+1} + \lambda_1\lambda_3Q_{n+2}Q_{m+2} + \lambda_2\lambda_3Q_{n+3}Q_{m+3} \\
&\quad + \lambda_3(Q_{n+3}Q_{m+2} - Q_{n+2}Q_{m+3})e_1 + \lambda_2(Q_{n+1}Q_{m+3} - Q_{n+3}Q_{m+1})e_2 \\
&\quad +\lambda_1(Q_{n+2}Q_{m+1} - Q_{n+1}Q_{m+2})e_3]\,.
\end{aligned}$$

Hence, we get what is desired. □

Theorem 3.10. *Let $\mathcal{Q}_n$ be the nth Horadam 3PGQ. For every $n > 0$, the following matrix properties are expressed:*

$$\begin{pmatrix} \mathcal{Q}_{n+1} \\ \mathcal{Q}_n \end{pmatrix} = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} \mathcal{Q}_1 \\ \mathcal{Q}_0 \end{pmatrix}.$$

Proof. By utilizing mathematical induction, we show the proof easily, so we omit it. □

Theorem 3.11 (Cassini Identity)**.** *Let $\mathcal{Q}_n$ be the nth Horadam 3PGQ. The following Cassini identity is satisfied:*

$$\mathcal{Q}_{n-1}\mathcal{Q}_{n+1} - \mathcal{Q}_n^2 = \frac{\left(ABx_1^{n-1}x_2^{n-1}\right)(x_2\,\widetilde{x}_1\,\widetilde{x}_2 - x_1\,\widetilde{x}_2\,\widetilde{x}_1)}{x_1 - x_2}.$$

Proof. By using equation (4) and (7), we can complete the proof easily. □

Definition 3.12. *Let $\mathcal{Q}_n$ be the nth Horadam 3PGQ. For $N_{\mathcal{Q}_n} > 0$ and $\lambda_1\lambda_2Q_{n+1}^2+\lambda_1\lambda_3Q_{n+2}^2+\lambda_2\lambda_3Q_{n+3}^2 \neq 0$, polar representation of $\mathcal{Q}_n$ is as follows:*

$$\mathcal{Q}_n = \sqrt{N_{\mathcal{Q}_n}}\left(\cos\theta + \hat{\mathcal{Q}}_n\sin\theta\right), \tag{9}$$

where

$$\hat{\mathcal{Q}}_n = \frac{1}{\sqrt{\lambda_1\lambda_2Q_{n+1}^2 + \lambda_1\lambda_3Q_{n+2}^2 + \lambda_2\lambda_3Q_{n+3}^2}}(Q_{n+1}, Q_{n+2}, Q_{n+3})$$

and

$$\cos\theta = \frac{Q_n}{\sqrt{N_{\mathcal{Q}_n}}}, \quad \sin\theta = \sqrt{\frac{\lambda_1\lambda_2Q_{n+1}^2 + \lambda_1\lambda_3Q_{n+2}^2 + \lambda_2\lambda_3Q_{n+3}^2}{N_{\mathcal{Q}_n}}}.$$

Here $\hat{\mathcal{Q}}_n$ is called Horadam 3-parameter generalized unit vector.

Theorem 3.13. *Let $\mathcal{Q}_n$ be the nth Horadam 3PGQ. Then, the matrix representation of $\mathcal{Q}_n$ can be written as follows:*

$$\mathcal{M}_{\mathcal{Q}_n}=\begin{pmatrix} Q_n & -\lambda_1\lambda_2 Q_{n+1} & -\lambda_1\lambda_3 Q_{n+2} & -\lambda_2\lambda_3 Q_{n+3} \\ Q_{n+1} & Q_n & -\lambda_3 Q_{n+3} & \lambda_3 Q_{n+2} \\ Q_{n+2} & \lambda_2 Q_{n+3} & Q_n & -\lambda_2 Q_{n+1} \\ Q_{n+3} & -\lambda_1 Q_{n+2} & \lambda_1 Q_{n+1} & Q_n \end{pmatrix}. \tag{10}$$

Here, the matrix $\mathcal{M}_{\mathcal{Q}_n}$ is called the fundamental matrix for Horadam 3PGQs.

Proof. By multiplying $\mathcal{Q}_n = Q_n+Q_{n+1}e_1+Q_{n+2}e_2+Q_{n+3}e_3$ with $1, e_1, e_2, e_3$ from the left side and using Table 1, we obtain:

$$\begin{aligned} \mathcal{Q}_n 1 &= Q_n + Q_{n+1}e_1 + Q_{n+2}e_2 + Q_{n+3}e_3, \\ \mathcal{Q}_n e_1 &= -\lambda_1\lambda_2 Q_{n+1} + Q_n e_1 + \lambda_2 Q_{n+3}e_2 - \lambda_1 Q_{n+2}e_3, \\ \mathcal{Q}_n e_2 &= -\lambda_1\lambda_3 Q_{n+2} - \lambda_3 Q_{n+3}e_1 + Q_n e_2 + \lambda_1 Q_{n+1}e_3, \\ \mathcal{Q}_n e_3 &= -\lambda_2\lambda_3 Q_{n+3} + \lambda_3 Q_{n+2}e_1 - \lambda_2 Q_{n+1}e_2 + Q_n e_3. \end{aligned}$$

Then, writing the coefficients of $\{1, e_1, e_2, e_3\}$ of the above equations as columns gives the matrix in equation (10). □

According to the values of $\lambda_{i\in\{1,2,3\}}$, the matrix $\mathcal{M}_{\mathcal{Q}_n}$ can be classified. For $\lambda_1 = 1, \lambda_2, \lambda_3 \in \mathbb{R}$, the fundamental matrix for Horadam 2PGQ is obtained. For $\lambda_1 = \lambda_2 = 1, \lambda_3 = -1$, then the fundamental matrix for Horadam split quaternions is obtained. Also, for $\lambda_1 = \lambda_2 = \lambda_3 = 1$, then the fundamental matrix for Horadam Hamilton quaternions is obtained.

Remark 3.14. *Let $\mathcal{Q}_n$ and $\mathcal{Q}_m$ be Horadam 3PGQs. Then, the following can be given:*

$$\mathcal{M}_{\mathcal{Q}_n}\left(\mathcal{Q}_m^*\right)^T = \mathcal{M}_{\mathcal{Q}_m}\left(\mathcal{Q}_n^*\right)^T = (\mathcal{Q}_n\mathcal{Q}_m)^*,$$

where the superscript * represents column matrix forms. Hence, here $\mathcal{Q}_n^* = \begin{pmatrix} Q_n & Q_{n+1} & Q_{n+2} & Q_{n+3} \end{pmatrix}^T$, $\mathcal{Q}_m^* = \begin{pmatrix} Q_m & Q_{m+1} & Q_{m+2} & Q_{m+3} \end{pmatrix}^T$ *and*

$$(\mathcal{Q}_n\mathcal{Q}_m)^* = \begin{pmatrix} Q_nQ_m - \lambda_1\lambda_2 Q_{n+1}Q_{m+1} - \lambda_1\lambda_3 Q_{n+2}Q_{m+2} - \lambda_2\lambda_3 Q_{n+3}Q_{m+3} \\ Q_nQ_{m+1} + Q_{n+1}Q_m + \lambda_3 Q_{n+2}Q_{m+3} - \lambda_3 Q_{n+3}Q_{m+2} \\ Q_nQ_{m+2} + Q_{n+2}Q_m - \lambda_2 Q_{n+1}Q_{m+3} + \lambda_2 Q_{n+3}T_{m+1} \\ Q_nQ_{m+3} + Q_nQ_{m+3} + \lambda_1 Q_{n+1}Q_{n+2} - \lambda_1 Q_{n+2}Q_{m+1} \end{pmatrix}.$$

Thanks to the Şentürk and Ünal [49], we can obtain the following definition:

Definition 3.15. *Let $\mathcal{Q}_n$ be the nth Horadam 3PGQ. For all $n \geq 0$, the following mathematical equations are satisfied:*

✳ *The determinant of $\mathcal{M}_{\mathcal{Q}_n}$:* $\det\left(\mathcal{M}_{\mathcal{Q}_n}\right) = N_{\mathcal{Q}_n}^2$.
✳ *The characteristic polynomial of $\mathcal{M}_{\mathcal{Q}_n}$:*

$$P_{\mathcal{M}_{\mathcal{Q}_n}}(t) = \left(t^2 - 2tQ_n + Q_n^2 + \lambda_1\lambda_2 Q_{n+1}^2 + \lambda_1\lambda_3 Q_{n+2}^2 + \lambda_2\lambda_3 Q_{n+3}^2\right)^2.$$

$\ast$ *The characteristic equation of* $\mathcal{M}_{\mathcal{Q}_n}$*:*

$\det\left(\mathcal{M}_{\mathcal{Q}_n} - tI_4\right) = 0$

$\Leftrightarrow P_{\mathcal{M}_{\mathcal{Q}_n}}(t) = \left(t^2 - 2tQ_n + Q_n^2 + \lambda_1\lambda_2 Q_{n+1}^2 + \lambda_1\lambda_3 Q_{n+2}^2 + \lambda_2\lambda_3 Q_{n+3}^2\right)^2 = 0.$

$\ast$ *The eigenvalues of* $\mathcal{M}_{\mathcal{Q}_n}$*:*

$$\mathcal{T}_{1,2} = Q_n + \sqrt{-\lambda_1\lambda_2 Q_{n+1}^2 - \lambda_1\lambda_3 Q_{n+2}^2 - \lambda_2\lambda_3 Q_{n+3}^2},$$
$$\mathcal{T}_{3,4} = Q_n - \sqrt{-\lambda_1\lambda_2 Q_{n+1}^2 - \lambda_1\lambda_3 Q_{n+2}^2 - \lambda_2\lambda_3 Q_{n+3}^2}.$$

$\ast$ *Multiplication of the eigenvalues of* $\mathcal{M}_{\mathcal{Q}_n}$*:*

$$\mathcal{T}_{1,2}\mathcal{T}_{3,4} = Q_n^2 + \lambda_1\lambda_2 Q_{n+1}^2 + \lambda_1\lambda_3 Q_{n+2}^2 + \lambda_2\lambda_3 Q_{n+3}^2 = N_{\mathcal{Q}_n}.$$

$\ast$ *The eigenvectors corresponding to the eigenvalue* $\mathcal{T}_{1,2}$ *of* $\mathcal{M}_{\mathcal{Q}_n}$*:*

$$\begin{pmatrix} \frac{\lambda_1 Q_{n+2}\sqrt{-\lambda_1\lambda_2 Q_{n+1}^2 - \lambda_1\lambda_3 Q_{n+2}^2 - \lambda_2\lambda_3 Q_{n+3}^2} - \lambda_1\lambda_2 Q_{n+1}Q_{n+3}}{\lambda_1 Q_{n+2}^2 + \lambda_2 Q_{n+3}^2} \\ \frac{Q_{n+3}\sqrt{-\lambda_1\lambda_2 Q_{n+1}^2 - \lambda_1\lambda_3 Q_{n+2}^2 - \lambda_2\lambda_3 Q_{n+3}^2} + \lambda_1 Q_{n+1}Q_{n+2}}{\lambda_1 Q_{n+2}^2 + \lambda_2 Q_{n+3}^2} \\ 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{\lambda_2 Q_{n+3}\sqrt{-\lambda_1\lambda_2 Q_{n+1}^2 - \lambda_1\lambda_3 Q_{n+2}^2 - \lambda_2\lambda_3 Q_{n+3}^2} + \lambda_1\lambda_2 Q_{n+1}Q_{n+2}}{\lambda_1 Q_{n+2}^2 + \lambda_2 Q_{n+3}^2} \\ -\frac{Q_{n+2}\sqrt{-\lambda_1\lambda_2 Q_{n+1}^2 - \lambda_1\lambda_3 Q_{n+2}^2 - \lambda_2\lambda_3 Q_{n+3}^2} - \lambda_2 Q_{n+1}Q_{n+3}}{\lambda_1 Q_{n+2}^2 + \lambda_2 Q_{n+3}^2} \\ 0 \\ 1 \end{pmatrix}.$$

$\ast$ *The eigenvectors corresponding to the eigenvalue* $\mathcal{T}_{3,4}$ *of* $\mathcal{M}_{\mathcal{Q}_n}$*:*

$$\begin{pmatrix} \frac{\lambda_1 Q_{n+2}\sqrt{-\lambda_1\lambda_2 Q_{n+1}^2 - \lambda_1\lambda_3 Q_{n+2}^2 - \lambda_2\lambda_3 Q_{n+3}^2} + \lambda_1\lambda_2 Q_{n+1}Q_{n+3}}{\lambda_1 Q_{n+2}^2 + \lambda_2 Q_{n+3}^2} \\ -\frac{Q_{n+3}\sqrt{-\lambda_1\lambda_2 Q_{n+1}^2 - \lambda_1\lambda_3 Q_{n+2}^2 - \lambda_2\lambda_3 Q_{n+3}^2} - \lambda_1 Q_{n+1}Q_{n+2}}{\lambda_1 Q_{n+2}^2 + \lambda_2 Q_{n+3}^2} \\ 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} -\frac{\lambda_2 Q_{n+3}\sqrt{-\lambda_1\lambda_2 Q_{n+1}^2 - \lambda_1\lambda_3 Q_{n+2}^2 - \lambda_2\lambda_3 Q_{n+3}^2} - \lambda_1\lambda_2 Q_{n+1}Q_{n+2}}{\lambda_1 Q_{n+2}^2 + \lambda_2 Q_{n+3}^2} \\ \frac{Q_{n+2}\sqrt{-\lambda_1\lambda_2 Q_{n+1}^2 - \lambda_1\lambda_3 Q_{n+2}^2 - \lambda_2\lambda_3 Q_{n+3}^2} + \lambda_2 Q_{n+1}Q_{n+3}}{\lambda_1 Q_{n+2}^2 + \lambda_2 Q_{n+3}^2} \\ 0 \\ 1 \end{pmatrix}.$$

Example 3.16. *Let $\mathcal{J}_4$ be the 4th Jacobsthal 3PGQ.*

✳ *According to equation (9), polar representation of $\mathcal{J}_4$ is as follows:*

$$\mathcal{J}_4 = \sqrt{25 + 121\lambda_1\lambda_2 + 441\lambda_1\lambda_3 + 1849\lambda_2\lambda_3}\left(\cos\theta + \hat{\mathcal{J}}_4 \sin\theta\right).$$

✳ *Moreover, Jacobsthal 3-parameter generalized unit vector is written as:*

$$\hat{\mathcal{J}}_4 = \frac{(5, 21, 43)}{\sqrt{121\lambda_1\lambda_2 + 441\lambda_1\lambda_3 + 1849\lambda_2\lambda_3}},$$

where

$$\begin{cases} \cos\theta = \dfrac{5}{\sqrt{25 + 121\lambda_1\lambda_2 + 441\lambda_1\lambda_3 + 1849\lambda_2\lambda_3}}, \\ \sin\theta = \sqrt{\dfrac{121\lambda_1\lambda_2 + 441\lambda_1\lambda_3 + 1849\lambda_2\lambda_3}{25 + 121\lambda_1\lambda_2 + 441\lambda_1\lambda_3 + 1849\lambda_2\lambda_3}}. \end{cases}$$

✳ *Also, the following matrix can be constructed as follows:*

$$\mathcal{M}_{\mathcal{J}_4} = \begin{pmatrix} 5 & -11\lambda_1\lambda_2 & -21\lambda_1\lambda_3 & -43\lambda_2\lambda_3 \\ 11 & 5 & -43\lambda_3 & 21\lambda_3 \\ 21 & 43\lambda_2 & 5 & -11\lambda_2 \\ 43 & -21\lambda_1 & 11\lambda_1 & 5 \end{pmatrix}.$$

✳ *The determinant of $\mathcal{M}_{\mathcal{J}_4}$ is written as:*

$$\det\left(\mathcal{M}_{\mathcal{J}_4}\right) = \left(25 + 121\lambda_1\lambda_2 + 441\lambda_1\lambda_3 + 1849\lambda_2\lambda_3\right)^2 = \left(N_{\mathcal{J}_4}\right)^2.$$

✳ *The characteristic polynomial of $\mathcal{M}_{\mathcal{J}_4}$ is given:*

$$P_{\mathcal{J}_4}(t) = \left(t^2 - 10t + 25 + 121\lambda_1\lambda_2 + 441\lambda_1\lambda_3 + 1849\lambda_2\lambda_3\right)^2.$$

✳ *The eigenvalues of $\mathcal{M}_{\mathcal{J}_4}$ are determined:*

$$\mathcal{T}_{1,2} = 5 + \sqrt{-121\lambda_1\lambda_2 - 441\lambda_1\lambda_3 - 1849\lambda_2\lambda_3}$$

and

$$\mathcal{T}_{3,4} = 5 - \sqrt{-121\lambda_1\lambda_2 - 441\lambda_1\lambda_3 - 1849\lambda_2\lambda_3}.$$

✳ *The eigenvectors corresponding to $\mathcal{T}_{1,2}$ are expressed:*

$$\begin{pmatrix} -\frac{\lambda_1\left(473\lambda_2+21\sqrt{-121\lambda_1\lambda_2-441\lambda_1\lambda_3-1849\lambda_2\lambda_3}\right)}{441\lambda_1+1849\lambda_2} & -\frac{-231\lambda_1+43\sqrt{-121\lambda_1\lambda_2-441\lambda_1\lambda_3-1849\lambda_2\lambda_3}}{441\lambda_1+1849\lambda_2} & 1 & 0 \end{pmatrix}^T$$

and

$$\begin{pmatrix} \frac{\lambda_2\left(231\lambda_1+43\sqrt{-121\lambda_1\lambda_2-441\lambda_1\lambda_3-1849\lambda_2\lambda_3}\right)}{441\lambda_1+1849\lambda_2} & -\frac{-253\lambda_2-21\sqrt{-121\lambda_1\lambda_2-441\lambda_1\lambda_3-1849\lambda_2\lambda_3}}{441\lambda_1+1849\lambda_2} & 0 & 1 \end{pmatrix}^T.$$

✳ *The eigenvectors corresponding to $\mathcal{T}_{3,4}$ are obtained:*

$$\begin{pmatrix} \frac{473\lambda_1\lambda_2+21\lambda_1\sqrt{-121\lambda_1\lambda_2-441\lambda_1\lambda_3-1849\lambda_2\lambda_3}}{441\lambda_1+1849\lambda_2} & -\frac{-231\lambda_2-43\sqrt{-121\lambda_1\lambda_2-441\lambda_1\lambda_3-1849\lambda_2\lambda_3}}{441\lambda_1+1849\lambda_2} & 1 & 0 \end{pmatrix}^T$$

and

$$\begin{pmatrix} -\frac{-231\lambda_1\lambda_2+43\lambda_2\sqrt{-121\lambda_1\lambda_2-441\lambda_1\lambda_3-1849\lambda_2\lambda_3}}{441\lambda_1+1849\lambda_2} & -\frac{-473\lambda_2-21\sqrt{-121\lambda_1\lambda_2-441\lambda_1\lambda_3-1849\lambda_2\lambda_3}}{441\lambda_1+1849\lambda_2} & 0 & 1 \end{pmatrix}^T.$$

✳ *The multiplication of the eigenvalues of $\mathcal{M}_{\mathcal{J}_4}$ is given:*

$$\mathcal{T}_{1,2}\mathcal{T}_{3,4} = 25 + 121\lambda_1\lambda_2 + 441\lambda_1\lambda_3 + 1849\lambda_2\lambda_3.$$

4. Conclusion

In this study, we determine Horadam 3PGQ by taking the coefficients of 3PGQ as Horadam numbers and also examine some special cases of them. Also, we obtain Binet formulas, generating function, exponential generating function, Poisson generating function, sum formulas, Cassini identity, polar representation, and matrix equation. Then, we get the determinant, characteristic polynomial, characteristic equation, eigenvalues, and eigenvectors for the matrix representation of Horadam 3PGQ.

In our future study, we plan to introduce the 3PGQ with generalized Tribonacci number components, as the choice of the special recurrence sequence type can be modified.

References

[1] M. Akyiğit, H. H. Kösal, and M. Tosun, *Fibonacci quaternions.* Adv. Appl. Clifford Algebras **23** (2013), no. 3, 535–545.

[2] M. Akyiğit, H. H. Kösal, and M. Tosun, *Generalized Fibonacci quaternions*, Adv. Appl. Clifford Algebras **24** (2014), no. 3, 631–641.

[3] G. Bilgici, *Fibonacci 3-parameter generalized quaternions*, Avrupa Bilim ve Teknoloji Dergisi **41** (2022), 357–361.

[4] G. Bilgici, Ü. Tokeşer, and Z. Ünal, *k-Fibonacci and k-Lucas generalized quaternions*, Konuralp Journal of Mathematics **5** (2017), no. 2, 102–113.

[5] G. Bilgici, *Jacobsthal 3-parameter generalized quaternions*, Advanced and Contemporary Studied in Natural Science and Mathematics, 46, 2024.

[6] D. Bród, *On some properties of split Horadam quaternions*, Acta Universitatis Sapientiae Mathematica **12** (2020), no. 2, 260–271.

[7] P. Catarino, *A note on $h(x)$-Fibonacci quaternion polynomials*, Chaos, Solitons & Fractals **77** (2015), 1–5.

[8] P. Catarino, *The modified Pell and modified k-Pell quaternions and octonions*, Adv. Appl. Clifford Algebras. **26** (2016), 577–590.

[9] R. Chaker and A. Boua, *Some results on generalized quaternions algebra with generalized Fibonacci quaternions*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **LXIX** (2023), no. 2, 233–246.

[10] C. B. Çimen and A. İpek, *On Pell quaternions and Pell-Lucas quaternions*, Adv. Appl. Clifford Algebras. **26** (2016), 39–51.

[11] J. Cockle, *On systems of algebra involving more than one imaginary; and on equations of the fifth degree*, Philosophical Magazine **35** (1849), no. 238, 434–437.

[12] L. E. Dickson, *On the theory of numbers and generalized quaternions*, Amer. J. Math. **46** (1924), 1–16.

[13] C. Flaut and D. Savin, *Quaternion algebras and generalized Fibonacci-Lucas quaternions*, Adv. Appl. Clifford Algebras. **25** (2015), 853–862.

[14] L. W. Griffiths, *Generalized quaternion algebras and the theory of numbers*, Amer. J. Math. **50** (1928), 303–314.

[15] S. Halıcı and A. Karataş, *On a generalization for Fibonacci quaternions*, Chaos, Solitons & Fractals **98** (2017), 178–182.

[16] S. Halıcı, *On Fibonacci quaternions*, Adv. Appl. Clifford Algebras. **22** (2012), no. 2, 321–327.

[17] W. R. Hamilton, *On quaternions; or on a new system of imaginaries in algebra*, Philos. Mag. **25** (1844), no. 3, 489–495.

[18] W. R. Hamilton, *Lectures on Quaternions*, Hodges and Smith, Dublin, 1853.

[19] W. R. Hamilton, *Elements of Quaternions*, Chelsea Pub. Com. New York, 1969.

[20] P. Haukkane, *A note on Horadam's sequence*, Fibonacci Quart. **40** (2002), no. 4, 358–361.

[21] A. F. Horadam, *Complex Fibonacci numbers and Fibonacci quaternions*, Am. Math. Month. **70** (1963), no. 3, 289–291.

[22] A. F. Horadam, *Generating functions for powers of a certain generalized sequence of numbers*, Duke Math. J. **32** (1965), 437–446.

[23] A. F. Horadam, *A generalized Fibonacci sequence*, Am. Math. Month. **68** (1961), no. 5, 455–459.

[24] A. F. Horadam, *Special properties of the sequence* $w_n(a,b;p,q)$, Fibonacci Quart. **5** (1967), no. 5, 424–434.

[25] A. F. Horadam, *Quaternion recurrence relations*, Ulam Quart. **2** (1993), no. 2, 23–33.

[26] A. F. Horadam, *Pell identities*, Fibonacci Quart. **9** (1971), no. 3, 245–263.

[27] A. F. Horadam, *Jacobsthal representation numbers*, Fibonacci Quart. **34** (1996), 40–54.

[28] A. F. Horadam, *Jacobsthal and Pell curves*, Fibonacci Quart. **26** (1988), 77–83.

[29] A. F. Horadam, *Basic properties of a certain generalized sequence of numbers*, Fibonacci Quart. **3** (1965), no. 3, 161–176.

[30] M. R. Iyer, *A note on Fibonacci quaternions*, Fibonacci Quart. **7** (1969), no. 3, 225–229.

[31] M. Jafari and Y. Yaylı, *Generalized quaternions and rotation in 3-space* $E^3_{\alpha\beta}$, TWMS J. Pure Appl. Math. **6** (2015), no. 2, 224–232.

[32] M. Jafari and Y. Yaylı, *Generalized quaternions and their algebraic properties*, Commun. Fac. Sci. Ank. Series A1 **64**(2015), no. 1, 15–27.

[33] M. Jafari, *Genelleştirilmiş Hamilton operatörleri ve Lie grupları*, Ankara University, PhD Thesis, 2012.

[34] D. Kalman, *Generalized Fibonacci numbers by matrix methods*, Fibonacci Quart. **20** (1982), no. 1, 73–76.

[35] C. Kızılateş and İ. Y. Kibar, *On 3-parameter quaternions with higher order generalized Fibonacci numbers components*, The Journal of Analysis (2024), 1–14.

[36] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons, Canada, 2001.

[37] P. Larcombe, O. Bagdasar, and E. Fennessey, *Horadam sequences: A survey*, Bull ICA. **67** (2013), 49–72.

[38] A. B. Mamagani and M. Jafari, *On properties of generalized quaternion algebra*. Journal of Novel Applied Sciences **2** (2013), no. 12, 683–689.

[39] H. Mortazaasl and M. Jafari, *A study of semi-quaternions algebra in semi-Euclidean 4-space*, Mathematical Science and Applications E-Notes **2** (2013), no. 1, 20–27.

[40] E. Polatlı, C. Kızılateş, and S. Kesim, *On split k-Fibonacci and k-Lucas quaternions*, Adv. Appl. Clifford Algebr. **26** (2016), no. 1, 353–362.

[41] E. Polatlı and S. Kesim, *On quaternions with generalized Fibonacci and Lucas number components*, Adv. Diff. Eqs. **169**, (2015), 1–8.

[42] E. Polatlı, *A generalization of Fibonacci and Lucas quaternions*, Adv. Appl. Clifford Algebras **26** (2016), no. 2, 719–730.

[43] H. Pottmann and J. Wallner, *Computational Line Geometry*, Springer-Verlag Berlin Heidelberg, New York, 2001.

[44] B. Rosenfeld, *Geometry of Lie Groups*, Kluwer Academic Publishers, 1997.

[45] D. Savin, C. Flaut, and C. Ciobanu, *Some properties of the symbol algebras*, Carpathian J. Math. **25** (2009), no. 2, 239–245.

[46] T. D. Şentürk, A. Daşdemir, G. Bilgici, and Z. Ünal, *On unrestricted Horadam generalized quaternions*, Utilitas Mathematica **110**, (2019), 89–98.

[47] T. D. Şentürk, G. Bilgici, A. Daşdemir, and Z. Ünal, *A study on Horadam hybrid numbers*, Turk. J. Math. **44** (2020), no. 4, Article 10, 1212–1221.

[48] T. D. Şentürk, *3-Parametreli Genelleştirilmiş Kuaterniyonlar*, Kastamonu University, PhD Thesis, 2020.

[49] T. D. Şentürk and Z. Ünal, 3-parameter generalized quaternions, Computational Methods and Function Theory **22** (2022), 575–608.

[50] N. J. A. Sloane, The online encyclopedia of integer sequences, 1964, available online at `http//oeis.org/`.

[51] Y. Soykan, *On generalized* (r,s)*-numbers*, Int. J. Adv. Appl. Math. Mech. **8** (2020), no. 1, 1–14.

[52] Y. Soykan, *On summing formulas for Horadam numbers*. Asian Journal of Advanced Research and Reports **8** (2020), no. 1, 45–61.

[53] Y. Soykan, *On summing formulas for generalized Fibonacci and Gaussian generalized Fibonacci numbers*, Advances in Research **20** (2019), no. 2, 1–15.

[54] M. N. S. Swamy, *On generalized Fibonacci quaternions*. Fibonacci Quart. **11** (1973), no. 5, 547–549.

[55] A. Szynal-Liana and I. Włoch, *A note on Jacobsthal quaternions*, Adv. Appl. Clifford Algebras **26** (2016), no. 1, 441–447.

[56] E. Tan, S. Yılmaz, and M. Şahin, *On a new generalization of Fibonacci quaternions*, Chaos, Solitons & Fractals **82** (2016), 1–4.

[57] Ü. Tokeşer, Z. Ünal, and G. Bilgici, *Split Pell and Pell-Lucas quaternions*, Adv. Appl. Clifford Algebras **27** (2017), no. 2, 1881–1893.

[58] T. Unger and N. Markin, *Quadratic forms and space-time block codes from generalized quaternion and biquaternion algebras*, IEEE Trans. Inform. Theory **57** (2011), no. 9, 6148–6156.

[59] Z. Ünal, *Dual Fibonacci ve Lucas* 3*-parametreli genelleştirilmiş kuaterniyonları*, In 2^{nd} International Conference on Scientific and Academic Research **1** (2023), 34–36.

[60] S. Yüce and F. Torunbalcı Aydın, *Generalized dual Fibonacci quaternions*, Applied Mathematics E-Notes **16** (2016), no. 30, 276–289.

[61] S. Yüce and F. Torunbalcı Aydın, *A new aspect of dual Fibonacci quaternions*, Adv. Appl. Clifford Algebras **26** (2016), 873–884.

Zehra İşbilir
Department of Mathematics,
Faculty of Arts and Sciences,
Düzce University,
Düzce, 81620, Türkiye.
E-mail: zehraisbilir@duzce.edu.tr

Nurten Gürses
Department of Mathematics,
Faculty of Arts and Sciences,
Yildiz Technical University,
İstanbul, 34220, Türkiye.
E-mail: nbayrak@yildiz.edu.tr