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TRIPOTENCY OF LINEAR COMBINATIONS OF A QUADRATIC MATRIX AND AN ARBITRARY MATRIX

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Abstract. We give necessary and sufficient conditions for tripotency of the linear combination of the form $a\mathbf{Q} + b\mathbf{A}$, under some certain conditions imposed on $\mathbf{Q}$ and $\mathbf{A}$, where $\mathbf{Q}$ is a nonzero quadratic matrix, $\mathbf{A}$ is a nonzero arbitrary matrix and a, b are nonzero complex numbers. Moreover, some examples illustrating the main results are given.

1. Introduction

Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. $\mathbb{C}^{m,n}$ and $\mathbb{C}^n$ denote the sets of $m \times n$ complex matrices and $n \times n$ complex matrices for positive integers m, n.

A matrix $\mathbf{M} \in \mathbb{C}^n$ is called an involutive, an idempotent, a tripotent, and a k–potent matrix if $\mathbf{M}^2 = \mathbf{I}_n$, $\mathbf{M}^2 = \mathbf{M}$, $\mathbf{M}^3 = \mathbf{M}$, and $\mathbf{M}^k = \mathbf{M}$, respectively, where $k \geq 2$ is a positive integer, $\mathbf{I}_n$ stands for the identity matrix of order n. Moreover, for $\alpha, \beta, \eta \in \mathbb{C}$, a matrix $\mathbf{M} \in \mathbb{C}^n$ is called an $\{\alpha, \beta, \eta\}$–cubic matrix and an $\{\alpha, \beta\}$–quadratic matrix if $(\mathbf{M} - \alpha\mathbf{I}_n)(\mathbf{M} - \beta\mathbf{I}_n)(\mathbf{M} - \eta\mathbf{I}_n) = \mathbf{0}$ and $(\mathbf{M} - \alpha\mathbf{I}_n)(\mathbf{M} - \beta\mathbf{I}_n) = \mathbf{0}$, respectively, where $\mathbf{0}$ stands for a zero matrix of appropriate size [17]. It is worth pointing out that quadratic matrices are a large class of matrices, including idempotent, involutive, nilpotent ($\mathbf{M}^2 = \mathbf{0}$, $\mathbf{M} \in \mathbb{C}^n$), etc. matrices. Similarly, tripotent and quadratic matrices are some subclasses of cubic matrices. Mainly, the above mentioned matrices and the problems in this paper related to these matrices should be of interest not only from the algebraic point of view but also from the role these type matrices play in applied sciences such as statistical theory [7, 10, 11], quantum mechanics [1, 5], digital image encryption [20]. These explain the intense interest in these types of matrices in the literature.

Note that an idempotent matrix and an involutive matrix are an $\{\alpha, \beta\}$–quadratic matrix for $\{\alpha, \beta\} = \{0, 1\}$ and $\{\alpha, \beta\} = \{-1, 1\}$, respectively. Likewise, for $\{\alpha, \beta, \eta\} = \{-1, 0, 1\}$, a tripotent matrix is an $\{\alpha, \beta, \eta\}$–cubic matrix.

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One can find detailed information about quadratic matrices in [2, 12]. A useful one of them is given below. From Theorem 2.1 in [12], an $\{\alpha, \beta\}$–quadratic matrix $\mathbf{Q} \in \mathbb{C}^n$ can be written as

$$\mathbf{Q} = \mathbf{U}\left(\alpha \mathbf{I}_p \oplus \beta \mathbf{I}_{n-p}\right)\mathbf{U}^{-1}, \tag{1}$$

where $\alpha, \beta \in \mathbb{C}$ and $\alpha \neq \beta$, $p \in \{0, 1, \ldots, n\}$, the symbol $\oplus$ denotes the direct sum of matrices, and $\mathbf{U}$ is a nonsingular matrix.

Consider a linear combination of the form

$$\mathbf{T} = a\mathbf{Q} + b\mathbf{A},\ \ \mathbf{Q}, \mathbf{A} \in \mathbb{C}^n,\ \ a, b \in \mathbb{C}^*. \tag{2}$$

The tripotency of the linear combinations of the form (2) has been the subject to many papers when $\mathbf{Q}$ and $\mathbf{A}$ are k–potent matrices (for example, [3, 4, 13, 14, 15, 21, 22]). Moreover, the problem of being a special type of matrix of the linear combination has been studied many times when at least one of $\mathbf{Q}$ and $\mathbf{A}$ is a tripotent matrix [3, 4, 6, 8, 9, 13, 14, 15, 21, 22] or a quadratic matrix [8, 16, 17, 18, 19].

Recently, there are some studies concerning that the $\mathbf{T}$ in (2) is a special type of matrix when one of $\mathbf{Q}$ and $\mathbf{A}$ is a special type matrix and the other is an arbitrary matrix. One of them is the paper of Liu et al. that characterize the involutiveness of the matrix $\mathbf{T}$ when $\mathbf{Q}$ is a quadratic or a tripotent matrix and $\mathbf{A}$ is an arbitrary matrix, under some certain conditions [8]. Moreover, the results about the involutiveness of the linear combination are given in [17] when $\mathbf{A}$ is an arbitrary matrix and $\mathbf{Q}$ is a quadratic matrix under some different conditions. Furthermore, the necessary and sufficient conditions to $a\mathbf{Q}+b\mathbf{A}$ be idempotent are obtained when $\mathbf{A}$ is an arbitrary matrix and $\mathbf{Q}$ is a quadratic or a cubic matrix [16].

This paper's purpose is to present the necessary and sufficient conditions for tripotency of the linear combination in (2) when $\mathbf{A}$ is an arbitrary matrix and $\mathbf{Q}$ is a quadratic matrix, under some specific conditions related to them. Moreover, there are several types of examples at the end of the paper exemplifying the main results.

2. Main Results

We establish the results of tripotency of the linear combinations of the form (2) in this section. The arbitrary matrix $\mathbf{A}$, which provides the conditions in the hypotheses of theorems given below, has been a zero matrix (this contradicts the assumptions of theorems below) except certain values of the scalars α or β. In fact, since the arbitrary matrix $\mathbf{A}$ has to be a zero matrix when $\alpha \neq 1$, $\beta \neq 1$, and $\beta \neq 0$, the following results are striking.

Theorem 2.1. *Let* $\mathbf{T}$ *be a linear combination of the form* $\mathbf{T} = a\mathbf{Q} + b\mathbf{A}$ *where* $\mathbf{Q} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *is an* $\{\alpha, \beta\}-$ *quadratic matrix,* $\mathbf{A} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *is an arbitrary matrix and* $\beta \in \mathbb{C}$, $\alpha, a, b \in \mathbb{C}^*$, *and* $\alpha \neq \beta$. *Then* $\mathbf{QAQ} = \mathbf{AQ}$ *and*

$\mathbf{T}$ *is a tripotent matrix if and only if there exists a nonsingular matrix* $\mathbf{V} \in \mathbb{C}^n$ *such that*

$$\mathbf{Q} = \mathbf{V} \begin{pmatrix} \alpha \mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \beta \mathbf{I}_{n-p} \end{pmatrix} \mathbf{V}^{-1} \tag{3}$$

and $\mathbf{A}$ *satisfies one of the following cases.*

(a) $\beta = 1$,
(a $-$ i) $a\alpha = 1$,

$$\mathbf{A} = \mathbf{V} \begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\alpha-1}{\alpha b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} \\ \mathbf{Z}_2 & \mathbf{0} & \frac{-\alpha-1}{\alpha b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} & \mathbf{0} & \frac{-1}{\alpha b}\mathbf{I}_{n-p-s-t} \end{pmatrix} \mathbf{V}^{-1}, \tag{4}$$

(a $-$ ii) $a\alpha = -1$,

$$\mathbf{A} = \mathbf{V} \begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{Z}_1 & \frac{\alpha+1}{\alpha b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{-\alpha+1}{\alpha b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} & \mathbf{0} & \frac{1}{\alpha b}\mathbf{I}_{n-p-s-t} \end{pmatrix} \mathbf{V}^{-1}, \tag{5}$$

being $\mathbf{Z}_1 \in \mathbb{C}^{s,p}$, $\mathbf{Z}_2 \in \mathbb{C}^{t,p}$, *and* $\mathbf{Z}_3 \in \mathbb{C}^{(n-p-s-t),p}$ *arbitrary.*
(b) $\beta \neq 1$,
(b$_1$) $\beta = 0$ *and* $\alpha = 1$,

$$\mathbf{A} = \mathbf{V} \begin{pmatrix} \frac{1-a}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Y}_2 & \mathbf{Y}_3 \\ \mathbf{0} & \frac{-1-a}{b}\mathbf{I}_r & \mathbf{0} & \mathbf{Y}_4 & \mathbf{0} & \mathbf{Y}_6 \\ \mathbf{0} & \mathbf{0} & \frac{-a}{b}\mathbf{I}_{p-q-r} & \mathbf{Y}_7 & \mathbf{Y}_8 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{1}{b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-s-t} \end{pmatrix} \mathbf{V}^{-1}, \tag{6}$$

being $\mathbf{Y}_2 \in \mathbb{C}^{q,t}$, $\mathbf{Y}_3 \in \mathbb{C}^{q,(n-p-s-t)}$, $\mathbf{Y}_4 \in \mathbb{C}^{r,s}$, $\mathbf{Y}_6 \in \mathbb{C}^{r,(n-p-s-t)}$, $\mathbf{Y}_7 \in \mathbb{C}^{(p-q-r),s}$, *and* $\mathbf{Y}_8 \in \mathbb{C}^{(p-q-r),t}$ *arbitrary.*
(b$_2$) $\beta = 0$, $\alpha \neq 1$, *and*

(b$_2 - i$) $a\alpha = 1$,

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{Y}'_2 & \mathbf{Y}'_3 \\ \mathbf{0} & \frac{1}{b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\frac{1}{b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-s-t} \end{pmatrix}\mathbf{V}^{-1}, \tag{7}$$

(b$_2 - ii$) $a\alpha = -1$,

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{Y}'_1 & \mathbf{0} & \mathbf{Y}'_3 \\ \mathbf{0} & \frac{1}{b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\frac{1}{b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-s-t} \end{pmatrix}\mathbf{V}^{-1}, \tag{8}$$

being $\mathbf{Y}'_1 \in \mathbb{C}^{p,s}$, $\mathbf{Y}'_2 \in \mathbb{C}^{p,t}$, *and* $\mathbf{Y}'_3 \in \mathbb{C}^{p,(n-p-s-t)}$ *arbitrary.*

(b$_3$) $\beta \neq 0$, $\alpha = 1$, *and*

(b$_3 - i$) $a\beta = 1$,

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} \frac{\beta-1}{\beta b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-\beta-1}{\beta b}\mathbf{I}_r & \mathbf{0} & \mathbf{Y}''_2 \\ \mathbf{0} & \mathbf{0} & -\frac{1}{\beta b}\mathbf{I}_{p-q-r} & \mathbf{Y}''_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p} \end{pmatrix}\mathbf{V}^{-1}, \tag{9}$$

(b$_3 - ii$) $a\beta = -1$,

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} \frac{\beta+1}{\beta b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{Y}''_1 \\ \mathbf{0} & \frac{-\beta+1}{\beta b}\mathbf{I}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{\beta b}\mathbf{I}_{p-q-r} & \mathbf{Y}''_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p} \end{pmatrix}\mathbf{V}^{-1}, \tag{10}$$

being $\mathbf{Y}''_1 \in \mathbb{C}^{q,(n-p)}$, $\mathbf{Y}''_2 \in \mathbb{C}^{r,(n-p)}$, *and* $\mathbf{Y}''_3 \in \mathbb{C}^{(p-q-r),(n-p)}$ *arbitrary.*

Proof. From (1), there exist $p \in \{0, 1, \ldots, n\}$ and a nonsingular matrix $\mathbf{U} \in \mathbb{C}^n$ such that

$$\mathbf{Q} = \mathbf{U}\left(\alpha\mathbf{I}_p \oplus \beta\mathbf{I}_{n-p}\right)\mathbf{U}^{-1}.$$

Since $\mathbf{A}$ is an arbitrary matrix, it can be written as $\mathbf{A} = \mathbf{U}\begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{pmatrix}\mathbf{U}^{-1}$, where $\mathbf{X} \in \mathbb{C}^{p}$. By the facts that $\mathbf{QAQ} = \mathbf{AQ}$, $\alpha \neq 0$, it can be written

$$\alpha\mathbf{X} = \mathbf{X}, \quad \alpha\beta\mathbf{Y} = \beta\mathbf{Y}, \quad \beta\mathbf{Z} = \mathbf{Z}, \quad \beta^2\mathbf{W} = \beta\mathbf{W}. \tag{11}$$

Now let us assume that $\mathbf{T}$ is a tripotent matrix, then we have the following equalities

(12)

$$(a\alpha\mathbf{I}_p + b\mathbf{X})^3 + ab^2(2\alpha+\beta)\mathbf{YZ} + b^3(\mathbf{XYZ} + \mathbf{YZX} + \mathbf{YWZ}) = a\alpha\mathbf{I}_p + b\mathbf{X},$$

$$a^2b\left(\alpha^2+\alpha\beta+\beta^2\right)\mathbf{Y} + ab^2\left((2\alpha+\beta)\mathbf{XY} + (\alpha+2\beta)\mathbf{YW}\right) + b^3\left(\mathbf{X}^2\mathbf{Y} + \mathbf{XYW} + \mathbf{YZY} + \mathbf{YW}^2\right) = b\mathbf{Y},$$

$$a^2b\left(\alpha^2+\alpha\beta+\beta^2\right)\mathbf{Z} + ab^2\left((2\alpha+\beta)\mathbf{ZX} + (\alpha+2\beta)\mathbf{WZ}\right) + b^3\left(\mathbf{ZX}^2 + \mathbf{ZYZ} + \mathbf{WZX} + \mathbf{W}^2\mathbf{Z}\right) = b\mathbf{Z},$$

$$(a\beta\mathbf{I}_{n-p} + b\mathbf{W})^3 + ab^2(\alpha+2\beta)\mathbf{ZY} + b^3(\mathbf{ZXY} + \mathbf{ZYW} + \mathbf{WZY}) = a\beta\mathbf{I}_{n-p} + b\mathbf{W}.$$

Depending on the values of the scalar β, the following cases should be examined.

(i) Let us assume that $\beta = 1$. $\mathbf{X} = \mathbf{0}$ and $\mathbf{Y} = \mathbf{0}$ is obtained by taking the first two equations in (11). Reorganizing (12) yields directly the equalities below

$$\begin{aligned} &(a\alpha\mathbf{I}_p)^3 = a\alpha\mathbf{I}_p, \quad (a\mathbf{I}_{n-p} + b\mathbf{W})^3 = a\mathbf{I}_{n-p} + b\mathbf{W}, \\ &a^2b\left(\alpha^2+\alpha+1\right)\mathbf{Z} + ab^2(\alpha+2)\mathbf{WZ} + b^3\mathbf{W}^2\mathbf{Z} = b\mathbf{Z}. \end{aligned} \tag{13}$$

First let us consider the second and first equations in (13). $a\mathbf{I}_{n-p} + b\mathbf{W}$ is a tripotent matrix and $a\alpha \in \{-1, 1\}$ since α, $a \neq 0$. Since a tripotent matrix is a $\{-1, 0, 1\}-$ cubic matrix, from the Lemma 1.1 in [16], there exist $s, t \in \{0, 1, \ldots, n-p\}$, $s + t \leq n - p$ and a nonsingular matrix $\mathbf{S} \in \mathbb{C}^{(n-p)}$ such that

$$\mathbf{W} = \mathbf{S}\left(\frac{1-a}{b}\mathbf{I}_s \oplus \frac{-1-a}{b}\mathbf{I}_t \oplus \frac{-a}{b}\mathbf{I}_{n-p-s-t}\right)\mathbf{S}^{-1}. \tag{14}$$

Let

$$\mathbf{Z} = \mathbf{S}\begin{pmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \\ \mathbf{Z}_3 \end{pmatrix}, \tag{15}$$

where $\mathbf{Z}_1 \in \mathbb{C}^{s,p}$ and $\mathbf{Z}_2 \in \mathbb{C}^{t,p}$. Substituting (14) and (15) into the third equation in (13) leads to

$$\begin{pmatrix} a\alpha(a\alpha+1)\mathbf{Z}_1 \\ a\alpha(a\alpha-1)\mathbf{Z}_2 \\ (a\alpha-1)(a\alpha+1)\mathbf{Z}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}.$$

We have known from the above that $a\alpha \in \{-1, 1\}$. If $a\alpha = 1$, then $\mathbf{Z}_1 = \mathbf{0}$ and the matrix $\mathbf{Z}$ is

$$\mathbf{Z} = \mathbf{S}\begin{pmatrix} \mathbf{0} \\ \mathbf{Z}_2 \\ \mathbf{Z}_3 \end{pmatrix}. \tag{16}$$

If $a\alpha = -1$, then $\mathbf{Z}_2 = \mathbf{0}$. So the matrix $\mathbf{Z}$ is

$$\mathbf{Z} = \mathbf{S}\begin{pmatrix} \mathbf{Z}_1 \\ \mathbf{0} \\ \mathbf{Z}_3 \end{pmatrix}. \tag{17}$$

We can write

$$\mathbf{Q} = \mathbf{U}\left(\alpha\mathbf{I}_p \oplus \mathbf{I}_{n-p}\right)\mathbf{U}^{-1} = \mathbf{U}\left(\mathbf{I}_p \oplus \mathbf{S}\right)\left(\alpha\mathbf{I}_p \oplus \mathbf{I}_{n-p}\right)\left(\mathbf{I}_p \oplus \mathbf{S}^{-1}\right)\mathbf{U}^{-1}.$$

In view of (14), (16) and (14), (17) we obtain that

$$\mathbf{A} = \mathbf{U}\left(\mathbf{I_p} \oplus \mathbf{S}\right)\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\alpha-1}{\alpha b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} \\ \mathbf{Z}_2 & \mathbf{0} & \frac{-\alpha-1}{\alpha b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} & \mathbf{0} & \frac{-1}{\alpha b}\mathbf{I}_{n-p-s-t} \end{pmatrix}\left(\mathbf{I_p} \oplus \mathbf{S^{-1}}\right)\mathbf{U^{-1}}$$

and

$$\mathbf{A} = \mathbf{U}\left(\mathbf{I_p} \oplus \mathbf{S}\right)\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{Z}_1 & \frac{\alpha+1}{\alpha b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{-\alpha+1}{\alpha b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} & \mathbf{0} & \frac{1}{\alpha b}\mathbf{I}_{n-p-s-t} \end{pmatrix}\left(\mathbf{I_p} \oplus \mathbf{S^{-1}}\right)\mathbf{U^{-1}},$$

respectively. One can easily see that the matrices $\mathbf{A}$ are the same as (4) and (5), respectively.

(ii) Let us assume that $\beta \neq 1$. $\mathbf{Z} = \mathbf{0}$ is obtained from (11). Reorganizing (12) leads directly to the equalities below

(18)
$$\left(a\alpha\mathbf{I}_p + b\mathbf{X}\right)^3 = a\alpha\mathbf{I}_p + b\mathbf{X}, \quad \left(a\beta\mathbf{I}_{n-p} + b\mathbf{W}\right)^3 = a\beta\mathbf{I}_{n-p} + b\mathbf{W},$$
$$a^2b\left(\alpha^2 + \alpha\beta + \beta^2\right)\mathbf{Y} + ab^2\left(\left(2\alpha + \beta\right)\mathbf{XY} + \left(\alpha + 2\beta\right)\mathbf{YW}\right) + b^3\left(\mathbf{X^2Y} + \mathbf{XYW} + \mathbf{YW^2}\right) = b\mathbf{Y}.$$

Clearly, $a\alpha\mathbf{I}_p + b\mathbf{X}$ and $a\beta\mathbf{I}_{n-p} + b\mathbf{W}$ are tripotent matrices from the first two equations in (18). Therefore, there exist $q, r \in \{0, 1, \ldots, p\}$, $q + r \leq p$, $s, t \in \{0, 1, \ldots, n-p\}$, $s + t \leq n - p$ and nonsingular matrices $\mathbf{S}_1 \in \mathbb{C}^p$, $\mathbf{S}_2 \in \mathbb{C}^{(n-p)}$ such that

$$\mathbf{X} = \mathbf{S}_1\left(\frac{1-a\alpha}{b}\mathbf{I}_q \oplus \frac{-1-a\alpha}{b}\mathbf{I}_r \oplus \frac{-a\alpha}{b}\mathbf{I}_{p-q-r}\right)\mathbf{S}_1{}^{-1}, \tag{19}$$

$$\mathbf{W} = \mathbf{S}_2\left(\frac{1-a\beta}{b}\mathbf{I}_s \oplus \frac{-1-a\beta}{b}\mathbf{I}_t \oplus \frac{-a\beta}{b}\mathbf{I}_{n-p-s-t}\right)\mathbf{S}_2{}^{-1}. \tag{20}$$

Let us write $\mathbf{Y}$ as

$$\mathbf{Y} = \mathbf{S}_1 \begin{pmatrix} \mathbf{Y}_1 & \mathbf{Y}_2 & \mathbf{Y}_3 \\ \mathbf{Y}_4 & \mathbf{Y}_5 & \mathbf{Y}_6 \\ \mathbf{Y}_7 & \mathbf{Y}_8 & \mathbf{Y}_9 \end{pmatrix} {\mathbf{S}_2}^{-1}, \tag{21}$$

where $\mathbf{Y}_1 \in \mathbb{C}^{q,s}$ and $\mathbf{Y}_5 \in \mathbb{C}^{r,t}$. Substituting (19), (20), and (21) into the third equation in (18), it is obtained that $(2\mathbf{Y}_1 \oplus 2\mathbf{Y}_5 \oplus -\mathbf{Y}_9) = \mathbf{0}$. So $\mathbf{Y}$ turns to

$$\mathbf{Y} = \mathbf{S}_1 \begin{pmatrix} \mathbf{0} & \mathbf{Y}_2 & \mathbf{Y}_3 \\ \mathbf{Y}_4 & \mathbf{0} & \mathbf{Y}_6 \\ \mathbf{Y}_7 & \mathbf{Y}_8 & \mathbf{0} \end{pmatrix} {\mathbf{S}_2}^{-1}. \tag{22}$$

Let us define $\mathbf{V} := \mathbf{U}\,(\mathbf{S}_1 \oplus \mathbf{S}_2)$. So we obtain

$$\begin{aligned} \mathbf{Q} &= \mathbf{U}\,(\alpha\mathbf{I}_p \oplus \beta\mathbf{I}_{n-p})\,\mathbf{U}^{-1} = \mathbf{V}\left({\mathbf{S}_1}^{-1} \oplus {\mathbf{S}_2}^{-1}\right)(\alpha\mathbf{I}_p \oplus \beta\mathbf{I}_{n-p})\,(\mathbf{S}_1 \oplus \mathbf{S}_2)\,\mathbf{V}^{-1} \\ &= \mathbf{V}\,(\alpha\mathbf{I}_p \oplus \beta\mathbf{I}_{n-p})\,\mathbf{V}^{-1}. \end{aligned}$$

Taking (19), (20), (22), and $\mathbf{Z} = \mathbf{0}$ into account, we obtain

$$\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Y}_2 & \mathbf{Y}_3 \\ \mathbf{0} & \frac{-1-a\alpha}{b}\mathbf{I}_r & \mathbf{0} & \mathbf{Y}_4 & \mathbf{0} & \mathbf{Y}_6 \\ \mathbf{0} & \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q-r} & \mathbf{Y}_7 & \mathbf{Y}_8 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1-a\beta}{b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-1-a\beta}{b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-a\beta}{b}\mathbf{I}_{n-p-s-t} \end{pmatrix}. \tag{23}$$

The above matrices $\mathbf{Q}$ and $\mathbf{A}$ provide the necessary and sufficient conditions for the tripotency of the matrix $\mathbf{T}$. Nevertheless, these are necessary to satisfy the condition $\mathbf{QAQ} = \mathbf{AQ}$ but not sufficient. Therefore, the proof should be continued in more detail according to the values of α and β.

(ii-1) Firstly, let $\beta = 0$ and $\alpha = 1$. Substituting these values of α and β into (23), clearly, the matrix $\mathbf{A}$ is just as in (6).

(ii-2) Let $\beta = 0$ and $\alpha \neq 1$. From (11), it is clear that $\mathbf{X} = \mathbf{0}$. Then the equations of (18) turn to

$$(a\alpha\mathbf{I}_p)^3 = a\alpha\mathbf{I}_p, \quad (b\mathbf{W})^3 = b\mathbf{W}, \quad a^2b\alpha^2\mathbf{Y} + ab^2\alpha\mathbf{YW} + b^3\mathbf{YW}^2 = b\mathbf{Y}. \tag{24}$$

Substituting the value of the scalar β into (20) yields

$$\mathbf{W} = \mathbf{S}_2\left(\frac{1}{b}\mathbf{I}_s \oplus -\frac{1}{b}\mathbf{I}_t \oplus \mathbf{0}_{n-p-s-t}\right){\mathbf{S}_2}^{-1}. \tag{25}$$

Let $\mathbf{Y}_1'$, $\mathbf{Y}_2'$, and $\mathbf{Y}_3'$ denote the first, second, and third block-columns of ${\mathbf{S}_1}^{-1}\mathbf{Y}\mathbf{S}_2$ in (22). So, we can write

$$\mathbf{Y} = \mathbf{S}_1 \begin{pmatrix} \mathbf{Y}_1' & \mathbf{Y}_2' & \mathbf{Y}_3' \end{pmatrix} {\mathbf{S}_2}^{-1}. \tag{26}$$

Substituting (25) and (26) into the last equation in (24),

$$\begin{pmatrix} (\alpha a+1)\,\alpha a\mathbf{Y}_1' & (\alpha a-1)\,\alpha a\mathbf{Y}_2' & (\alpha a+1)\,(\alpha a-1)\,\mathbf{Y}_3' \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

can be obtained. Moreover, it is obvious that $\alpha a \in \{-1,1\}$ from the first equation in (24). Therefore, if $\alpha a = 1$, $\mathbf{Y}$ turns to $\mathbf{Y} = \mathbf{S}_1 \begin{pmatrix} \mathbf{0} & \mathbf{Y}_2' & \mathbf{Y}_3' \end{pmatrix} {\mathbf{S}_2}^{-1}$ or if $\alpha a = -1$, $\mathbf{Y}$ reduces to $\mathbf{Y} = \mathbf{S}_1 \begin{pmatrix} \mathbf{Y}_1' & \mathbf{0} & \mathbf{Y}_3' \end{pmatrix} {\mathbf{S}_2}^{-1}$. So, these forms of $\mathbf{Y}$ together with (25) constitute the matrix $\mathbf{A}$ as in (7) and (8), respectively.

(ii-3) Let $\beta \neq 0$ and $\alpha = 1$. From (11), it is clear that $\mathbf{W} = \mathbf{0}$. By rewriting the equations of (18), we obtain

$$\begin{aligned} &(a\mathbf{I}_p + b\mathbf{X})^3 = a\mathbf{I}_p + b\mathbf{X}, \quad (a\beta\mathbf{I}_{n-p})^3 = a\beta\mathbf{I}_{n-p}, \\ &a^2 b\left(1+\beta+\beta^2\right)\mathbf{Y} + ab^2\,(2+\beta)\,\mathbf{X}\mathbf{Y} + b^3\mathbf{X}^2\mathbf{Y} = b\mathbf{Y}. \end{aligned} \tag{27}$$

Substituting the value of α into (19) gives

$$\mathbf{X} = \mathbf{S}_1 \left(\frac{1-a}{b}\mathbf{I}_q \oplus \frac{-1-a}{b}\mathbf{I}_r \oplus -\frac{a}{b}\mathbf{I}_{p-q-r} \right) {\mathbf{S}_1}^{-1}. \tag{28}$$

Let $\mathbf{Y}_1''$, $\mathbf{Y}_2''$, and $\mathbf{Y}_3''$ denote the first, second, and third block rows of ${\mathbf{S}_1}^{-1}\mathbf{Y}\mathbf{S}_2$ in (22). Then

$$\mathbf{Y} = \mathbf{S}_1 \begin{pmatrix} \mathbf{Y}_1'' \\ \mathbf{Y}_2'' \\ \mathbf{Y}_3'' \end{pmatrix} {\mathbf{S}_2}^{-1}. \tag{29}$$

Substituting (28) and (29) into the last equation in (27),

$$\begin{pmatrix} a\beta\,(a\beta+1)\,\mathbf{Y}_1'' \\ a\beta\,(a\beta-1)\,\mathbf{Y}_2'' \\ (a\beta-1)\,(a\beta+1)\,\mathbf{Y}_3'' \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}.$$

can be obtained. From the second equation in (27), $a\beta \in \{-1,1\}$. So $\mathbf{Y}$ reduces to $\mathbf{Y} = \mathbf{S}_1 \begin{pmatrix} \mathbf{0} \\ \mathbf{Y}_2'' \\ \mathbf{Y}_3'' \end{pmatrix} {\mathbf{S}_2}^{-1}$ when $a\beta = 1$ or $\mathbf{Y} = \mathbf{S}_1 \begin{pmatrix} \mathbf{Y}_1'' \\ \mathbf{0} \\ \mathbf{Y}_3'' \end{pmatrix} {\mathbf{S}_2}^{-1}$ when $a\beta = -1$. These forms of $\mathbf{Y}$ together with (28) constitute the matrix $\mathbf{A}$ as in (9) and (10), respectively.

(ii-4) Let $\beta \neq 0$ and $\alpha \neq 1$. From (11), the matrices $\mathbf{X}, \mathbf{Y}, \mathbf{W}$ are zero. This means that $\mathbf{A}$ is a zero matrix, which contradicts the hypothesis. The first part of the proof is completed. Conversely, the sufficiency part is obvious. $\square$

In the above theorem, the condition $\mathbf{QAQ} = \mathbf{AQ}$ has been imposed on the matrices $\mathbf{Q}$ and $\mathbf{A}$. If the condition $\mathbf{Q}^2\mathbf{AQ} = \mathbf{Q}^2\mathbf{A}$ is substituted for the former condition, the equations of (11) turn to

$$\alpha\mathbf{X} = \mathbf{X}, \quad \beta\mathbf{Y} = \mathbf{Y}, \quad \alpha\beta^2\mathbf{Z} = \beta^2\mathbf{Z}, \quad \beta^3\mathbf{W} = \beta^2\mathbf{W}.$$

Here, as in Theorem 2.1, an investigation can be made depending on the values of the scalar β. For instance, if $\beta = 1$, the matrices $\mathbf{X}$ and $\mathbf{Z}$ are zero matrices of appropriate size. Hence, $\mathbf{Y}$ and $\mathbf{W}$ can be easily found. When $\beta \neq 1$, we have $\mathbf{Y} = \mathbf{0}$, so the matrices $\mathbf{X}$, $\mathbf{Z}$, and $\mathbf{W}$ can be obtained just like before. Thus, $\mathbf{A}$ can be obtained as in the following theorem.

Theorem 2.2. *Let $\mathbf{T}$ be a linear combination of the form $\mathbf{T} = a\mathbf{Q} + b\mathbf{A}$ where $\mathbf{Q} \in \mathbb{C}^n \backslash \{\mathbf{0}\}$ is an $\{\alpha, \beta\}-$ quadratic matrix, $\mathbf{A} \in \mathbb{C}^n \backslash \{\mathbf{0}\}$ is an arbitrary matrix and $\beta \in \mathbb{C}$, $\alpha, a, b \in \mathbb{C}^*$, and $\alpha \neq \beta$. Then $\mathbf{Q}^2\mathbf{AQ} = \mathbf{Q}^2\mathbf{A}$ and $\mathbf{T}$ is a tripotent matrix if and only if there exists a nonsingular matrix $\mathbf{V} \in \mathbb{C}^n$ such that*

$$\mathbf{Q} = \mathbf{V}\begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I}_{n-p} \end{pmatrix}\mathbf{V}^{-1}$$

and $\mathbf{A}$ satisfies one of the following cases.

(a) $\beta = 1$,

(a $-$ i) $\alpha a = 1$,

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{Y}_2 & \mathbf{Y}_3 \\ \mathbf{0} & \frac{\alpha-1}{\alpha b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{-\alpha-1}{\alpha b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-1}{\alpha b}\mathbf{I}_{n-p-s-t} \end{pmatrix}\mathbf{V}^{-1},$$

(a $-$ ii) $\alpha a = -1$,

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{Y}_1 & \mathbf{0} & \mathbf{Y}_3 \\ \mathbf{0} & \frac{\alpha+1}{\alpha b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{-\alpha+1}{\alpha b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{\alpha b}\mathbf{I}_{n-p-s-t} \end{pmatrix}\mathbf{V}^{-1},$$

being $\mathbf{Y}_1 \in \mathbb{C}^{p,s}$, $\mathbf{Y}_2 \in \mathbb{C}^{p,t}$, and $\mathbf{Y}_3 \in \mathbb{C}^{p,(n-p-s-t)}$ arbitrary.

(b) $\beta \neq 1$,

(b$_1$) $\beta = 0$, *and* $\alpha = 1$,

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} \frac{1-a}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-1-a}{b}\mathbf{I}_r & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\frac{a}{b}\mathbf{I}_{p-q-r} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2 & \mathbf{Z}_3 & \frac{1}{b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} \\ \mathbf{Z}_4 & \mathbf{0} & \mathbf{Z}_6 & \mathbf{0} & -\frac{1}{b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{Z}_7 & \mathbf{Z}_8 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-s-t} \end{pmatrix}\mathbf{V}^{-1},$$

being $\mathbf{Z}_2 \in \mathbb{C}^{s,r}$, $\mathbf{Z}_3 \in \mathbb{C}^{s,(p-q-r)}$, $\mathbf{Z}_4 \in \mathbb{C}^{t,q}$, $\mathbf{Z}_6 \in \mathbb{C}^{t,(p-q-r)}$, $\mathbf{Z}_7 \in \mathbb{C}^{(n-p-s-t),q}$, *and* $\mathbf{Z}_8 \in \mathbb{C}^{(n-p-s-t),r}$ *arbitrary.*

(b$_2$) $\beta = 0$, $\alpha \neq 1$, *and*

(b$_2 - i$) $\alpha a = 1$,

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} \\ \mathbf{Z}'_2 & \mathbf{0} & -\frac{1}{b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{Z}'_3 & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-s-t} \end{pmatrix}\mathbf{V}^{-1},$$

(b$_2 - ii$) $\alpha a = -1$,

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{Z}'_1 & \frac{1}{b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\frac{1}{b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{Z}'_3 & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-s-t} \end{pmatrix}\mathbf{V}^{-1},$$

being $\mathbf{Z}'_1 \in \mathbb{C}^{s,p}$, $\mathbf{Z}'_2 \in \mathbb{C}^{t,p}$, *and* $\mathbf{Z}'_3 \in \mathbb{C}^{(n-p-s-t),p}$ *arbitrary.*

(b$_3$) $\beta \neq 0$, $\alpha = 1$, *and*

(b$_3 - i$) $a\beta = 1$,

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} \frac{\beta-1}{\beta b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-\beta-1}{\beta b}\mathbf{I}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\frac{1}{\beta b}\mathbf{I}_{p-q-r} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}''_2 & \mathbf{Z}''_3 & \mathbf{0}_{n-p} \end{pmatrix}\mathbf{V}^{-1},$$

$(\mathrm{b}_3 - ii)$ $a\beta = -1$,

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} \frac{\beta+1}{\beta b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-\beta+1}{\beta b}\mathbf{I}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{\beta b}\mathbf{I}_{p-q-r} & \mathbf{0} \\ \mathbf{Z}_1'' & \mathbf{0} & \mathbf{Z}_3'' & \mathbf{0}_{n-p} \end{pmatrix}\mathbf{V}^{-1},$$

being $\mathbf{Z}_1'' \in \mathbb{C}^{(n-p),q}$, $\mathbf{Z}_2'' \in \mathbb{C}^{(n-p),r}$, *and* $\mathbf{Z}_3'' \in \mathbb{C}^{(n-p),(p-q-r)}$ *arbitrary.*

The proof of Theorem 2.2 is similar to the proof of Theorem 2.1. In the examples below, we try to find the coefficients of the matrices $\mathbf{Q}$ and $\mathbf{A}$ in the linear combination $\mathbf{T}$ in Theorems 2.1 and 2.2 for the matrix $\mathbf{T}$ to be tripotent.

Example 2.3. *Let*

$$\mathbf{Q} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 2 & 1 & -2 & 2 & 0 \\ 2 & 0 & -3 & 4 & 0 \\ 2 & 0 & -2 & 3 & 0 \\ 0 & 0 & -4 & 4 & 1 \end{pmatrix}$$

and

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 2 & 0 \\ 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & -2 & 2 & 1 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 9 & -4 & -1 & 12 & -4 \\ 7 & -3 & -1 & 9 & -3 \\ 7 & -3 & -1 & 9 & -3 \\ 8 & -4 & -2 & 11 & -3 \end{pmatrix}.$$

It is obvious that $\mathbf{Q}$ *is a* $\{1,-1\}$*–quadratic matrix and* $\mathbf{Q}\mathbf{A}_i\mathbf{Q} = \mathbf{A}_i\mathbf{Q}$, $i = 1, 2$. *If* ${\mathbf{T}_1}^3 = \mathbf{T}_1$ *and* ${\mathbf{T}_2}^3 = \mathbf{T}_2$ *are solved, then the pairs of all ordered pair* (a_i, b_i), *are obtained as* $(a_1, b_1) \in \{(-1,1), (-1,2), (1,-2), (1,-1)\}$ *and* $(a_2, b_2) \in \emptyset$. *Although* $\mathbf{A}_1$ *satisfies the aforementioned form of the matrix* $\mathbf{A}$, $\mathbf{A}_2$ *does not match the desired form. These can be verified with (for example)*

$$\mathbf{V} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & -2 & 1 & 0 \\ 1 & 0 & -1 & 2 & 1 \\ 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & -1 & 2 & 2 \end{pmatrix}$$

that diagonalize the matrix $\mathbf{Q}$.

Example 2.4. *Let* $\mathbf{Q} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ -1 & 1 & 0 \end{pmatrix}$. *Let us find all* $a \in \mathbb{C}^*$ *and all* $\mathbf{A} \in \mathbb{C}^3$ *such that* $\mathbf{Q}^2\mathbf{A}\mathbf{Q} = \mathbf{Q}^2\mathbf{A}$ *and* $a\mathbf{Q} + \mathbf{A}$ *is tripotent. Note that*

$$\mathbf{Q} = \mathbf{V} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{V}^{-1} \quad \text{and} \quad \mathbf{V} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

Let $\beta = 1$ *and* $\alpha = 2$. *Then, from part* (a) *of Theorem 2.2, we get* $a = 1/\alpha = 1/2$ *or* $a = -1/\alpha = -1/2$ *where* $p = 1$, $s \in \{0, 1, 2\}$, $t \in \{0, 1, 2\}$, *and* $s + t \leq 2$. *Furthermore, depending on the appearing and disappearing blocks of* $\mathbf{V}^{-1}\mathbf{A}\mathbf{V}$, *we can write the following possible cases.*

If $a = 1/2$ $(\alpha a = 1)$,

t/s	$s = 0$	$s = 1$	$s = 2$
$t = 0$	$\begin{pmatrix} 0 & c_1 & c_2 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & c_3 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$
$t = 1$	$\begin{pmatrix} 0 & c_4 & c_5 \\ 0 & -3/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & c_6 \\ 0 & 1/2 & 0 \\ 0 & 0 & -3/2 \end{pmatrix}$	
$t = 2$	$\begin{pmatrix} 0 & c_7 & c_8 \\ 0 & -3/2 & 0 \\ 0 & 0 & -3/2 \end{pmatrix}$		

where c_i *is an arbitrary complex number,* $i = 1, 2, \ldots, 8$.

If $a = -1/2$ $(\alpha a = -1)$,

t/s	$s=0$	$s=1$	$s=2$
$t=0$	$\begin{pmatrix} 0 & d_1 & d_2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$	$\begin{pmatrix} 0 & d_3 & d_4 \\ 0 & 3/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$	$\begin{pmatrix} 0 & d_5 & d_6 \\ 0 & 3/2 & 0 \\ 0 & 0 & 3/2 \end{pmatrix}$
$t=1$	$\begin{pmatrix} 0 & 0 & d_7 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$	$\begin{pmatrix} 0 & d_8 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}$	
$t=2$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}$		

where d_i is an arbitrary complex number, $i = 1, 2, \ldots, 8$.

Example 2.5. *For the matrices*

$$\mathbf{Q}_1 = \begin{pmatrix} 1 & 0 & 3/2 & -3/2 \\ 0 & 1 & 3/2 & -3/2 \\ 0 & 0 & -1/2 & 3/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{A}_1 = \begin{pmatrix} -3 & 0 & -3 & 3 \\ 2 & -1 & 3 & -5 \\ -1 & 0 & 0 & -2 \\ -1 & 0 & 0 & -2 \end{pmatrix}$$

and

$$\mathbf{Q}_2 = \begin{pmatrix} 3/2 & 2 & 3 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/2 & 2 & 3 & 0 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} 3 & 3 & 2 & -3 \\ 4 & 14 & 18 & -4 \\ -2 & -6 & -7 & 2 \\ 4 & 12 & 16 & -4 \end{pmatrix}$$

linear combinations $-2\mathbf{Q}_1 - \mathbf{A}_1$ and $-2\mathbf{Q}_2 + \mathbf{A}_2$ become tripotent. These can be considered as examples for the parts $(\mathrm{a} - i)$ *of Theorem 2.1 and* $(\mathrm{a} - ii)$ *of Theorem 2.2 together with*

$$\mathbf{V}_1 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & -2 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \text{ and } \mathbf{V}_2 = \begin{pmatrix} -1 & 0 & -2 & -2 \\ 0 & 2 & 2 & 1 \\ 0 & -1 & -1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix},$$

respectively. Similarly, examples for the other parts of Theorem 2.1 and Theorem 2.2 can be given as above. For example, the matrices given by (30), (31) *and* (32), (33) *provide that the linear combinations $\mathbf{Q} + 2\mathbf{A}$, $3\mathbf{Q} - 2\mathbf{A}$ and $3\mathbf{Q} - 2\mathbf{A}$, $-\frac{1}{2}\mathbf{Q} - 2\mathbf{A}$ are tripotent. It follows that they correspond to the parts* (b_1), $(\mathrm{b}_2 - i)$ *of Theorem 2.1 and* (b_1), $(\mathrm{b}_2 - ii)$ *of Theorem 2.2, respectively.*

$$(30)\quad \begin{pmatrix} -1 & 0 & 1 & -1 & 1 & -1 \\ -2 & 1 & 1 & 0 & 2 & 0 \\ -1 & 1 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & -2 & -1 \\ 2 & -1 & 0 & -1 & -2 & -1 \\ 1 & 0 & -1 & 1 & 1 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 3 & -4 & 5 & -1 & -2 \\ 2 & -5 & 6 & -6 & 0 & 2 \\ 0 & -1 & 3 & -3 & 1 & 1 \\ -2 & 4 & -3 & 3 & 1 & -1 \\ -4 & 12 & -13 & 14 & 0 & -5 \\ 0 & -3 & 4 & -5 & 1 & 2 \end{pmatrix},$$
$$\begin{pmatrix} 6 & -17 & 33/2 & -33/2 & -1/2 & 6 \\ 7 & -37/2 & 31/2 & -15 & -2 & 9/2 \\ 2 & -4 & 1/2 & 1/2 & -5/2 & -1 \\ -9/2 & 13 & -27/2 & 14 & -1/2 & -5 \\ -9/2 & 11 & -17/2 & 8 & 3/2 & -2 \\ -5 & 14 & -27/2 & 27/2 & 1/2 & -5 \end{pmatrix};$$

$$(31)\quad \begin{pmatrix} 0 & -1 & -2 & -1 \\ -1 & 2 & 0 & 0 \\ 1 & -1 & -1 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2/3 & -1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & 2/3 & 2/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -3/2 & 3/2 & 3/2 & 5/2 \\ 3 & -2 & -2 & -4 \\ -3 & 2 & 2 & 9/2 \\ -1 & 1 & 1 & 3/2 \end{pmatrix};$$

$$(32)\quad \begin{pmatrix} 0 & 2 & 2 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 & -2 \\ -1 & -1 & 1 & 2 & 0 & 2 \\ 0 & -1 & 0 & -1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -8 & 10 & -12 & -8 & -8 & -8 \\ -4 & 5 & -6 & -4 & -4 & -4 \\ -2 & 2 & -2 & -2 & -2 & -2 \\ 2 & -2 & 4 & 4 & 3 & 2 \\ 0 & 0 & -2 & -2 & -1 & 0 \\ 5 & -6 & 7 & 5 & 5 & 5 \end{pmatrix},$$
$$\begin{pmatrix} -41/2 & 25 & -31 & -23 & -22 & -21 \\ 11/2 & -11/2 & 27/2 & 27/2 & 19/2 & 11/2 \\ 4 & -5 & 10 & 8 & 6 & 4 \\ -59/2 & 69/2 & -91/2 & -35 & -32 & -59/2 \\ 91/2 & -105/2 & 131/2 & 101/2 & 95/2 & 91/2 \\ 15/2 & -9 & 16 & 29/2 & 23/2 & 8 \end{pmatrix};$$

$$(33)\quad \begin{pmatrix} 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & -2 & 1 & 1 \\ 1 & 1 & -2 & 0 \end{pmatrix}, \begin{pmatrix} 6 & -2 & -4 & -2 \\ 6 & -2 & -4 & -2 \\ 0 & 0 & 0 & 0 \\ 6 & -2 & -4 & -2 \end{pmatrix}, \begin{pmatrix} -1/2 & 1/2 & 0 & 0 \\ -3 & 1 & 2 & 1 \\ 5/2 & 0 & -5/2 & -3/2 \\ -7/2 & 1/2 & 3 & 2 \end{pmatrix}.$$

The matrices in (30), (31), (32), (33) *are given in the order of* $\mathbf{V}$, $\mathbf{Q}$, $\mathbf{A}$.

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