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RICCI SOLITONS AND RICCI BI-CONFORMAL VECTOR FIELDS ON THE MODEL SPACE Sol_1^4

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Abstract. In the present paper, we classify the Ricci solitons and the Ricci bi-conformal vector fields on the model space Sol_1^4. Also, we show that which of them are gradient vector fields and Killing vector fields.

1. Introduction

Conformal vector fields have a fundamental role in geometry and physics. In geometry, a conformal vector field is a vector field that preserves angles between curves. Conformal vector fields also arise naturally in the study of Einstein's theory of general relativity, where they correspond to symmetries of spacetime.

A conformal vector field is a smooth vector field X on a Riemannian manifold (M, g) if a smooth function like f that named a potential function, exists on M that satisfies $\mathcal{L}_X g = fg$, where $\mathcal{L}_X g$ is the Lie derivative of g with respect X. So if the potential function $f = 0$, X is a Killing vector field. We say that X is a gradient conformal vector field, if X is a gradient of a smooth function. A conformal vector field explain completely in [6, 7]. If the following equations hold for some smooth functions α and β and any vector fields Y, Z, then the vector field X is called a Ricci bi-conformal vector field:

$$(\mathcal{L}_X g)(Y, Z) = \alpha g(Y, Z) + \beta S(Y, Z), \tag{1}$$

and

$$(\mathcal{L}_X S)(Y, Z) = \alpha S(Y, Z) + \beta g(Y, Z), \tag{2}$$

where S is the Ricci tensor of M. Note, that Garcia-Parrado and Senovilla introduced bi-conformal vector fields [10], then De et al. defined Ricci bi-conformal vector fields in [5]. In [1, 2, 3] have been studied Ricci bi-conformal vector fields on Siklos spacetimes, homogeneous Gödel-type spacetimes, and Lorentzian five-dimensional two-step nilpotent Lie groups, respectively.

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One of the most important and attractive topics in physics and geometry is study of the Ricci solitons, that they were introduced by Hamilton [12], are natural generalization of Einstein metrics. Its applications were investigated in various fields of sciences such as physics [11], biology, chemistry [13], and economics [14]. On a pseudo-Riemannian manifold (M, g), it is defined by

$$\mathcal{L}_X g + S = \lambda g, \tag{3}$$

where X is a smooth vector field on M, and λ is a real number [4]. See [19] for further reading.

If the group of isometries of (M, g) acts transitivity on M, the connected pseudo-Riemannian manifold (M, g) is named to be a homogeneous. A Thurston geometry (G, X) is a homogeneous space where X is connected and simply connected, suppose G be a group and it acts transitively on X with compact point stabilizers such that G is not contained in any larger group of diffeomorphisms of X, and there is at least one compact manifold modeled on (G, X). Thurston geometry is a subset of Riemannian homogeneous spaces, that studied in dimension three for three-manifolds. So the possible Riemannian structures of compact orientable three-manifolds are similar to the uniformization theorem for surfaces that are compact and orientable. We can decompose any three-manifold into pieces and each of them admits a Riemannian metric locally isometric to one of eight three-dimensional model spaces, the Thurston geometries $\mathbb{R}^3, \mathbb{S}^3, \mathbb{H}^3, \mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, \tilde{SL}(2, \mathbb{R}), Nil^3$ and Sol^3. Eight three-dimensional Thurston spaces explain completely in [15, 16]. The model space (Sol_1^4, g) is one of the four-dimensional Thurston geometries. Filipkiewicz in [9] listed 19 types of Thurston geometries in dimension four. According to Wall [17], the space (Sol_1^4, g) belongs to 14 spaces among these model spaces that admit complex structure compatible with the geometric structure, for more information study [8].

The paper is organized as follows: In Section 2, we recall some necessary concepts on (Sol_1^4, g) which be used throughout this paper. In Section 3, we calculate the Ricci solitons and we talk about a theorem of this equation on this space and we discuss about the existence of Ricci solitons, also, in Section 4, we investigate the Ricci bi-conformal vector fields on (Sol_1^4, g) spaces and we prove which of them are gradient vector fields and Killing vector fields.

2. The model space Sol_1^4

2.1. Lie Group

The primary manifold of the model space Sol_1^4 is $\mathbb{R}^4(x, y, z, t)$ with the group operation

$$(x_1, y_1, z_1, t_1) * (x_2, y_2, z_2, t_2) = (x_1 + e^{t_1} x_2, y_1 + e^{-t_1} y_2, z_1 + z_2 + e^{-t_1} x_1 y_2, t_1 + t_2).$$

This operation is deduced from the matrix multiplications by the following definition

$$(x,y,z,t) := \begin{pmatrix} 1 & 0 & e^{-t}x & z \\ 0 & e^t & 0 & x \\ 0 & 0 & e^{-t} & y \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The neutral element of (4) is $(0,0,0,0)$. The inverse element of (x,y,z,t) is given by

$$(x,y,z,t)^{-1} = (-e^{-t}x, -e^{t}y, -z+xy, -t). \tag{4}$$

2.2. Metric and Basis

Using the inverse translation (5), by pullback of coordinate defferentials,

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & -x & xy-z \\ 0 & e^{-t} & 0 & -e^{-t}x \\ 0 & 0 & e^t & -e^t y \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & e^{-t}(dx-xdt) & dz \\ 0 & e^t dt & 0 & dx \\ 0 & 0 & -e^{-t}dt & dy \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & e^{-t}dx & dz-xdy \\ 0 & dt & 0 & e^{-t}dx \\ 0 & 0 & -dt & e^t dy \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{5}$$

The left invariant Riemannian metric g of Sol_1^4 is obtained as follows

$$g = e^{-2t}dx^2 + e^{2t}dy^2 + (dz - xdy)^2 + dt^2, \tag{6}$$

Therefore, the metrically dual left invariant basis vector fields are considered as

$$e_1 = e^t\frac{\partial}{\partial x},\ e_2 = e^{-t}(\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}),\ e_3 = \frac{\partial}{\partial z},\ e_4 = \frac{\partial}{\partial t}. \tag{7}$$

So basis vector fields are satisfied the following brackets:

$$\begin{aligned} &[e_1,e_3] = [e_2,e_3] = [e_3,e_4] = 0,\ [e_1,e_2] = e_3,\ [e_1,e_4] = -e_1, \\ &[e_2,e_4] = e_2. \end{aligned}$$

The Levi-Civita connection of manifold (M,g) is shown by ∇. The curvature tensor R of (M,g) can be defined as follows

$$R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

and we define the Ricci tensor S by $S(X,Y) = tr(Z \to R(X,Z)Y)$. The non-zero components of Levi-Civita connection are calculated by

$$\nabla_{e_i}e_j = \begin{pmatrix} e_4 & \frac{1}{2}e_3 & -\frac{1}{2}e_2 & -e_1 \\ -\frac{1}{2}e_3 & -e_4 & \frac{1}{2}e_1 & e_2 \\ -\frac{1}{2}e_2 & \frac{1}{2}e_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and non-zero components of Ricci tensor is determined by

$$S = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}. \tag{8}$$

For any vector field $X = X^k e_k$ by $(\mathcal{L}_X g)(e_i, e_j) = g(\nabla_{e_i} X, e_j) + g(e_i, \nabla_{e_j} X)$ the Lie derivative of the metric g with respect to the vector field X (see [18]), is given by

$$\begin{aligned}
(\mathcal{L}_X g)_{11} &= -2X^4 + 2e_1X^1, \\
(\mathcal{L}_X g)_{12} &= e_1X^2 + e_2X^1, \\
(\mathcal{L}_X g)_{13} &= X^2 + e_1X^3 + e_3X^1, \\
(\mathcal{L}_X g)_{14} &= X^1 + e_1X^4 + e_4X^1, \\
(\mathcal{L}_X g)_{22} &= 2X^4 + 2e_2X^2, \\
(\mathcal{L}_X g)_{23} &= -X^1 + e_2X^3 + e_3X^2, \\
(\mathcal{L}_X g)_{24} &= -X^2 + e_2X^4 + e_4X^2, \\
(\mathcal{L}_X g)_{33} &= 2e_3X^3, \\
(\mathcal{L}_X g)_{34} &= e_3X^4 + e_4X^3, \\
(\mathcal{L}_X g)_{44} &= 2e_4X^4.
\end{aligned} \tag{9}$$

Further, using the formula $(\mathcal{L}_X S)(e_i, e_j) = X(S(e_i, e_j)) - S(\mathcal{L}_X e_i, e_j) - S(e_i, \mathcal{L}_X e_j)$ the Lie derivative of the Ricci tensor in direction X (see [18]), is determined by

$$\begin{aligned}
(\mathcal{L}_X S)_{11} &= X^4 - e_1X^1, \\
(\mathcal{L}_X S)_{12} &= -\frac{1}{2}e_2X^1 - \frac{1}{2}e_1X^2, \\
(\mathcal{L}_X S)_{13} &= \frac{1}{2}X^2 + \frac{1}{2}e_1X^3 - \frac{1}{2}e_3X^1, \\
(\mathcal{L}_X S)_{14} &= -2e_1X^4 - \frac{1}{2}X^1 - \frac{1}{2}e_4X^1, \\
(\mathcal{L}_X S)_{22} &= -e_2X^2 - X^4, \\
(\mathcal{L}_X S)_{23} &= \frac{1}{2}e_2X^3 - \frac{1}{2}X^1 - \frac{1}{2}e_3X^2, \\
(\mathcal{L}_X S)_{24} &= -2e_2X^4 + \frac{1}{2}X^2 - \frac{1}{2}e_4X^2, \\
(\mathcal{L}_X S)_{33} &= e_3X^3, \\
(\mathcal{L}_X S)_{34} &= -2e_3X^4 + \frac{1}{2}e_4X^3, \\
(\mathcal{L}_X S)_{44} &= -4e_4X^4.
\end{aligned} \tag{10}$$

3. Ricci solitons on the model space Sol_1^4

In this section, we solve the equation (3) on the model space Sol_1^4. Substituting (8) and (9) into (3), the following equations are obtained

$$2e_1X^1 - 2X^4 - \frac{1}{2} = \lambda, \tag{11}$$

$$2X^4 + 2e_2X^2 - \frac{1}{2} = \lambda, \tag{12}$$

$$2e_3X^3 + \frac{1}{2} = \lambda, \tag{13}$$

$$2e_4X^4 - 2 = \lambda, \tag{14}$$

$$e_2X^1 + e_1X^2 = 0, \tag{15}$$

$$X^2 + e_3X^1 + e_1X^3 = 0, \tag{16}$$

$$X^1 + e_4X^1 + e_1X^4 = 0, \tag{17}$$

$$-X^1 + e_3X^2 + e_2X^3 = 0, \tag{18}$$

$$-X^2 + e_4X^2 + e_2X^4 = 0, \tag{19}$$

$$e_4X^3 + e_3X^4 = 0. \tag{20}$$

By taking integral of the equation (14) yields

$$X^4 = \frac{\lambda+2}{2}t + F(x,y,z), \tag{21}$$

for some smooth function F. The following equation is deduced by integrating of the equation (11)

$$X^1 = (\frac{2\lambda+1}{4})e^{-t}x + (\frac{\lambda+2}{2})e^{-t}xt + e^{-t}\int F(x,y,z)dx + G(y,z,t), \tag{22}$$

for some smooth function G. Next, by taking integration of equation (15), arrived at

$$\begin{aligned} X^2 = &-e^{-2t}(\partial_y G(y,z,t)x + \frac{x^2}{2}\partial_z G(y,z,t)) \\ &-e^{-3t}\int\int \partial_y F(x,y,z)dxdx - e^{-3t}\int(x\int \partial_z F(x,y,z)dx)dx + K(y,z,t), \end{aligned} \tag{23}$$

for some smooth function K. Integrating of the equation (13), X^3 is deduced as

$$X^3 = (\frac{2\lambda-1}{4})z + L(x,y,t), \tag{24}$$

for some smooth function L. Substituting (7), (24), and (21) into (20), we obtain the following relation

$$\partial_t L(x,y,t) = -\partial_z F(x,y,z), \tag{25}$$

by derivation of the equation (25) with respect to t, we get

$$\partial_{tt} L(x, y, t) = 0, \tag{26}$$

also by derivation of the equation (25) with respect to z, we have

$$\partial_{zz} F(x, y, z) = 0, \tag{27}$$

then by taking integration of (26) and (27), the following relations are obtained

$$\begin{aligned} L(x, y, t) &= A(x, y)t + B(x, y), \\ F(x, y, z) &= C(x, y)z + D(x, y), \end{aligned} \tag{28}$$

for some smooth functions A, B, C, D. From equation (25), we get $C(x, y) = -A(x, y)$, so (28) can be rewritten as follow

$$F(x, y, z) = -A(x, y)z + D(x, y).$$

Consequently, by substituting (7), (21), and (22) in (17), we have

$$G(y, z, t) + \partial_t G(y, z, t) + e^t \partial_x F(x, y, z) + \frac{\lambda + 2}{2} e^{-t} x = 0, \tag{29}$$

by derivation of the equation (29) with respect to x, then derivation it with respect to t, $\lambda = -2$ is received. So (29) is considered as follows

$$G(y, z, t) + \partial_t G(y, z, t) + e^t \partial_x F(x, y, z) = 0. \tag{30}$$

Derivating the equations (30) with respect to x, we obtain

$$\partial_{xx} F(x, y, z) = 0, \tag{31}$$

therefore from (28) and (31), the following relations are deduced

$$\begin{aligned} C(x, y) &= A_1(y)x + A_2(y), \\ D(x, y) &= D_1(y)x + D_2(y), \end{aligned}$$

for some smooth functions A_1, A_2, D_1, and D_2. Also, from (30), $G(y, z, t)$ can be calculated as follows

$$G(y, z, t) = -\frac{e^t}{2}(-A_1(y)z + D_1(y)) + e^{-t}\phi(y, z),$$

for some smooth function ϕ. By substituting X^1, X^2, and X^3 in (16) and by differentiating with respect to x, the following relation is concluded

$$B(x, y) = B_1(y)x + B_2(y),$$

for some smooth functions B_1 and B_2. This implies that equation (16) is a polynomial with respect to x. Thus, the following equations are obtained

$$A_1(y) = 0,$$

$$K(y,z,t) + e^{-t}\partial_z\phi(y,z) + e^t B_1(y) = 0, \tag{32}$$

$$-e^{-t}A_2(y) + \frac{e^{-t}}{2}D_1^{'}(y) - e^{-3t}\partial_y\phi(y,z) = 0, \tag{33}$$

$$D_1^{'}(y) + 2A_2(y) = 0, \tag{34}$$

$$A_2^{'}(y) + D_2^{'}(y) + \partial_z\phi(y,z) = 0. \tag{35}$$

From (33), we have

$$\frac{1}{2}D_1^{'}(y) - A_2(y) = 0, \tag{36}$$

$$\partial_y\phi(y,z) = 0, \tag{37}$$

and (37) yield

$$\phi(y,z) = A_3(z), \tag{38}$$

for some smooth function A_3. Thus, (34) and (36) yield

$$D_1^{'}(y) = 0,$$

and by integrating it, we have

$$D_1(y) = b_2,$$

for some smooth constant b_2. Also, from (32) and (34), we get

$$A_2(y) = 0. \tag{39}$$

The equations (35) and (39) yield

$$D_2^{'}(y) + \partial_z\phi(y,z) = 0. \tag{40}$$

Now, from (38) and (40), we have

$$A_3(z) = b_1 z + b_6,$$

for some smooth constants b_1 and b_6. Therefore, we have

$$D_2(y) = -b_1 y + b_3,$$

for some smooth constant b_3. The equation (18) yields

$$b_1 = b_2 = 0,$$
$$B_1(y) = \frac{1}{2}y - b_3 y + b_4,$$
$$B_2(y) = b_6 y + b_5,$$

for some smooth constants b_4 and b_5. Using (21) and (23), we conclude that the equation (19) is valid. Therefore, from all of these obtained parameters, X^1, X^2, X^3, X^4 are listed as follows

$$X^1 = -\frac{3}{4}e^{-t}x - e^{-t}(b_3x - b_6),$$
$$X^2 = -e^t(\frac{1}{2}y - b_3y + b_4),$$
$$X^3 = -\frac{5}{4}z + (\frac{1}{2}y - b_3y + b_4)x + b_5,$$
$$X^4 = b_3.$$

But X^2 and X^4 not satisfied in (12). Therefore, we have the following theorem:

Theorem 3.1. *There is no Ricci soliton on* (Sol_1^4, g).

4. Ricci bi-conformal vector fields on the model space Sol_1^4

In this section, we solve the equation (1) and (2) on the model space Sol_1^4. Substituting (6), (8), and (9) into (1), the following system is obtained

$$2e_1X^1 - 2X^4 = \alpha - \frac{1}{2}\beta, \tag{41}$$
$$e_1X^2 + e_2X^1 = 0, \tag{42}$$
$$X^2 + e_1X^3 + e_3X^1 = 0, \tag{43}$$
$$X^1 + e_1X^4 + e_4X^1 = 0, \tag{44}$$
$$2X^4 + 2e_2X^2 = \alpha - \frac{1}{2}\beta, \tag{45}$$
$$-X^1 + e_2X^3 + e_3X^2 = 0, \tag{46}$$
$$-X^2 + e_2X^4 + e_4X^2 = 0, \tag{47}$$
$$2e_3X^3 = \alpha + \frac{1}{2}\beta, \tag{48}$$
$$e_3X^4 + e_4X^3 = 0, \tag{49}$$
$$2e_4X^4 = \alpha - 2\beta. \tag{50}$$

Also, substituting (6), (8), and (10) into (2), the following system is obtained

$$X^4 - e_1 X^1 = -\frac{\alpha}{2} + \beta, \tag{51}$$

$$-\frac{1}{2} e_1 X^2 - \frac{1}{2} e_2 X^1 = 0, \tag{52}$$

$$\frac{1}{2} X^2 + \frac{1}{2} e_1 X^3 - \frac{1}{2} e_3 X^1 = 0, \tag{53}$$

$$-2 e_1 X^4 - \frac{1}{2} X^1 - \frac{1}{2} e_4 X^1 = 0, \tag{54}$$

$$-e_2 X^2 - X^4 = -\frac{\alpha}{2} + \beta, \tag{55}$$

$$-\frac{1}{2} X^1 - \frac{1}{2} e_3 X^2 + \frac{1}{2} e_2 X^3 = 0, \tag{56}$$

$$\frac{1}{2} X^2 - 2 e_2 X^4 - \frac{1}{2} e_4 X^2 = 0, \tag{57}$$

$$e_3 X^3 = \frac{\alpha}{2} + \beta, \tag{58}$$

$$-2 e_3 X^4 + \frac{1}{2} e_4 X^3 = 0, \tag{59}$$

$$-4 e_4 X^4 = -2\alpha + \beta. \tag{60}$$

In following, we solve the above equations. From (41) and (51), we have

$$\begin{aligned} &\beta = 0, \\ &-X^4 + e_1 X^1 = \frac{\alpha}{2}, \end{aligned} \tag{61}$$

from equations (42) and (52), we get

$$e_1 X^2 + e_2 X^1 = 0, \tag{62}$$

and the equations (43) and (53) yields

$$e_3 X^1 = 0, \tag{63}$$

$$X^2 + e_1 X^3 = 0, \tag{64}$$

also, using equations (44) and (54), we deduce

$$e_1 X^4 = 0, \tag{65}$$

$$X^1 + e_4 X^1 = 0. \tag{66}$$

Now, (45) and (55) yields

$$X^4 + e_2 X^2 = \frac{\alpha}{2}. \tag{67}$$

From equations (46) and (56), we have

$$\begin{aligned} &e_3 X^2 = 0, \\ &-X^1 + e_2 X^3 = 0. \end{aligned} \tag{68}$$

Then we use the equations (47) and (57), so we have

$$e_2X^4 = 0, \quad -X^2 + e_4X^2 = 0. \tag{69}$$

From equations (48) and (58), we get

$$e_3X^3 = \frac{\alpha}{2}, \tag{70}$$

also, the equations (49) and (59) yields

$$e_4X^3 = 0, \tag{71}$$

$$e_3X^4 = 0. \tag{72}$$

Using equations (50) and (60), we get

$$e_4X^4 = \frac{\alpha}{2}. \tag{73}$$

From (65), (69), (71), and (72), we have

$$X^4 = F(t), \tag{74}$$

for some smooth function F. Therefore, from the equation (73), we obtain

$$\alpha = 2F'(t). \tag{75}$$

Substituting (74) and (75) into (61), we obtain

$$\partial_x X^1 = e^{-t}(F(t) + F'(t)). \tag{76}$$

Integrating of the equation (76), X^1 is deduced as

$$X^1 = e^{-t}(F(t) + F'(t))x + G(y, z, t), \tag{77}$$

for some smooth function G. By substituting (77) into (66), we get

$$F'(t) + F''(t) = 0, \quad G(y, z, t) + \partial_t G(y, z, t) = 0. \tag{78}$$

The equations (70) and (71) yields

$$F'(t) = 0,$$

thus

$$F(t) = a_1, \tag{79}$$

for some constant a_1. Therefore the equations (75) and (79) yields

$$\alpha = 0.$$

Integrating of the equations (70) and (71), X^3 is obtained as

$$X^3 = K(x, y), \tag{80}$$

for some smooth function K. From the equation (78)

$$G(y, z, t) = L(y, z)e^{-t}, \tag{81}$$

for some smooth function L. Now, we can rewrite X^1 as

$$X^1 = e^{-t} a_1 x + L(y,z) e^{-t}. \tag{82}$$

By substituting (82) into (63)

$$L(y,z) = L_1(y),$$

for some smooth function L_1, and (82) becomes

$$X^1 = e^{-t}(a_1 x + L_1(y)). \tag{83}$$

From the equation (64), X^2 is deduced as

$$X^2 = -e^t \partial_x K(x,y). \tag{84}$$

By substituting (83) and (84) into (62)

$$\begin{aligned} &K(x,y) = A(y)x + B(y),\\ &L_1(y) = a_2, \end{aligned}$$

for some smooth functions $A(y)$ and $B(y)$, and for some constant a_2. Now, from (67), we have

$$A(y) = a_1 y + a_3,$$

for some constant a_3. Thus, X^2 is obtained as

$$X^2 = -e^t(a_1 y + a_3).$$

By substituting (80) and (83) into (68), we get

$$B(y) = a_2 y + a_4,$$

for some constant a_4. Subsequently, $X^1, X^2, X^3, X^4, \alpha$ and β becomes

$$\begin{aligned} &X^1 = e^{-t}(a_1 x + a_2),\\ &X^2 = -e^t(a_1 y + a_3),\\ &X^3 = (a_1 y + a_3)x + (a_2 y + a_4),\\ &X^4 = a_1,\\ &\alpha = \beta = 0. \end{aligned}$$

Therefore, we have the following theorem.

Theorem 4.1. *The vector field X on (Sol_1^4, g) where g given by (6), is a Ricci bi-conformal vector field if and only if*

$$X = (a_1x+a_2)\frac{\partial}{\partial x} - (a_1y+a_3)(\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}) + ((a_1y+a_3)x + (a_2y+a_4))\frac{\partial}{\partial z} + a_1\frac{\partial}{\partial t}.$$

Now, consider $X = \nabla f$ on (M,g) with potential function f. Therefore,

$$\nabla f = e_1 f e_1 + e_2 f e_2 + e_3 f e_3 + e_4 f e_4.$$

Thus, the Ricci bi-conformal vector field X on (M, g) is gradient vector field as ∇f if and only if

$$\begin{aligned} &\partial_x f = e^{-2t}(a_1 x + a_2), \\ &\partial_y f = (-e^{2t} - x^2)(a_1 y + a_3) + (a_2 y + a_4)x, \\ &\partial_z f = (a_1 y + a_3)x + (a_2 y + a_4), \\ &\partial_t f = a_1. \end{aligned} \tag{85}$$

The derivation of the fourth equation of (85) with respect to x implies that $\partial_x \partial_t f = 0$. So by deriving the first equation with respect to t gives that $\partial_t \partial_x f = -2e^{-2t}(a_1 x + a_2)$, thus, $a_1 = a_2 = 0$. Now, the third equation becomes $\partial_z f = a_3 x + a_4$. By deriving the first and the third equations of (85) with respect to z and x, respectively, yield that $\partial_z \partial_x f = 0$ and $\partial_x \partial_z f = a_3 x + a_4$, therefore $a_1 = a_3 = 0$. The derivation of the first equation of (85) with respect to y concludes that $\partial_y \partial_x f = 0$ and the derivation of the second equation with respect to x gives that $\partial_x \partial_y f = -2x(a_1 y + a_3) + (a_2 y + a_4)$, thus $a_1 = a_2 = a_3 = a_4 = 0$. Thus (85) becomes

$$\partial_x f = \partial_y f = \partial_z f = \partial_t f = 0.$$

The direct integration leads to the following

$$f(x, y, z, t) = c.$$

for some constant c. At the end we can state:

Corollary 4.2. *Any Ricci bi-conformal vector field X on (Sol_1^4, g) is gradient vector field with potential function f if $f(x, y, z, t) = c$, for some constant c.*

Corollary 4.3. *Any Ricci bi-conformal vector field X on (Sol_1^4, g) is Killing vector field.*

Declarations

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Conflict of interests

We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

Ethics approval and consent to participate

Not applicable.

Consent for publication

Not applicable.

Availability of data and material

All data generated or analysed during this study are included in this published article.

Author's contributions

All authors contributed equally in the preparation of this manuscript.

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