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EIGENVALUES AND CONGRUENCES FOR THE WEIGHT 3 PARAMODULAR NONLIFTS OF LEVELS 61, 73, AND 79

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Abstract. We use Borcherds products to give a new construction of the weight 3 paramodular nonlift eigenform f_N for levels $N = 61, 73, 79$. We classify the congruences of f_N to Gritsenko lifts. We provide techniques that compute eigenvalues to support future modularity applications. Our method does not compute Hecke eigenvalues from Fourier coefficients but instead uses elliptic modular forms, specifically the restrictions of Gritsenko lifts and their images under the slash operator to modular curves.

1. Introduction

Let $\mathcal{S}_3(\mathrm{K}(N))$ denote the space of weight 3 paramodular cusp forms of level N. We compute the nonlift newforms $f_N \in \mathcal{S}_3(\mathrm{K}(N))$ for the prime levels $N = 61, 73, 79$, first computed in [26], by a simple new construction that expresses them as Borcherds products, thereby making their Hecke eigenvalues at bad primes accessible. This new construction also lets us use the integrality of Borcherds product Fourier expansions to prove congruences of Fourier coefficients and of eigenvalues between lifts and nonlifts. Each congruence holds between f_N and a Gritsenko lift g; the congruence holds over the number ring $\mathcal{O}_K$ of the number field $K = \mathbb{Q}(a)$, with $a \in \overline{\mathbb{Z}}$ the $\mathrm{T}(2)$-eigenvalue of g. We find all such congruences. The methods of orthogonal modular forms [4] provide more powerful and versatile methods of proving congruences among eigenvalues for weights $k \geq 3$ but do not address congruences among Fourier coefficients. Also we give speedups for computing Hecke operators. These speedups were necessary for the computations of [3], where they were only partly explained, and they are necessary for the computations of this article. This article gives the speedups for bad prime Hecke operator computations for the first time.

After this introduction and a section that gives some background, Propositions 3.1, 3.5, and 3.6, for $N = 61, 73, 79$, construct the nonlift newform f_N that combines with specified Gritsenko lifts to span $\mathcal{S}_3(\mathrm{K}(N))$, all these basis elements having Fourier coefficients in $\mathbb{Z}$. Each f_N is a finite integer linear

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combination of the Gritsenko lifts plus one more term of the form $m\mathfrak{f}$ where $m \in \mathbb{Z}$ and $\mathfrak{f} = G_i G_j / G_k$ with three of the Gritsenko lifts, and each summand of f_N has Fourier coefficients in $\mathbb{Z}$. All the Gritsenko lifts are also Borcherds products.

In the next section, Theorems 4.3, 4.4, and 4.5 construct congruences for $N = 61, 73, 79$. The theorems are similar, but the levels have distinct features. Each theorem makes reference to eigenvalues of the Hecke operator T(2), usually denoted a.

- For $N = 61$, the polynomial of T(2) is irreducible of degree 6 and a is any of its roots. With $K = \mathbb{Q}(a)$, an $\mathcal{O}_K$-linear combination $g(a)$ of the aforementioned Gritsenko lifts/Borcherds products is an eigenform congruent to f_{61} modulo the $\mathcal{O}_K$-ideal $\mathfrak{a} = \langle 43, a+7 \rangle$, and $\mathfrak{a}$ is the only such ideal. This congruence is proved in [4], along with new types of congruences.
- For $N = 73$, the polynomial of T(2) factors into terms of degrees 1 and 7. The linear term gives rise to an eigenform g_1, an integer linear combination of the relevant Gritsenko lifts/Borcherds products, having T(2)-eigenvalue 9, congruent to f_{73} modulo $3\mathbb{Z}$, and $3\mathbb{Z}$ is the only such ideal. With a any root of the degree-7 polynomial and $K = \mathbb{Q}(a)$, an $\mathcal{O}_K$-linear combination $g_7(a)$ of the Gritsenko lifts/Borcherds products is an eigenform congruent to f_{73} modulo the $\mathcal{O}_K$-ideals $\mathfrak{a} = \langle 3, a \rangle$ and $\mathfrak{b} = \langle 13, a+6 \rangle$, and these are the only such ideals.
- For $N = 79$, the polynomial of T(2) factors into terms of degrees 2 and 5. Similarly to the previous two theorems, these give rise to $g_2(a)$ congruent to f_{79} modulo $\mathfrak{a} = \langle 2, a+1 \rangle$ and this is the only such ideal, and $g_5(b)$ congruent to f_{79} modulo $\mathfrak{b} = \langle 8, w \rangle$ where w is a helpful algebraic integer and $\mathfrak{b}$ is the only such ideal.

Finally, the work of Section 5 leads to the speedups for computing the action of Hecke operators even at bad primes, given in Propositions 5.3 and 5.6. Using these speedups, we were able to compute eigenvalues and Euler polynomials at levels 61, 73, and 79 that confirm values reported by Rama and Tornaría at the companion web page [28] to [27]; a link to our code at GitHub is at [35].

These computations are carried out by the technique of restricting paramodular forms to modular curves, as in [3], with this technique facilitated by our new representations of the nonlifts f_N. The speedups of this article can support restriction computations in weight 2 where the systematic methods of orthogonal modular forms do not apply.

For the interested reader, we give a partial narrative of recent results related to this project.

Golyshev and von Straten [6] recently discovered a Calabi–Yau threefold with conductor $N = 79$. This paper was originally motivated by the idea of supporting a proof of its modularity that would proceed by showing the equivalence of the associated Galois representations, as in [3]; in fact all nonlift paramodular weight 3 cusp forms with rational eigenvalues are candidates for such proofs, and this paper works with the first three cases of prime level. The first appearance of the aforementioned f_N-eigenvalues occurred in the work of Ash, Gunnells, and McConnell [1], who were searching for an element of the cohomology space $H^5(\Gamma_0(N);\mathbb{C})$ that genuinely arises from SL(4). Here $\Gamma_0(N) \subseteq \mathrm{SL}_4(\mathbb{Z})$ is the subgroup of elements having bottom row congruent to $(0,0,0,*)$ mod N. Although they did not find such an element, for $N = 61, 73, 79$ they did see 2 and 3-Euler factors that they believed originated from degree 2 Siegel modular cusp forms.

Paramodular nonlift newforms in $\mathcal{S}_3(\mathrm{K}(N))$ were constructed as holomorphic quotients of Gritsenko lifts in [26] for $N = 61, 73, 79$. Eigenvalues were computed directly from the Fourier coefficients, and enough Fourier coefficients were computed to give the $2, 3$, and 5-Euler factors. The existence of f_{61}, for example, follows from the dimension formula for prime levels due to Ibukiyama [16], namely $\dim \mathcal{S}_3(\mathrm{K}(61)) = 7$, and the dimension of lifts $\dim \mathrm{J}_{3,61}^{\mathrm{cusp}} = 6$ from [5]. Actually, 61 is the lowest level, prime or composite, for which $\mathcal{S}_3(\mathrm{K}(N))$ contains nonlifts; the first such levels are $61, 69, 73, 76, 79, 82, 85, 87, 89$.

Golyshev and Mellit developed an experimental method, relying on the existence of a functional equation, to directly search for L-series. In 2010, Mellit found the first 53 Dirichlet coefficients of a degree 4 L-series with conductor 61 and matched the initial coefficients with the Euler factors for $L(s, f_{61}, \mathrm{spin})$ in the arXiv version of [26].

Supported by increasingly broad dimension formulae, Ibukiyama proposed conjectures relating scalar [14, 15] and vector [17, 19] paramodular forms for GSp(4) of weight $k \geq 3$ to algebraic modular forms on the compact twist $\mathrm{GU}(2, B)$, where B is a definite quaternion algebra. A form of the conjecture in [17] has been proven by van Hoften [34]. The conjecture of Ibukiyama and Kitayama in [19] has been proven by Rösner and Weissauer in [31], and broadened in [4] to allow the discriminant of the quaternion algebra B to properly divide the level N. The main result of Dummigan, Pacetti, Rama, and Tornaría in [4] is to give a correspondence, influenced by [18], between algebraic modular forms for $\mathrm{GU}(2, B)$ and orthogonal modular forms for a carefully chosen quinary lattice. As a consequence, the Hecke eigenvalues of paramodular newforms and certain orthogonal modular forms agree for weight $k \geq 3$ and levels N such that $p \| N$ for some prime p. Fast methods for computing eigenvalues of orthogonal modular forms for prime levels N were developed by a collaboration of Hein, Ladd, and Tornaría [13,21]. Rama and Tornaría [27] extended these methods to orthogonal modular forms with characters involving the spinor norm, and they conjectured a Hecke invariant isomorphism for prime N between $S_3\,(K(N))$ and a direct sum of spaces of orthogonal modular forms with trivial and nontrivial characters. This work was a motivation for [31]. Hein [13] computed Euler factors for $p < 100$ and prime levels $N \leq 197$. The appendix of [21] contains Euler factors for primes $3 \leq p \leq 31$ and prime levels N ranging from

61 to 359. In [27] the 2, 3, and 5-Euler factors are given for squarefree levels $N < 1000$. An even larger data set of eigenvalues of orthogonal modular forms for levels N meeting the condition of [4] has been given by Assaf, Ladd, Rama, Tornaría, and Voight in [2]. This includes the eigenvalues of $T(p^j)$ for similitudes $p^j < 200$, $1 \le j \le 2$, for weight three paramodular newforms.

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2. Background

The set of positive integers is denoted $\mathbb{N}$, the set of nonnegative integers $\mathbb{N}_0$. The integer ring of a number field K is denoted $\mathcal{O}_K$. The $\mathcal{O}_K$-ideal generated by any set $S \subseteq \mathcal{O}_K$ is denoted $\langle S\rangle$, but we usually write $p\mathcal{O}_K$ for $\langle p\rangle$ when p is a rational prime. The ideal norm and the Galois norm from K to $\mathbb{Q}$ are both denoted N, so that $\mathrm{N}(\langle a\rangle) = |\,\mathrm{N}(a)|$ for all $a \in \mathcal{O}_K$.

2.1. Paramodular forms

2.1.1. *Definitions, Fourier series representation.* The degree 2 symplectic group Sp(2) of 4×4 matrices is defined by the condition $g'Jg = J$, where the prime denotes matrix transpose and J is the skew form $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ with each block 2×2. The Klingen parabolic subgroup of Sp(2) is

$$\mathrm{P}_{2,1} = \{ \left(\begin{array}{cc|cc} * & 0 & * & * \\ * & * & * & * \\ \hline * & 0 & * & * \\ 0 & 0 & 0 & * \end{array}\right) \},$$

with either line of three zeros forcing the remaining two because the matrices are symplectic. For any positive integer N, the paramodular group $\mathrm{K}(N)$ of degree 2 and level N is the group of rational symplectic matrices that stabilize the column vector lattice $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus N\mathbb{Z}$. In coordinates,

$$\mathrm{K}(N) = \{ \left(\begin{array}{cc|cc} * & *N & * & * \\ * & * & * & */N \\ \hline * & *N & * & * \\ *N & *N & *N & * \end{array}\right) \in \mathrm{Sp}_2(\mathbb{Q}) : \text{all } * \text{ entries integral} \}.$$

Let $\mathcal{H}_2$ denote the Siegel upper half space of 2×2 symmetric complex matrices that have positive definite imaginary part, generalizing the complex upper half plane $\mathcal{H}$. Elements of this space are written

$$\Omega = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathcal{H}_2,$$

with $\tau, \omega \in \mathcal{H}$, $z \in \mathbb{C}$, and $\mathrm{Im}(\Omega) > 0$. Also, letting $\mathrm{e}(w) = e^{2\pi i w}$ for $w \in \mathbb{C}$, our standard notation throughout is

$$q = \mathrm{e}(\tau), \quad \zeta = \mathrm{e}(z), \quad \xi = \mathrm{e}(\omega).$$

The real symplectic group $\mathrm{Sp}_2(\mathbb{R})$ acts on $\mathcal{H}_2$ as fractional linear transformations, $g(\Omega) = (a\Omega + b)(c\Omega + d)^{-1}$ for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and the automorphy factor is $j(g, \Omega) = \det(c\Omega + d)$. Fix an integer k. Any function $f : \mathcal{H}_2 \longrightarrow \mathbb{C}$ and any real symplectic matrix $g \in \mathrm{Sp}_2(\mathbb{R})$ combine to form another such function through the weight k slash operator,

$$(f|_k g)(\Omega) = j(g, \Omega)^{-k} f(g(\Omega)).$$

When k is well established, we freely write $f|g$ rather than $f|_k g$. A paramodular form of weight k and level N is a holomorphic function $f : \mathcal{H}_2 \longrightarrow \mathbb{C}$ that is $|_k \mathrm{K}(N)$-invariant. The space of weight k, level N paramodular forms is denoted $\mathcal{M}_k(\mathrm{K}(N))$.

A paramodular form of level N has a Fourier expansion

$$f(\Omega) = \sum_{t \in \mathcal{X}_2(N)^{\mathrm{semi}}} a(t; f)\, \mathrm{e}(\langle t, \Omega \rangle)$$

with all $a(t; f) \in \mathbb{C}$, where the index set is

$$\mathcal{X}_2(N)^{\mathrm{semi}} = \{ \left(\begin{smallmatrix} n & r/2 \\ r/2 & mN \end{smallmatrix}\right) : n, m \in \mathbb{N}_0, r \in \mathbb{Z}, 4nmN - r^2 \geq 0 \}$$

and $\langle t, \Omega \rangle = \mathrm{tr}(t\Omega)$. For any subring R of $\mathbb{C}$ we let $\mathcal{M}_k(\mathrm{K}(N))(R)$ denote the R-module of $\mathcal{M}_k(\mathrm{K}(N))$-elements whose Fourier coefficients all lie in R. This notation also applies to all subspaces of $\mathcal{M}_k(\mathrm{K}(N))$ to be introduced below, e.g., $\mathcal{S}_k(\mathrm{K}(N))^+(\mathbb{Z})$ is the $\mathbb{Z}$-module of paramodular cusp forms that are Fricke eigenfunctions having eigenvalue 1 and all Fourier coefficients in $\mathbb{Z}$. An element of $\mathcal{M}_k(\mathrm{K}(N))(R)$ is said to have *unit content* if the ideal generated by its Fourier coefficients is all of R.

The Siegel Φ operator takes any holomorphic function that has a Fourier series of the form $f(\Omega) = \sum_t a(t; f)\, \mathrm{e}(\langle t, \Omega \rangle)$, summing over rational positive semidefinite 2×2 matrices t, to the function $(\Phi f)(\tau) = \lim_{\lambda \to +\infty} f(\left(\begin{smallmatrix} \tau & 0 \\ 0 & i\lambda \end{smallmatrix}\right))$. A paramodular form f in $\mathcal{M}_k(\mathrm{K}(N))$ is called a cusp form if $\Phi(f|_k g) = 0$ for all $g \in \mathrm{Sp}_2(\mathbb{Q})$. This is a finite condition because it only needs to be checked for one representative g of each double coset in Helmut Reefschläger's decomposition ([29], and see Theorem 1.2 of [25]), in which a superscript asterisk denotes matrix inverse-transpose,

$$\mathrm{Sp}_2(\mathbb{Q}) = \bigsqcup_{m \in \mathbb{N}: m | N} \mathrm{K}(N) u(\alpha_m)\, \mathrm{P}_{2,1}(\mathbb{Q}), \qquad \alpha_m = \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right), \; u(\alpha) = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^* \end{smallmatrix}\right).$$

A paramodular form is a cusp form if and only if its Fourier expansion is supported on $\mathcal{X}_2(N)$, defined by the strict inequality $4nmN - r^2 > 0$; this characterization of cusp forms does not hold in general for groups commensurable with $\mathrm{Sp}_2(\mathbb{Z})$, but it does hold for $\mathrm{K}(N)$ because the representatives in Reefschläger's decomposition have block diagonal form. The space of paramodular cusp forms is denoted $\mathcal{S}_k(\mathrm{K}(N))$.

2.1.2. *Symmetric and antisymmetric forms.* The elliptic Fricke involution

$$\alpha_N = \frac{1}{\sqrt{N}}\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} : \tau \longmapsto -\frac{1}{N\tau}$$

normalizes the level N Hecke subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$, and it squares to -1 as a matrix, hence to the identity as a transformation. The corresponding paramodular Fricke involution is

$$\mu_N = \begin{pmatrix} \alpha_N^* & 0 \\ 0 & \alpha_N \end{pmatrix} : \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \longmapsto \begin{pmatrix} \omega N & -z \\ -z & \tau/N \end{pmatrix}.$$

The paramodular Fricke involution normalizes the paramodular group $\mathrm{K}(N)$, and it squares to the identity as a transformation. The space $\mathcal{S}_k(\mathrm{K}(N))$ decomposes as the direct sum of the Fricke eigenspaces for the two eigenvalues ± 1, $\mathcal{S}_k(\mathrm{K}(N)) = \mathcal{S}_k(\mathrm{K}(N))^+ \oplus \mathcal{S}_k(\mathrm{K}(N))^-$. We let ϵ denote either eigenvalue. A paramodular Fricke eigenform is called *symmetric* if $(-1)^k\epsilon = +1$, and *antisymmetric* if $(-1)^k\epsilon = -1$.

2.1.3. *Atkin–Lehner involutions.* Let N be a positive integer, and let c be a positive divisor of N such that $\gcd(c, N/c) = 1$. In this article N is always squarefree, so c can be any positive divisor of N. For any integers $\alpha, \beta, \gamma, \delta$ such that $\alpha\delta c - \beta\gamma N/c = 1$, an elliptic c-Atkin–Lehner matrix is

$$\alpha_c = \frac{1}{\sqrt{c}}\begin{pmatrix} \alpha c & \beta \\ \gamma N & \delta c \end{pmatrix}.$$

Especially, for $c = 1$ we may take $\alpha, \delta = 1$ and $\beta, \gamma = 0$ to get the identity matrix, and for $c = N$ we may take $\alpha, \delta = 0$ and $\beta, \gamma = \mp 1$ to get the Fricke involution matrix $\alpha_N = \frac{1}{\sqrt{N}}\left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$. By quick calculations, the inverse of any α_c is another $\tilde{\alpha}_c$, any product $\tilde{\alpha}_c\alpha_c$ lies in $\Gamma_0(N)$ and so the set of all $\tilde{\alpha}_c$ lies in the coset $\Gamma_0(N)\alpha_c$, and this coset also lies in the set of all $\tilde{\alpha}_c$, making them equal. Consequently, α_c squares into $\Gamma_0(N)$ and normalizes $\Gamma_0(N)$. A paramodular c-Atkin–Lehner matrix is

$$\mu_c = u(\alpha_c^*) = \begin{pmatrix} \alpha_c^* & 0 \\ 0 & \alpha_c \end{pmatrix}.$$

The inverse of any μ_c is another $\tilde{\mu}_c$, and any product $\tilde{\mu}_c\mu_c$ lies in $\mathrm{K}(N)$, so that the set of all $\tilde{\mu}_c$ lies in the coset $\mathrm{K}(N)\mu_c$ and they all give the same action on paramodular forms, although now the containment is proper. Again μ_c squares into $\mathrm{K}(N)$, and a blockwise check shows that μ_c normalizes $\mathrm{K}(N)$. For $c = 1$ we take $\mu_1 = 1_4$. For $c = N$, the paramodular Fricke involution is μ_N from the previous paragraph.

2.2. Hecke operators

The real symplectic group lies in the larger group $\mathrm{GSp}_2^+(\mathbb{R})$ defined by the condition $g'Jg = m(g)J$ for some $m(g) \in \mathbb{R}_{>0}$. The weight k slash operator extends to $(f|_k g)(\Omega) = m(g)^{2k-3} j(g,\Omega)^{-k} f(g(\Omega))$ for $g \in \mathrm{GSp}_2^+(\mathbb{R})$. This is the arithmetic normalization of the slash operator, as compared to the analytic normalization that has $m(g)^k$ instead, compare Schmidt [32]. Three Hecke operators figure in this article, defined in the usual way by double cosets. In particular, let $\mathrm{T}(a,b,c,d)$ abbreviate the double coset $\mathrm{K}(N)\,\mathrm{diag}(a,b,c,d)\mathrm{K}(N)$.

- $\mathrm{T}(p) = \mathrm{T}(1,1,p,p)$. Because the double coset here is also $\mathrm{T}(p,p,1,1)$, used to define the operator $\mathrm{T}_{0,1}(p)$ from [32], the two operators are equal and we make no reference to the notation $\mathrm{T}_{0,1}(p)$ in this article. In general, $\mathrm{T}(a,b,c,d)$, $\mathrm{T}(c,b,a,d)$, $\mathrm{T}(a,d,c,b)$, $\mathrm{T}(c,d,a,b)$ are all equal, i.e., we may exchange the first and third entries, or the second and fourth, or both pairs.
- $\mathrm{T}_1(p^2) = \mathrm{T}(1,p,p^2,p)$. This is a standard operator, used in [3].
- $\mathrm{T}_{1,0}(p^2) = \mathrm{T}(p,p^2,p,1)$. This operator is denoted $\mathrm{T}_{1,0}(p)$ in [32]. We write it $\mathrm{T}_{1,0}(p^2)$ to indicate its multiplier. The operators $\mathrm{T}_1(p^2)$ and $\mathrm{T}_{1,0}(p^2)$ are conjugate under the Fricke operator, $\mu_N^{-1}\mathrm{T}_1(p^2)\mu_N = \mathrm{T}_{1,0}(p^2)$. In general, $\mu_N^{-1}\mathrm{T}(a,b,c,d)\mu_N = \mathrm{T}(b,a,d,c)$, and then the right side has three other names as explained just above.

In this article we use $\mathrm{T}_1(p^2)$ when $p \nmid N$ and $\mathrm{T}_{1,0}(p^2)$ when $p\|N$.

2.2.1. *Fourier–Jacobi expansion.* The Fourier–Jacobi expansion of a paramodular cusp form $f \in \mathcal{S}_k(\mathrm{K}(N))$ is

$$f(\Omega) = \sum_{m\geq 1} \phi_m(f)(\tau,z)\xi^{mN}, \quad \Omega = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix},$$

with Fourier–Jacobi coefficients

$$\phi_m(f)(\tau,z) = \sum_{t=\begin{pmatrix} n & r/2 \\ r/2 & mN \end{pmatrix} \in \mathcal{X}_2(N)} a(t;f) q^n \zeta^r.$$

The Fourier–Jacobi coefficients are also written

$$\phi_m(f)(\tau,z) = \sum_{n,r:4nmN-r^2>0} c\left(n,r;\phi_m\right) q^n \zeta^r.$$

Each Fourier–Jacobi coefficient $\phi_m(f)$ lies in the space $\mathrm{J}_{k,mN}^{\mathrm{cusp}}$ of weight k, index mN Jacobi cusp forms, whose dimension is known. Jacobi forms will briefly be reviewed next.

2.3. Jacobi forms

For the theory of Jacobi forms, see [5,10,33]. Let k be an integer and let m be a nonnegative integer. The complex vector spaces of weight k, index m Jacobi forms, Jacobi cusp forms, and weakly holomorphic Jacobi forms consist of holomorphic functions $g : \mathcal{H} \times \mathbb{C} \longrightarrow \mathbb{C}$ that have Fourier series representations

$$g(\tau, z) = \sum_{n,r} c\,(n, r; g)\, q^n \zeta^r$$

with all $c\,(n, r; g) \in \mathbb{C}$, and that satisfy transformation laws and constraints on the support. With the usual notation $\gamma(\tau) = (a\tau + b)/(c\tau + d)$ and $j(\gamma, \tau) = c\tau + d$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$, the transformation laws are

- $g(\gamma(\tau), z/j(\gamma, \tau)) = j(\gamma, z)^k \mathrm{e}(mcz^2/j(\gamma, \tau))g(\tau, z)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,
- $g(\tau, z + \lambda\tau + \mu) = \mathrm{e}(-m\lambda^2\tau - 2m\lambda z)g(\tau, z)$ for all $\lambda, \mu \in \mathbb{Z}$.

To describe the constraints on the support, associate to any integer pair (n, r) the discriminant

$$D = D(n, r) = 4nm - r^2.$$

The principal part of g is $\sum_{n<0} g_n(\zeta)q^n$, where $g_n(\zeta) = \sum_r c\,(n, r; g)\,\zeta^r$, and the singular part is $\sum_{D(n,r)\le 0} c\,(n, r; g)\, q^n\zeta^r$.

- For the space $\mathrm{J}_{k,m}$ of Jacobi forms, if $m > 0$ then the sum is taken over integers n and r such that $D \ge 0$, so that in particular $n \ge 0$, and if $m = 0$ then the sum is taken over $n \in \mathbb{N}_0$ and $r = 0$, and we have elliptic modular forms.
- For the space $\mathrm{J}_{k,m}^{\mathrm{cusp}}$ of Jacobi cusp forms, if $m > 0$ then the sum is taken over integers n and r such that $D > 0$, so that in particular $n > 0$, and if $m = 0$ then the sum is taken over $n \in \mathbb{N}$ and $r = 0$, and we have elliptic cusp forms.
- For the space $\mathrm{J}_{k,m}^{!}$ of weakly holomorphic Jacobi forms the sum is taken over integers $n \gg -\infty$ and r. For positive index m, the conditions $n \gg -\infty$ and $D \gg -\infty$ are equivalent.

When all the Fourier coefficients lie in a ring we also append the ring to the notation; for example, $\mathrm{J}_{0,m}^{!}(\mathbb{Z})$ denotes the $\mathbb{Z}$-module of weight 0, index m weakly holomorphic forms with integral Fourier coefficients.

2.4. Theta blocks

The theory of theta blocks is due to Gritsenko, Skoruppa, and Zagier [12]. Recall the Dedekind eta function $\eta : \mathcal{H} \longrightarrow \mathbb{C}$ and the odd Jacobi theta function $\vartheta : \mathcal{H} \times \mathbb{C} \longrightarrow \mathbb{C}$,

$$\begin{aligned} \eta(\tau) &= q^{1/24} \prod_{n\ge 1} (1 - q^n), \\ \vartheta(\tau, z) &= \sum_{n\in\mathbb{Z}} (-1)^n q^{(n+1/2)^2/2} \zeta^{n+1/2} \\ &= q^{1/8}(\zeta^{1/2} - \zeta^{-1/2}) \prod_{n\ge 1} (1 - q^n\zeta)(1 - q^n\zeta^{-1})(1 - q^n). \end{aligned}$$

For any $d \in \mathbb{N}$ define ϑ_d to be $\vartheta_d(\tau, z) = \vartheta(\tau, dz)$. Given $k, \ell, d_1, \dots, d_\ell \in \mathbb{N}$, the resulting theta block is defined to be

$$\mathrm{TB}_k[d_1, \dots, d_\ell] = \eta^{2k-\ell} \prod_{j=1}^{\ell} \vartheta_{d_j}.$$

We will use the following result from [12].

Theorem 2.1 ([12]). *Let $k, m, \ell, d_1, \dots, d_\ell \in \mathbb{N}$. Let $\bar{B}_2(x) = B_2(x - \lfloor x \rfloor)$ where $B_2(x) = x^2 - x + 1/6$ is the second Bernoulli polynomial. Then $\mathrm{TB}_k[d_1, \dots, d_\ell] \in \mathrm{J}_{k,m}^{\mathrm{cusp}}$ if and only if $12 \mid k + \ell$, $2m = \sum_{j=1}^{\ell} d_j^2$, and $\frac{k}{12} + \frac{1}{2} \sum_{j=1}^{\ell} \bar{B}_2(d_j x) > 0$ for $0 \le x \le 1$; the positivity needs to be checked only for $x \in [0, 1/2] \cap \frac{1}{2m}\mathbb{Z}$.*

2.5. Gritsenko lifts

The Gritsenko lift, or additive lift, [7] is an injection

$$\mathrm{Grit} : \mathrm{J}_{k,N}^{\mathrm{cusp}} \longrightarrow \mathcal{S}_k(\mathrm{K}(N))^{\epsilon} \quad \text{where } \epsilon = (-1)^k.$$

Its definition uses the Eichler–Zagier [5] index raising operator $V_\ell : \mathrm{J}_{k,m} \longrightarrow \mathrm{J}_{k,m\ell}$, extended to weakly holomorphic Jacobi forms [10]. The Gritsenko lift of $\phi \in \mathrm{J}_{k,N}^{\mathrm{cusp}}$ is

$$\mathrm{Grit}(\phi) \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix} \right) = \sum_{m=1}^{\infty} (\phi | V_m)(\tau, z) \mathrm{e}(m\omega).$$

2.6. Borcherds products

The Borcherds product theorem, quoted here from [23], is a special case of Theorem 3.3 of [11], which in turn is quoted from [8, 10] and relies on the work of Richard Borcherds. In the theorem, $\sigma_0(m)$ denotes the number of positive divisors of the positive integer m.

Theorem 2.2. *Let N be a positive integer. Let $\psi \in \mathrm{J}_{0,N}^{!}$ be a weakly holomorphic weight 0, index N Jacobi form, having Fourier expansion*

$$\psi(\tau, z) = \sum_{\substack{n, r \in \mathbb{Z} \\ n \gg -\infty}} c(n, r) q^n \zeta^r \quad \text{where } q = \mathrm{e}(\tau),\ \zeta = \mathrm{e}(\zeta).$$

Define

$$A = \frac{1}{24} c(0,0) + \frac{1}{12} \sum_{r \ge 1} c(0, r), \quad B = \frac{1}{2} \sum_{r \ge 1} r\, c(0, r),$$

$$C = \frac{1}{2} \sum_{r \ge 1} r^2 c(0, r), \qquad D_0 = \sum_{n \le -1} \sigma_0(|n|) c(n, 0).$$

Suppose that the following conditions hold:

(1) *$c(n, r) \in \mathbb{Z}$ for all integer pairs (n, r) such that $4nN - r^2 \le 0$,*

(2) *$A \in \mathbb{Z}$,*

(3) $\sum_{i\geq 1} c(i^2nm, ir) \geq 0$ *for all primitive integer triples* (n,m,r) *such that* $4nmN - r^2 < 0$ *and* $m \geq 0$.

Then for weight $k = \frac{1}{2}c(0,0)$ *and Fricke eigenvalue* $\epsilon = (-1)^{k+D_0}$ *the Borcherds product* $\mathrm{Borch}(\psi)$ *lies in* $\mathcal{M}_k(\mathrm{K}(N))^\epsilon$. *For sufficiently large* λ, *for* $\Omega = \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) \in \mathcal{H}_2$ *and* $\xi = \mathrm{e}(\omega)$, *the Borcherds product has the following convergent product expression on the subset* $\{\,\mathrm{Im}(\Omega) > \lambda I_2\,\}$ *of* $\mathcal{H}_2$:

$$\mathrm{Borch}(\psi)(\Omega) = q^A\zeta^B\xi^C \prod_{\substack{n,m,r\in\mathbb{Z},\ m\geq 0 \\ \text{if } m=0 \text{ then } n\geq 0 \\ \text{if } m=n=0 \text{ then } r<0}} (1 - q^n\zeta^r\xi^{mN})^{c(nm,r)}.$$

Also, let $\lambda(r) = c(0,r)$ *for* $r \in \mathbb{N}_0$, *and recall the corresponding theta block,*

$$\mathrm{TB}(\lambda)(\tau,z) = \eta(\tau)^{\lambda(0)} \prod_{r\geq 1} (\vartheta_r(\tau,z)/\eta(\tau))^{\lambda(r)} \qquad \text{where } \vartheta_r(\tau,z) = \vartheta(\tau, rz).$$

On $\{\,\mathrm{Im}(\Omega) > \lambda I_2\,\}$, *the Borcherds product is a rearrangement of a convergent infinite series,*

$$\mathrm{Borch}(\psi)(\Omega) = \mathrm{TB}(\lambda)(\tau,z)\xi^C \exp\left(-\,\mathrm{Grit}(\psi)(\Omega)\right).$$

In the theorem, the divisor of the Borcherds product $\mathrm{Borch}(\psi)$ is a sum of Humbert surfaces with multiplicities, the multiplicities necessarily nonnegative for holomorphy. Let $\mathrm{K}(N)^+$ denote the group generated by $\mathrm{K}(N)$ and the paramodular Fricke involution μ_N. The sum in item (3) of the theorem is the multiplicity of the following Humbert surface in the divisor, in which $t_o = \left(\begin{smallmatrix} n & r/2 \\ r/2 & mN \end{smallmatrix}\right)$ and $D = 4nmN - r^2 < 0$,

$$H_N(t_o) = H_N(-D, r) = \mathrm{K}(N)^+\{\,\Omega \in \mathcal{H}_2 : \langle \Omega, t_o\rangle = 0\,\} \subseteq \mathrm{K}(N)^+\backslash\mathcal{H}_2.$$

This surface depends only on the discriminant D and on r, so that we may take t_o with $m = 1$, and furthermore, by work of Gritsenko and Hulek [9], it depends only on the residue class of r modulo $2N$.

3. Construction of the nonlift newforms f_N

The following proposition, other than its last statement, was proven in [26] using the method of integral closure. Here we give a new proof using Borcherds products, as indicated in the added last statement. This new proof works directly in the weight 3 space with no need to span the weight 6 space as the integral closure method did. As a byproduct of this construction, f_{61} is clearly congruent to a Gritsenko lift modulo 43, because each $G[i]$ and also $G[1]G[6]/G[2]$ in the equality just below have integral Fourier coefficients.

Proposition 3.1. *There is a nonlift Hecke eigenform* $f_{61} \in \mathcal{S}_3(\mathrm{K}(61))^-(\mathbb{Z})$ *with unit content, given by a rational function of Gritsenko lifts,*

$$f_{61} = -9G[1] - 2G[2] + 22G[3] + 9G[4] - 10G[5] + 19G[6] - 43\frac{G[1]G[6]}{G[2]}.$$

Here $G[j] = \mathrm{Grit}\,(\mathrm{TB}_3(D_j))$ *for* $j = 1, \ldots, 6$ *are Gritsenko lifts of theta blocks, with*

$$\begin{aligned} D_1 &= [2,2,2,3,3,3,3,5,7] & D_2 &= [2,2,2,2,3,4,4,4,7] \\ D_3 &= [2,2,2,2,3,3,4,6,6] & D_4 &= [1,2,3,3,3,3,4,4,7] \\ D_5 &= [1,2,3,3,3,3,3,6,6] & D_6 &= [1,2,2,2,4,4,4,5,6]. \end{aligned}$$

The set $\{f_{61}, G[1], \ldots, G[6]\}$ *is a basis of* $\mathcal{S}_3(\mathrm{K}(61))$*. Each Gritsenko lift* $G[j]$ *is also a Borcherds product* $B[j] = \mathrm{Borch}(-(\mathrm{TB}_3(D_j)|V_2)/\mathrm{TB}_3(D_j))$*.*

The existence of f_{61} follows from the dimension formula of Ibukiyama for prime levels p in [16], stated below for convenience. A more general dimension formula for weights $k \geq 3$ and squarefree levels $N \geq 3$ is due to Ibukiyama and Kitayama [19].

Theorem 3.2 (Ibukiyama dimension formula)**.** *Let* $p \geq 5$ *be prime. Then*

$$\begin{aligned} \dim \mathcal{S}_3(\mathrm{K}(p)) = {} & \frac{1}{2880}(p^2-1) - 1 \\ & + \frac{1}{64}(p+1)\left(1 - \left(\frac{-1}{p}\right)\right) + \frac{5}{192}(p-1)\left(1 + \left(\frac{-1}{p}\right)\right) \\ & + \frac{1}{72}(p+1)\left(1 - \left(\frac{-3}{p}\right)\right) + \frac{1}{36}(p-1)\left(1 + \left(\frac{-3}{p}\right)\right) \\ & + \frac{1}{8}\left(1 - \left(\frac{2}{p}\right)\right) + \frac{1}{5}\left(1 - \left(\frac{5}{p}\right)\right) + \left\{\frac{1}{6} \text{ if } p \equiv 5 \bmod 12\right\}. \end{aligned}$$

Ibukiyama's formula gives $\dim \mathcal{S}_3(\mathrm{K}(61)) = 7$, and the dimension formula for Jacobi forms in [5],

$$\dim \mathrm{J}_{3,m}^{\mathrm{cusp}} = \sum_{j=1}^{m-1} \left(\dim \mathcal{S}_{2+2j}(\mathrm{SL}_2(\mathbb{Z})) - \mathrm{floor}\left(j^2/(4m)\right)\right),$$

gives $\dim \mathrm{J}_{3,61}^{\mathrm{cusp}} = 6$. We begin spanning $\mathcal{S}_3(\mathrm{K}(61))$ by using theta blocks to span $\mathrm{J}_{3,61}^{\mathrm{cusp}}$. With the D_j from Proposition 3.1, Theorem 2.1 shows that $\mathrm{TB}_3(D_j) \in \mathrm{J}_{3,61}^{\mathrm{cusp}}$ for $j = 1, \ldots, 6$. By computing Fourier coefficients [35], we see that the theta blocks $\mathrm{TB}_3(D_j)$ form a basis of $\mathrm{J}_{3,61}^{\mathrm{cusp}}$ and the Gritsenko lifts $G[j] = \mathrm{Grit}\,(\mathrm{TB}_3(D_j))$ a basis of the Gritsenko lift subspace in $\mathcal{S}_3(\mathrm{K}(61))$. We finish spanning $\mathcal{S}_3(\mathrm{K}(61))$ by using a special case of the Borcherds Products Everywhere theorem. The following result is Theorem 6.6 of [11] specialized to the case where the q-order of vanishing is 1, and so we call it a corollary.

Corollary 3.3. *Let* $k, N, \ell, d_1, \ldots, d_\ell \in \mathbb{N}$*. Assume* $\phi = \mathrm{TB}_k[d_1, \ldots, d_\ell] \in \mathrm{J}_{k,N}^{\mathrm{cusp}}$ *and* $k + \ell = 12$*. Let* $\psi = -\frac{\phi|V_2}{\phi}$*. Then* $\psi \in \mathrm{J}_{0,N}^{!}(\mathbb{Z})$*,* $\mathrm{Borch}(\psi) \in \mathcal{M}_k(\mathrm{K}(N))^{\epsilon}$ *for* $\epsilon = (-1)^k$*, and* $\mathrm{Borch}(\psi)$ *and* $\mathrm{Grit}(\phi)$ *have equal first and second Fourier–Jacobi coefficients.*

For $j = 1, \dots, 6$ set $\psi_j = -(\mathrm{TB}_3(D_j)|V_2)/\mathrm{TB}_3(D_j)$ and $B[j] = \mathrm{Borch}(\psi_j)$. Thus $\psi_j \in \mathrm{J}^!_{0,61}(\mathbb{Z})$ and $B[j] \in \mathcal{M}_3(\mathrm{K}(61))$ by the corollary. The divisor of $B[j]$ consists of Humbert surfaces, the multiplicity of $B[j]$ on $H_{61}\begin{pmatrix} n_o & r_o/2 \\ r_o/2 & 61m_o \end{pmatrix}$ being $\sum_{i\geq 1} c\left(i^2 n_o m_o, i r_o; \psi_j\right)$. The divisors of $B[1], B[2]$, and $B[6]$ in Table 1 show that $\mathfrak{f} = B[1]B[6]/B[2]$ is also holomorphic because the Humbert multiplicities in its divisor are all nonnegative, and so $\mathfrak{f} \in \mathcal{M}_3(\mathrm{K}(61))^-$. In fact $\mathfrak{f} \in \mathcal{S}_3(\mathrm{K}(61))$ because the divisor of $\mathfrak{f}$ contains $H_{61}(1,1)$ and $N = 61$ is prime, as the following lemma shows.

Lemma 3.4. *Let N be prime and let $f \in \mathcal{M}_k(\mathrm{K}(N))^{\pm}$ be a Fricke eigenform. If f vanishes on the Humbert surface $H_N(1,1)$ then f is a cusp form.*

Proof. The Humbert surface $H_N(1,1)$ is $\mathrm{K}(N)^+\hat{H}_N(t_o)$ where $t_o = \begin{pmatrix} 0 & 1/2 \\ 1/2 & N \end{pmatrix}$ and $\hat{H}_N(t_o) = \{\Omega \in \mathcal{H}_2 : \langle \Omega, t_o \rangle = 0\}$. Define $t[u] = u'tu$ for compatibly sized matrices t and u. To parametrize $\hat{H}_N(t_o)$, choose $\alpha \in \mathrm{SL}_2(\mathbb{R})$ having columns v_1 and v_2 such that $t_o[v_1] = t_o[v_2] = 0$; we may take either of $\alpha = \left(\begin{smallmatrix} N & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & -N \\ 0 & 1 \end{smallmatrix}\right)$. The parametrization is

$$W_\alpha : \mathcal{H} \times \mathcal{H} \longrightarrow \hat{H}_N(t_o), \qquad (\tau, \omega) \longmapsto \tau v_1 v_1' + \omega v_2 v_2' = \alpha \left(\begin{smallmatrix} \tau & 0 \\ 0 & \omega \end{smallmatrix}\right) \alpha'.$$

Recall that $u(\alpha) = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^* \end{smallmatrix}\right)$. The parametrization shows that because f vanishes on $\hat{H}_N(t_o)$, also $f|_k u(\alpha) \left(\begin{smallmatrix} \tau & 0 \\ 0 & \omega \end{smallmatrix}\right)$ vanishes on $\mathcal{H} \times \mathcal{H}$ and so clearly $\Phi(f|_k u(\alpha))(\tau) = \lim_{\lambda \to +\infty} (f|_k u(\alpha)) \left(\begin{smallmatrix} \tau & 0 \\ 0 & i\lambda \end{smallmatrix}\right)$ is 0 for all $\tau \in \mathcal{H}$.

As in section 2.1.1, the cusp form condition for f is that $\Phi(f|_k g) = 0$ for one representative g of each double coset $\mathrm{K}(N)u(\alpha_m)\,\mathrm{P}_{2,1}(\mathbb{Q})$ where $0 < m \mid N$ and $\alpha_m = \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$. Because N is prime here, the only cases are $m = 1, N$, with representatives $g = u(\alpha)$ for $\alpha = \left(\begin{smallmatrix} N & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & -N \\ 0 & 1 \end{smallmatrix}\right)$ from the previous paragraph; the former membership holds because $\alpha = \left(\begin{smallmatrix} N+1 & -N \\ -1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$ and u is a homomorphism, the latter because $u\left(\begin{smallmatrix} 1 & 2N \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{K}(N)$. The previous paragraph has shown that $\Phi(f|_k u(\alpha)) = 0$ for these two α, so the proof is complete. □

The Fourier coefficients [35] of $G[1], \dots, G[6]$, and $\mathfrak{f}$ show them to span $\mathcal{S}_3(\mathrm{K}(61))$, so that $\mathfrak{f}$ is a nonlift. Further, the Fourier coefficients show that each element of $\mathcal{S}_3(\mathrm{K}(61))$ is determined by its first and second Fourier–Jacobi coefficients, so Corollary 3.3 gives the *equality* of each $B[j] = \mathrm{Borch}\,(\psi_j)$ and $G[j] = \mathrm{Grit}\,(\mathrm{TB}_3(D_j))$, and the Borcherds products are Gritsenko lifts.

With $\mathcal{S}_3(\mathrm{K}(61))$ spanned, it is a matter of linear algebra to compute the action of $\mathrm{T}(2)$ on the basis, assuming we can compute enough Fourier coefficients, and then to obtain the nonlift eigenform $f_{61} = -9G[1]-2G[2]+22G[3]+9G[4]-10G[5]+19G[6]-43\mathfrak{f}$, a rational function of Gritsenko lifts. By the product expansion in Theorem 2.2, a Borcherds product $\mathrm{Borch}(\psi)$ has integral Fourier coefficients when ψ does, and noting that $\mathfrak{f}$ is also a Borcherds product because Borcherds product formation is an additive-to-multiplicative homomorphism, f_{61} has integral Fourier coefficients. Also f_{61} has unit content, as shown by

TABLE 1. Divisors of $B[1], B[2], B[6]$, and $\mathfrak{f}$ in $\mathcal{S}_3(\mathrm{K}(61))$

	$B[1]$	$B[2]$	$B[6]$	$\mathfrak{f} = B[1]B[6]/B[2]$
$H_{61}(1,1)$	9	9	9	9
$H_{61}(4,2)$	3	7	7	3
$H_{61}(5,35)$	7	7	2	2
$H_{61}(9,3)$	4	1	1	4
$H_{61}(13,47)$	3	0	0	3
$H_{61}(16,4)$	0	3	3	0
$H_{61}(20,52)$	4	3	1	2
$H_{61}(25,5)$	1	0	1	2
$H_{61}(36,6)$	0	0	1	1
$H_{61}(41,23)$	0	0	1	1
$H_{61}(49,7)$	1	1	0	0
$H_{61}(56,42)$	1	0	0	1
$H_{61}(65,59)$	0	1	1	0

the Fourier coefficients $a((\begin{smallmatrix}1&5\\5&61\end{smallmatrix}); f_{61}) = -75$ and $a((\begin{smallmatrix}1&6\\6&61\end{smallmatrix}); f_{61}) = 107$. This completes the new proof of Proposition 3.1.

The same argument works for $N = 73$ and 79; we only present the key elements. Here $\dim \mathcal{S}_3(\mathrm{K}(73)) = 9$ and the lift space dimension is $\dim \mathrm{J}_{3,73}^{\mathrm{cusp}} = 8$, and $\dim \mathcal{S}_3(\mathrm{K}(79)) = 8$ and the lift space dimension is $\dim \mathrm{J}_{3,79}^{\mathrm{cusp}} = 7$.

Proposition 3.5. *There is a nonlift Hecke eigenform $f_{73} \in \mathcal{S}_3(\mathrm{K}(73))^-(\mathbb{Z})$ with unit content, given by a rational function of Gritsenko lifts*

$$f_{73} = 9G[1] + 19G[2] + 2G[3] - 13G[4] + 34G[5] - 15G[6] - 12G[7] - 10G[8] - 39\frac{G[2]G[6]}{G[4]}.$$

Here $G[j] = \mathrm{Grit}\,(\mathrm{TB}_3(E_j))$ for $j = 1, \ldots, 8$ are Gritsenko lifts of theta blocks, with

$$E_1 = [2,3,3,3,3,4,4,5,7] \qquad E_2 = [2,3,3,3,3,3,5,6,6]$$
$$E_3 = [2,2,3,4,4,4,4,4,7] \qquad E_4 = [2,2,3,3,4,4,4,6,6]$$
$$E_5 = [2,2,3,3,3,5,5,5,6] \qquad E_6 = [2,2,2,4,4,4,5,5,6]$$
$$E_7 = [2,2,2,2,3,4,4,5,8] \qquad E_8 = [2,2,2,2,2,4,5,6,7].$$

The set $\{f_{73}, G[1], \ldots, G[8]\}$ is a basis of $\mathcal{S}_3(\mathrm{K}(73))$. Each Gritsenko lift $G[j]$ is also a Borcherds product $B[j] = \mathrm{Borch}(-(\mathrm{TB}_3(E_j)|V_2)/\mathrm{TB}_3(E_j))$.

Proof. The proof follows the pattern of the proof for $N = 61$ in Proposition 3.1. The Fourier coefficients $a(\begin{pmatrix}1&13/2\\13/2&73\end{pmatrix}; f_{73}) = 7$ and $a((\begin{smallmatrix}1&-7\\-7&73\end{smallmatrix}); f_{73}) = -6$ prove unit content, and see Table 2 for the holomorphy of f_{73}. □

TABLE 2. Divisors of $B[2], B[4], B[6]$, and $B[2]B[6]/B[4]$ in $\mathcal{S}_3(\mathrm{K}(73))$

	$B[2]$	$B[4]$	$B[6]$	$B[2]B[6]/B[4]$
$H_{73}(1,1)$	9	9	9	9
$H_{73}(4,2)$	3	7	7	3
$H_{73}(8,64)$	1	3	3	1
$H_{73}(9,3)$	7	4	1	4
$H_{73}(12,42)$	0	1	1	0
$H_{73}(16,4)$	0	3	3	0
$H_{73}(24,30)$	0	1	3	2
$H_{73}(25,5)$	1	0	2	3
$H_{73}(36,6)$	2	2	1	1
$H_{73}(37,57)$	2	0	0	2
$H_{73}(48,62)$	0	0	1	1
$H_{73}(65,49)$	1	0	0	1
$H_{73}(72,46)$	1	1	0	0
$H_{73}(73,73)$	0	2	2	0

The next proposition gives a simpler expression for f_{79} than was given in [26]. This simpler expression makes the congruence modulo 32 visible.

Proposition 3.6. *There is a nonlift Hecke eigenform* $f_{79} \in \mathcal{S}_3(\mathrm{K}(79))^-(\mathbb{Z})$ *with unit content, given by a rational function of Gritsenko lifts*

$$f_{79} = 4G[1] + 13G[2] - 15G[3] + 8G[4] + 5G[6] - 11G[7] - 32\frac{G[2]G[3]}{G[1]}.$$

Here $G[j] = \mathrm{Grit}\,(\mathrm{TB}_3(F_j))$ *for* $j = 1, \ldots, 7$ *are Gritsenko lifts of theta blocks with*

$$F_1 = [1,2,2,2,2,3,4,4,10] \qquad F_2 = [2,2,2,2,4,4,5,6,7]$$
$$F_3 = [1,1,1,1,2,3,4,5,10] \qquad F_4 = [2,2,2,2,2,4,4,5,9]$$
$$F_5 = [1,3,3,3,3,4,4,5,8] \qquad F_6 = [1,1,1,1,1,2,2,8,9]$$
$$F_7 = [1,2,2,3,3,3,4,5,9].$$

The set $\{f_{79}, G[1], \ldots, G[7]\}$ *is a basis of* $\mathcal{S}_3(\mathrm{K}(79))$*. Each Gritsenko lift* $G[j]$ *is also a Borcherds product* $B[j] = \mathrm{Borch}(-(\mathrm{TB}_3(F_j)|V_2)/\mathrm{TB}_3(F_j))$.

Proof. Again the proof follows from that of Proposition 3.1. The Fourier coefficients $a(\left(\begin{smallmatrix} 1 & 6 \\ 6 & 79 \end{smallmatrix}\right); f_{79}) = 58$ and $a(\left(\begin{smallmatrix} 1 & -7 \\ -7 & 79 \end{smallmatrix}\right); f_{79}) = 101$ prove unit content, and see Table 3 for the holomorphy of f_{79}. $\square$

TABLE 3. Divisors of $B[1], B[2], B[3]$, and $B[2]B[3]/B[1]$ in $\mathcal{S}_3(\mathrm{K}(79))$

	$B[1]$	$B[2]$	$B[3]$	$B[2]B[3]/B[1]$
$H_{79}(1,1)$	9	9	9	9
$H_{79}(4,2)$	7	7	3	3
$H_{79}(5,59)$	2	2	2	2
$H_{79}(8,18)$	1	1	1	1
$H_{79}(9,3)$	1	1	1	1
$H_{79}(13,31)$	1	0	1	0
$H_{79}(16,4)$	2	2	1	1
$H_{79}(20,40)$	1	1	0	0
$H_{79}(21,69)$	0	2	0	2
$H_{79}(25,5)$	1	1	2	2
$H_{79}(36,6)$	0	1	0	1
$H_{79}(40,44)$	0	0	1	1
$H_{79}(44,26)$	0	1	0	1
$H_{79}(45,19)$	0	0	1	1
$H_{79}(49,7)$	0	1	0	1
$H_{79}(65,67)$	1	1	0	0
$H_{79}(76,32)$	0	0	1	1
$H_{79}(80,78)$	1	1	0	0
$H_{79}(100,10)$	1	0	1	0

4. Classification of congruences to Gritsenko lifts

A few observations will repeatedly be handy for proving Fourier coefficient-wise congruences of eigenforms, and so we spell them out quickly for the reader's convenience even though they are very easy or well known.

- Let p be a rational prime. Let K be a number field. Consider an element a of $\mathcal{O}_K$ and the $\mathcal{O}_K$-ideal $\mathfrak{a} = \langle p, a\rangle$. If $p \mid \mathrm{N}(a)$ then $\mathfrak{a}$ is divisible by a prime ideal over p, and in particular $\mathfrak{a}$ is not all of $\mathcal{O}_K$. If $p \| \mathrm{N}(a)$ then $\mathfrak{a}$ is prime and $\mathrm{N}(\mathfrak{a}) = p$.
- Let $k, \ell \in \mathbb{N}$ be coprime. Let K be a number field, $\mathfrak{u} \subseteq \mathcal{O}_K$ an ideal, and $c \in \mathcal{O}_K$ an element. If $\ell c \in \mathfrak{u}$ then $c \in \mathfrak{u} + k\mathcal{O}_K$.
- The Kummer–Dedekind theorem (if $K = \mathbb{Q}(a)$ is a number field with $a \in \overline{\mathbb{Z}}$, and $\varphi(x) \in \mathbb{Z}[x]$ is the minimal polynomial of a, then for all $p \nmid [\mathcal{O}_K : \mathbb{Z}[a]]$ the factorization of φ modulo $p\mathbb{Z}[x]$ gives the factorization of p in $\mathcal{O}_K$) applies when $p^2 \nmid \mathrm{disc}(\varphi)$. We mention this because the condition that one power of p can divide $\mathrm{disc}(\varphi)$ seems not to be prominent in textbooks, and we use it several times below.

The next lemma gives the action of the standard Hecke operators on Fourier coefficients when the bad prime divides the level at most once.

Lemma 4.1. *Let $f \in \mathcal{S}_k(\mathrm{K}(N))$ with $k \geq 3$ and $N \in \mathbb{N}$. Let $p \in \mathbb{N}$ be prime with $p \nmid N$ or $p \| N$. If $p \nmid N$ then let* T *be either of* $\mathrm{T}(p), \mathrm{T}_1(p^2)$*, and if* $p\|N$ *then let* T *be either of* $\mathrm{T}(p), \mathrm{T}_{1,0}(p^2)$*. Then the Fourier coefficients of* $f|_k\mathrm{T}$ *are fixed* $\mathbb{Z}$*-linear combinations of the Fourier coefficients of* f*, independent of* f*.*

Proof. For $p \nmid N$ and $\mathrm{T} = \mathrm{T}(p), \mathrm{T}_1(p^2)$, see [3], pp.1165–1168.

For $p\|N$ and $\mathrm{T} = \mathrm{T}(p), \mathrm{T}_{1,0}(p^2)$, the result follows from the following formulas, which hold for $k \geq 0$ and clearly have integral coefficients for $k \geq 3$. Let $M \equiv (N/p)^{-1} \bmod p$ and let $a, c \in \mathbb{Z}$ be such that $ap - cN/p = 1$, so that $NM/p \equiv 1 \bmod p$ and $ap^2 - cN = p$. Then for any $t \in \mathcal{X}_2(N)$, recalling that $t[u] = u'tu$ for compatibly sized matrices t and u,

$$\begin{aligned} a(t; f|\mathrm{T}(p)) = {} & a(pt; f) + p^{k-2} \sum_{x \bmod p} a(\tfrac{1}{p}\, t\left[\left(\begin{smallmatrix} 1 & 0 \\ -x & p \end{smallmatrix}\right)\right]; f) \\ & + p^{k-2} \sum_{y \bmod p} a(\tfrac{1}{p}\, t\left[\left(\begin{smallmatrix} p & NMy \\ 0 & 1 \end{smallmatrix}\right)\right]; f) + p^{2k-3} a(\tfrac{1}{p}\, t; f) \\ & + p^{k-3} \begin{Bmatrix} p-1 & \text{if } p \mid 2t_{12} \\ -1 & \text{else} \end{Bmatrix} a(\tfrac{1}{p}\, t\left[\left(\begin{smallmatrix} ap & N \\ c & p \end{smallmatrix}\right)\right]; f), \end{aligned}$$

and

$$\begin{aligned} a(t; f|\mathrm{T}_{1,0}(p^2)) = {} & p^{k-3} \sum_{x \bmod p} a(t\left[\left(\begin{smallmatrix} 1 & 0 \\ -x & p \end{smallmatrix}\right)\right]; f) \\ & + p^{3k-6} \sum_{y \bmod p} a(t\left[\left(\begin{smallmatrix} 1 & NMy/p \\ 0 & 1/p \end{smallmatrix}\right)\right]; f) \\ & + p^{2k-6} \sum_{y \bmod p} \left\{\begin{matrix} p-1 & \text{if } p \left| \begin{matrix} 2t_{12}(1 + 2cN/p + 2cy) \\ + 2t_{22}c/p \end{matrix}\right. \\ -1 & \text{else} \end{matrix}\right\} \\ & \quad \cdot a(t\left[\left(\begin{smallmatrix} (cN+p+cNMy)/p & N+NMy \\ c/p & 1 \end{smallmatrix}\right)\right]; f) \\ & + p^{2k-6} \sum_{x,y \bmod p} \begin{Bmatrix} p-1 & \text{if } p \mid 2t_{12}My + t_{22}/N \\ -1 & \text{else} \end{Bmatrix} \\ & \quad \cdot a(t\left[\left(\begin{smallmatrix} 1+NMxy/p & NMy \\ x/p & 1 \end{smallmatrix}\right)\right]; f). \end{aligned}$$

These formulas are derived from the double coset formulas for $\mathrm{T}(p)$ and $\mathrm{T}_{1,0}(p^2)$ when $p\|N$, as given in [32]. The derivation relies on the following result: with p, N, M, a, c as above, also consider any $x \neq 0 \bmod p$, and let $\hat{x} = x^{-1} \bmod p$, so that $x\hat{x} = 1 \bmod p$. Let $S \in \mathrm{K}(N)$ be the matrix

$$S = \begin{pmatrix} ap & N & N\hat{x}/p & a\hat{x} \\ c & p & \hat{x} & c\hat{x}/p \\ cNMx/p & NMx & 1 + NMx\hat{x}/p & c(NMx\hat{x}/p - 1)/p \\ aNMx & N^2Mx/p & N(NMx\hat{x}/p - 1)/p & a + aNMx\hat{x}/p \end{pmatrix}$$

and let U be the blockwise upper triangular matrix

$$U = \begin{pmatrix} 1 & -N/p & -N\hat{x}/p & -\hat{x} \\ -c/p & a & -a\hat{x} & -c\hat{x}/p \\ 0 & 0 & ap & c \\ 0 & 0 & N & p \end{pmatrix}.$$

Then their product is the 0-dimensional cusp $C_0(NMx/p)$ as defined in [25] p.449.

$$SU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & NMx/p & 1 & 0 \\ NMx/p & 0 & 0 & 1 \end{pmatrix}.$$

From here we can upper triangularize all the matrices in the formulas from [32], and we get the above two Fourier coefficient formulas. The formulas from [32] give the analytic, scalar invariant normalization of the slash operator. So, beyond upper triangularizing, the first formula from [32] has to be multiplied by p^{k-3} and the second by p^{2k-6} for the arithmetic normalization. □

In the following lemma the ideal $\mathfrak{a}$ does not need to be prime.

Lemma 4.2. *Let $k \geq 3$ and N be positive integers. Let $p \in \mathbb{N}$ be prime with $p \nmid N$ or $p \| N$. If $p \nmid N$ then let T be either of $\mathrm{T}(p), \mathrm{T}_1(p^2)$, and if $p\|N$ then let T be either of $\mathrm{T}(p), \mathrm{T}_{1,0}(p^2)$. Let K be a number field. Consider two T-eigenforms lying in the same Fricke space, and having the same Atkin–Lehner sign ϵ_p if $p\|N$, these being $f \in \mathcal{S}_k(\mathrm{K}(N))^{\pm}(\mathbb{Z})$, having unit content, and $g \in \mathcal{S}_k(\mathrm{K}(N))^{\pm}(\mathcal{O}_K)$. Let $\mathfrak{a} \subseteq \mathcal{O}_K$ be an ideal, and assume that f and g have congruent Fourier coefficients modulo $\mathfrak{a}$,*

$$a(t; f) \equiv a(t; g) \bmod \mathfrak{a} \quad \text{for all } t \in \mathcal{X}_2(N).$$

Then f and g have congruent T-eigenvalues modulo $\mathfrak{a}$,

$$\lambda_f(\mathrm{T}) \equiv \lambda_g(\mathrm{T}) \bmod \mathfrak{a}.$$

If $p \nmid N$ and f and g are T-eigenforms for both $\mathrm{T} = \mathrm{T}(p)$ and $T = \mathrm{T}_1(p^2)$, or if $p\|N$ and f and g are T-eigenforms for both $\mathrm{T} = \mathrm{T}(p)$ and $T = \mathrm{T}_{1,0}(p^2)$, then f and g have congruent p-Euler polynomials modulo $\mathfrak{a}$.

Proof. By Lemma 4.1, each $a(t_o; f|_k\mathrm{T}) = \lambda_f(\mathrm{T})a(t_o; f)$ is a $\mathbb{Z}$-linear combination of the Fourier coefficients $a(t; f)$. Thus $\lambda_f(\mathrm{T})$ lies in $\mathbb{Q}$. Also the characteristic polynomial of T is monic in $\mathbb{Z}[x]$, making its roots algebraic integers, so in fact $\lambda_f(\mathrm{T})$ lies in $\mathbb{Z}$. Similarly $\lambda_g(\mathrm{T})$ lies in $\mathcal{O}_K$. For all $t \in \mathcal{X}_2(N)$, because $a(t; f)$ and $a(t; g)$ are equivalent modulo $\mathfrak{a}$, and because $a(t; f|_k\mathrm{T})$ and $a(t; g|_k\mathrm{T})$ are the same $\mathbb{Z}$-linear combination of the $a(u; f)$ and the $a(u; g)$, they are equivalent modulo $\mathfrak{a}$ as well. So, because f and g are eigenforms, $(\lambda_f(\mathrm{T}) - \lambda_g(\mathrm{T}))a(t; f) = a(t; f|_k\mathrm{T}) - a(t; g|_k\mathrm{T})$ lies in $\mathfrak{a}$. Because f has unit content some finite $\mathbb{Z}$-linear combination $\sum_t n_t a(t;f)$ is 1, and so the eigenvalue difference $\lambda_f(\mathrm{T}) - \lambda_g(\mathrm{T}) = \sum_t n_t(\lambda_f(\mathrm{T}) - \lambda_g(\mathrm{T}))a(t;f)$ lies in $\mathfrak{a}$. Thus $\lambda_f(\mathrm{T}) \equiv \lambda_g(\mathrm{T}) \bmod \mathfrak{a}$.

As for the p-Euler polynomials of f and g, for $p \nmid N$ let $f|_k\mathrm{T}(p) = \lambda_f(\mathrm{T}(p))f$ and $f|_k\mathrm{T}_1(p^2) = \lambda_f(\mathrm{T}_1(p^2))f$. Then the p-Euler polynomial is given in (4.2.16) of [3],

$$\begin{aligned} Q_p(f,x) = 1 &- \lambda_f(\mathrm{T}(p))x + (p\lambda_f(\mathrm{T}_1(p^2)) + p^{2k-5}(1+p^2))x^2 \\ &- p^{2k-3}\lambda_f(\mathrm{T}(p))x^3 + p^{4k-6}x^4, \end{aligned}$$

and this is determined modulo $\mathfrak{a}$ by $\lambda_f(\mathrm{T}(p))$ and $\lambda_f(\mathrm{T}_1(p^2))$ modulo $\mathfrak{a}$. For a prime $p\|N$ let $f|_k\mathrm{T}(p) = \lambda_f(\mathrm{T}(p))f$ and $f|_k\mathrm{T}_{1,0}(p^2) = \lambda_f(\mathrm{T}_{1,0}(p^2))f$, and recall that ϵ_p denotes the shared Atkin–Lehner sign of f and g. In this case the p-Euler polynomial is, see Johnson-Leung and Roberts [20], p.547,

$$Q_p(f,x) = 1 - (\lambda_f(\mathrm{T}(p)) + p^{k-3}\epsilon_p)x + (p\lambda_f(\mathrm{T}_{1,0}(p^2)) + p^{2k-3})x^2 + p^{3k-5}\epsilon_p x^3$$

and this is determined modulo $\mathfrak{a}$ by $\lambda_f(\mathrm{T}(p))$, $\lambda_f(\mathrm{T}_{1,0}(p^2))$, and ϵ_p modulo $\mathfrak{a}$. We remark that this last case can be proved without the formula for $a(t;f|\mathrm{T}_{1,0}(p^2))$ in Lemma 4.1, because for $N = p$ the bad Euler polynomial only depends upon $\lambda_f(\mathrm{T}(p))$ and ϵ_p. Indeed, we have the relation $p^{k-3}\lambda_f(\mathrm{T}(p))\epsilon_p + \lambda_f(\mathrm{T}_{1,0}(p^2)) + p^{2k-5} + p^{2k-6} = 0$. The reference for this relation for local representations is Roberts and Schmidt [30], p.248. □

With the needed supporting results in place, we can establish the main results of this section, at levels $N = 61, 73, 79$. Each nonlift eigenform f_N takes the form $c_o \cdot G + m\mathfrak{f}$ where c_o is a vector of rational integers, G is a vector of Gritsenko lifts, m is a rational integer, and $\mathfrak{f} \in \mathcal{S}_3(\mathrm{K}(N))$ has rational integer Fourier coefficients; specifically $m_{61}, m_{73}, m_{79} = -43, -39, -32$ from Propositions 3.1, 3.5, and 3.6. We work in the number field $K = \mathbb{Q}(a)$ where a is any root of an irreducible factor of the characteristic polynomial of $\mathrm{T}(2)$ on the Gritsenko lift space. With d_o a vector of elements of $\mathbb{Q}[a] = K$ computed by a machine search, and with G the vector of Gritsenko lifts $G[j]$ from earlier, the linear combination $g(a) = d_o \cdot G$ is a Gritsenko lift $\mathrm{T}(2)$ eigenform, denoted $g(a)$ because it depends on the chosen root a. To find the congruences between the nonlift eigenform f_N and a scalar multiple of $g(a)$ modulo an $\mathcal{O}_K$-ideal over a rational prime $p \mid m$, we seek to rescale d_o to a vector of algebraic integers that is p-minimal, meaning that the smallest exponent of p in the norms of the vector entries is as small as possible, we hope 0. To do so, we first rescale d_o by a positive rational integer to get entries in $\mathcal{O}_K$, then divide it by its entry whose norm is divisible by the lowest power of p, and then try to scale it back by a rational integer coprime to p to make its entries algebraic integers again. With d_o so rescaled, we test for $n = 1, \dots, p-1$ whether the entries of $c_o - nd_o$ are all algebraic integer multiples of some algebraic number w whose norm is a multiple of p; either w is the difference of the $\mathrm{T}(2)$-eigenvalues or it is one of the entries of $c_o - nd_o$. If so, then f_N and ng are congruent modulo the ideal $\langle m, w\rangle$. This strategy suffices to prove the three theorems to follow. The tables accompanying these theorems show the rescaled vectors d_o, and we give the values w that are not eigenvalue differences, to make our computations reproducible. Because such a w is an ideal generator at level 79, it occurs in the statement of the third theorem.

Theorem 4.3. *Let $\mathcal{G} \subseteq \mathcal{S}_3(\mathrm{K}(61))$ be the subspace of Gritsenko lifts. The characteristic polynomial of $\mathrm{T}(2)$ on $\mathcal{G}$ is irreducible over $\mathbb{Q}$,*

$$q(x) = x^6 - 29x^5 + 322x^4 - 1714x^3 + 4471x^2 - 5205x + 2026.$$

Let a be a root of q and let $K = \mathbb{Q}(a)$. With reference to the elements $d_j(a)$ of K in Table 4 and to the Gritsenko lifts $G[j]$ from Proposition 3.1, consider an element of $\mathcal{G}(K)$,

$$g(a) = \sum_{j=1}^{6} d_j(a) G[j].$$

Then $g(a)$ lies in $\mathcal{G}(\mathcal{O}_K)$, and it is a $\mathrm{T}(2)$-eigenform with eigenvalue a, and it is an eigenform of $\mathrm{T}(p)$ and $\mathrm{T}_{1,0}(p^2)$ for all primes p. The $\mathcal{O}_K$-ideal

$$\mathfrak{a} = \langle 43, a+7\rangle$$

is prime. The Fourier coefficients and the Euler polynomials of f_{61} and $g(a)$ are congruent modulo $\mathfrak{a}$. The ideal $\mathfrak{a}$ is the only (proper) $\mathcal{O}_K$-ideal that gives a congruence between the Euler polynomials of f_{61} and $g(a)$.

Table 4. Coefficients of Gritsenko lifts for $N = 61$

j	c_j	$d_j(a)$
1	-9	$\frac{515}{2} - \frac{2377}{4}a + 443a^2 - \frac{269}{2}a^3 + 17a^4 - \frac{3}{4}a^5$
2	-2	$\frac{1899}{8} - \frac{7813}{16}a + \frac{1223}{4}a^2 - \frac{649}{8}a^3 + \frac{39}{4}a^4 - \frac{7}{16}a^5$
3	22	$-\frac{305}{2} + \frac{1697}{4}a - 359a^2 + \frac{237}{2}a^3 - 16a^4 + \frac{3}{4}a^5$
4	9	$-396 + 855a - 596a^2 + 174a^3 - 22a^4 + a^5$
5	-10	$-140 + 215a - 97a^2 + 17a^3 - a^4$
6	19	-24

Proof. The Gritsenko lift space $\mathcal{G}$ has basis $\{G[j] : j = 1, \ldots, 6\}$. We can compute enough Fourier coefficients of the $G[j]$ (see [35]) to give the matrix of $\mathrm{T}(2)$ on $\mathcal{G}$ for the basis, acting from the left on column vectors of $\mathbb{C}^6$,

$$M = \begin{pmatrix} 9 & 0 & 4 & -1 & 4 & 1 \\ -6 & 1 & -4 & -4 & 0 & 0 \\ 3 & 0 & 8 & 2 & 2 & 0 \\ 3 & 4 & 0 & 11 & -4 & -2 \\ -8 & 0 & -8 & -1 & -3 & -1 \\ -1 & -4 & 2 & -4 & 0 & 3 \end{pmatrix}.$$

The characteristic polynomial $q(x) = \det(xI - M)$ is as stated.

Each $d_j(a)$ lies in $\mathcal{O}_K$, as can be confirmed by computer software, and each $G[j]$ lies in $\mathcal{G}(\mathbb{Z})$, so $g(a)$ lies in $\mathcal{G}(\mathcal{O}_K)$. The vector $(d_1(a), \dots, d_6(a))$, viewed as a column, lies in null $(aI - M)$, and so $g(a)$ is a T(2)-eigenform in $\mathcal{G}(\mathcal{O}_K)$ with eigenvalue a. The characteristic polynomial of T(2) is separable because its discriminant $\mathrm{disc}(q) = 2^{14} \cdot 3^6 \cdot 1892022169$ is nonzero. So the eigenspaces of T(2) are one-dimensional, and consequently $g(a)$ is an eigenform of every Hecke operator that commutes with T(2). The commutativity is automatic for $\mathrm{T}(p)$ and $\mathrm{T}_{1,0}(p^2)$ with $p \neq 2$. For $\mathrm{T}_{1,0}(4)$, the commutator $[\mathrm{T}(2), \mathrm{T}_{1,0}(4)]$ consists of level lowering operators by Proposition 6.21 in [30] and so $[\mathrm{T}(2), \mathrm{T}_{1,0}(4)] = 0$ on $\mathcal{S}_3(\mathrm{K}(61))$ because 61 is prime and $\mathcal{S}_3(\mathrm{K}(1)) = \{0\}$. Thus $g(a)$ is an eigenform of all the stated Hecke operators.

The reduction of $q(x)$ modulo 43 is

$$q(x) \equiv (x+7)(x+25)(x+30)(x^3 + 38x^2 + 13x + 12) \bmod 43,$$

and because $43 \nmid \mathrm{disc}(q)$, The Kummer–Dedekind theorem says that the $\mathcal{O}_K$-ideal $\mathfrak{a} = \langle 43, a+7 \rangle$ is prime, and more generally that $\mathfrak{a}$, $\langle 43, a+25 \rangle$, and $\langle 43, a+30 \rangle$ are the norm-43 ideals of $\mathcal{O}_K$.

The values c_j in Table 4 are such that $f_{61} = -43\mathfrak{f} + \sum_{j=1}^{6} c_j G[j]$ in Proposition 3.1. Thus

$$f_{61} - g(a) = -43\mathfrak{f} + \sum_{j=1}^{6} (c_j - d_j(a)) G[j].$$

For $j = 1, \dots, 6$ we compute that $1616(c_j - d_j(a)) \in \langle a+7 \rangle$. The second bullet from the beginning of this section with $k = 43$, $\ell = 1616$, $\mathfrak{u} = \langle a+7 \rangle$, and $c = c_j - d_j(a)$ gives $c_j - d_j(a) \in \langle 43, a+7 \rangle = \mathfrak{a}$. The Fourier coefficients of $43\mathfrak{f}$ lie in $\mathfrak{a}$ as well. Thus $f_{61} \equiv g(a) \bmod \mathfrak{a}$ at the level of Fourier coefficients. Because $\mathcal{S}_3(\mathrm{K}(61)) = \mathcal{S}_3(\mathrm{K}(61))^-$ and 61 is prime, Lemma 4.2 says that all p-Euler polynomials of f_{61} and $g(a)$ are congruent modulo $\mathfrak{a}$.

The matrix of T(3) on $\mathcal{G}$ is given in [35]. Using it along with the matrix of T(2) we compute two eigenvalue differences,

$$\begin{aligned} &\lambda_{g(a)}(\mathrm{T}(2)) - \lambda_{f_{61}}(\mathrm{T}(2)) = a + 7 \\ &\lambda_{g(a)}(\mathrm{T}(3)) - \lambda_{f_{61}}(\mathrm{T}(3)) = \tfrac{1495}{48} - \tfrac{4313}{96}a + \tfrac{691}{24}a^2 - \tfrac{349}{48}a^3 + \tfrac{19}{24}a^4 - \tfrac{1}{32}a^5 + 3. \end{aligned}$$

The norm of any ideal containing these eigenvalue differences divides their norms,

$$\begin{aligned} \mathrm{N}\left(\lambda_{g(a)}(\mathrm{T}(2)) - \lambda_{f_{61}}(\mathrm{T}(2))\right) &= 2^9 \cdot 43 \cdot 101 \\ \mathrm{N}\left(\lambda_{g(a)}(\mathrm{T}(3)) - \lambda_{f_{61}}(\mathrm{T}(3))\right) &= 5^2 \cdot 19 \cdot 43 \cdot 139, \end{aligned}$$

and therefore divides the greatest common divisor 43 of these norms, and therefore, because the ideal is proper, equals 43. So the ideal is one of the norm-43 ideals $\mathfrak{a} = \langle 43, a+7 \rangle, \langle 43, a+25 \rangle, \langle 43, a+30 \rangle$, and it contains $a+7$, so it is $\mathfrak{a}$. $\square$

The congruence of eigenvalues modulo a prime ideal above 43 is also proven in [4], which also contains a proof of a new type of congruence above 19, discovered by Buzzard and Golyshev, of f_{61} to a Yoshida lift.

Theorem 4.4. *Let $\mathcal{G} \subseteq \mathcal{S}_3(\mathrm{K}(73))$ be the subspace of Gritsenko lifts. The characteristic polynomial q of $\mathrm{T}(2)$ on $\mathcal{G}$ factors over $\mathbb{Q}$ as $q = q_1 q_7$, where its irreducible factors q_1 and q_7 are*

$$q_1(x) = x - 9,$$
$$q_7(x) = x^7 - 30x^6 + 357x^5 - 2157x^4 + 7034x^3 - 12145x^2 + 9964x - 2832.$$

With reference to the integers $d_{1,j}$ in Table 5 and to the Gritsenko lifts $G[j]$ from Proposition 3.5, consider an element of $\mathcal{G}(\mathbb{Z})$,

$$g_1 = \sum_{j=1}^{8} d_{1,j} G[j].$$

Then g_1 is a $\mathrm{T}(2)$-eigenform with eigenvalue 9, and it is an eigenform of $\mathrm{T}(p)$ and $\mathrm{T}_{1,0}(p^2)$ for all primes p. The Fourier coefficients and the Euler polynomials of f_{73} and g_1 are congruent modulo $3\mathbb{Z}$. The only (proper) $\mathbb{Z}$-ideal that gives a congruence between the Euler polynomials of f_{73} and g_1 is $3\mathbb{Z}$.

Let a be a root of q_7 and $K = \mathbb{Q}(a)$. With reference to the elements $d_{7,j}(a)$ of K in Table 5 and to the Gritsenko lifts $G[j]$ from Proposition 3.5, consider an element of $\mathcal{G}(K)$,

$$g_7(a) = \sum_{j=1}^{8} d_{7,j}(a) G[j].$$

Then $g_7(a)$ lies in $\mathcal{G}(\mathcal{O}_K)$, and it is a $\mathrm{T}(2)$-eigenform with eigenvalue a, and it is an eigenform of $\mathrm{T}(p)$ and $\mathrm{T}_{1,0}(p^2)$ for all primes p. The $\mathcal{O}_K$-ideals

$$\mathfrak{a} = \langle 3, a \rangle, \qquad \mathfrak{b} = \langle 13, a+6 \rangle$$

are prime. The Fourier coefficients and the Euler polynomials of f_{73} and $g_7(a)$ are congruent modulo $\mathfrak{a}$ and modulo $\mathfrak{b}$. The only prime-power $\mathcal{O}_K$-ideals that give congruences between the Euler polynomials of f_{73} and $g_7(a)$ are $\mathfrak{a}$ and $\mathfrak{b}$.

Proof. Similarly to the proof of Theorem 4.3, the matrix M of $\mathrm{T}(2)$ on $\mathcal{G}$ for the basis $\{G[j] : j = 1, \ldots, 8\}$ is

$$M = \begin{pmatrix} 8 & 0 & 3 & -4 & -2 & -2 & 4 & 2 \\ -4 & 1 & -1 & -2 & -3 & 0 & -4 & -5 \\ 0 & 0 & 6 & 0 & -1 & 0 & 0 & -1 \\ 2 & 2 & 5 & 8 & 3 & 2 & 2 & 1 \\ 2 & 4 & -1 & 6 & 8 & 8 & -4 & -5 \\ -4 & -4 & -8 & -4 & -1 & -4 & 2 & 7 \\ -2 & 0 & -3 & 4 & 3 & 2 & 1 & -1 \\ -3 & -4 & -7 & -6 & -4 & -8 & 4 & 11 \end{pmatrix}.$$

TABLE 5. Coefficients of Gritsenko lifts for $N = 73$

j	c_j	$d_{1,j}$	$d_{7,j}(a)$
1	9	-3	$-\frac{632307}{4}+\frac{8480021}{16}a-\frac{36446543}{64}a^2+\frac{16486531}{64}a^3 -\frac{442583}{8}a^4+\frac{357943}{64}a^5-\frac{13671}{64}a^6$
2	19	-2	$-\frac{247387}{2}+\frac{2906781}{8}a-\frac{9639367}{32}a^2+\frac{3516515}{32}a^3 -\frac{77947}{4}a^4+\frac{52719}{32}a^5-\frac{1703}{32}a^6$
3	2	-1	$\frac{134939}{4}-\frac{1626701}{16}a+\frac{6689239}{64}a^2-\frac{2996715}{64}a^3 +\frac{80271}{8}a^4-\frac{64895}{64}a^5+\frac{2479}{64}a^6$
4	-13	2	$-\frac{280607}{2}+\frac{3673513}{8}a-\frac{15573571}{32}a^2+\frac{6924839}{32}a^3 -\frac{181543}{4}a^4+\frac{142859}{32}a^5-\frac{5307}{32}a^6$
5	34	1	$\frac{411139}{4}-\frac{4124773}{16}a+\frac{16534783}{64}a^2-\frac{7232243}{64}a^3 +\frac{187815}{8}a^4-\frac{146535}{64}a^5+\frac{5399}{64}a^6$
6	-15	3	$\frac{850479}{4}-\frac{12504265}{16}a+\frac{51374171}{64}a^2-\frac{22172735}{64}a^3 +\frac{569099}{8}a^4-\frac{441443}{64}a^5+\frac{16243}{64}a^6$
7	-12	3	$\frac{648207}{4}-\frac{8826201}{16}a+\frac{37654939}{64}a^2-\frac{16649519}{64}a^3 +\frac{434027}{8}a^4-\frac{340515}{64}a^5+\frac{12643}{64}a^6$
8	-10	2	-10072

The characteristic polynomial $q(x) = \det(xI - M) = q_1(x)q_7(x)$ is as stated.

The vector $(d_{1,1}, \ldots, d_{1,8})$ lies in $\mathrm{null}(9I - M)$, and so g_1 is a T(2)-eigenform with eigenvalue 9. The characteristic polynomial q of T(2) is separable because its discriminant $\mathrm{disc}(q) = 2^{36} \cdot 3^3 \cdot 5^2 \cdot 13 \cdot 19^2 \cdot 37 \cdot 101 \cdot 30931$ is nonzero, and so g_1 is an eigenform of all the stated Hecke operators as in the proof of Theorem 4.3.

The values c_j in Table 5 are such that $f_{73} = -39\mathfrak{f} + \sum_{j=1}^{8} c_j G[j]$ in Proposition 3.5. Thus

$$f_{73} - g_1 = -39\mathfrak{f} + \sum_{j=1}^{8} (c_j - d_{1,j})G[j].$$

Table 5 shows that the coefficients $c_j - d_{1,j}$ all lie in $3\mathbb{Z}$, as do the Fourier coefficients of $-39\mathfrak{f}$, and so the Fourier coefficients of g_1 and f_{73} are congruent modulo $3\mathbb{Z}$. Because $\mathcal{S}_3(\mathrm{K}(73)) = \mathcal{S}_3(\mathrm{K}(73))^-$ and 73 is prime, Lemma 4.2 says that all p-Euler polynomials of f_{73} and g_1 are congruent modulo $3\mathbb{Z}$. Again the matrix of T(3) on $\mathcal{G}$ is given in [35]. The eigenvalue differences $\lambda_{g_1}(\mathrm{T}(p)) - \lambda_{f_{73}}(\mathrm{T}(p))$ for $p = 2, 3$ are

$$\lambda_{g_1}(\mathrm{T}(2)) - \lambda_{f_{73}}(\mathrm{T}(2)) = 9 + 6 = 15 = 3 \cdot 5,$$

$$\lambda_{g_1}(\mathrm{T}(3)) - \lambda_{f_{73}}(\mathrm{T}(3)) = 4 + 2 = 6 = 2 \cdot 3.$$

Any ideal containing these contains $3\mathbb{Z}$, so it is $3\mathbb{Z}$.

Now let a be a root of q_7 and $K = \mathbb{Q}(a)$. Each $d_{7,j}(a)$ lies in $\mathcal{O}_K$ and each $G[j]$ lies in $\mathcal{G}(\mathbb{Z})$, so $g_7(a)$ lies in $\mathcal{G}(\mathcal{O}_K)$. The vector $(d_{7,1}(a), \dots, d_{7,8}(a))$ lies in $\mathrm{null}(aI - M)$, and so $g_7(a)$ is a $\mathrm{T}(2)$-eigenform in $\mathcal{G}(\mathcal{O}_K)$ with eigenvalue a. Because the roots of q_7 are distinct, $g_7(a)$ is an eigenform of all the stated Hecke operators.

Because $\mathrm{disc}(q_7) = 2^{24} \cdot 3 \cdot 13 \cdot 19^2 \cdot 37 \cdot 101 \cdot 30931$ is divisible by 3 only once, the Kummer–Dedekind theorem says that the factorization of q_7 modulo 3,

$$q_7(x) \equiv x(x+2)^2(x^4 + 2x^3 + x + 1) \bmod 3,$$

determines the factorization $3\mathcal{O}_K = \langle 3, a\rangle\langle 3, a+2\rangle^2\langle 3, a^4 + 2a^3 + a + 1\rangle$, the first two prime ideals on the right side having norm 3 and the third having norm 3^4. Similarly the factorization of q_7 modulo 13,

$$q_7(x) \equiv (x+6)^2(x^5 + 10x^4 + 6x^3 + 11x^2 + 4x + 8) \bmod 13,$$

determines the factorization $13\mathcal{O}_K = \langle 13, a+6\rangle^2\langle 13, a^5 + 10a^4 + 6a^3 + 11a^2 + 4a + 8\rangle$. Here $\langle 3, a\rangle$ and $\langle 13, a+6\rangle$ are the ideals $\mathfrak{a}$ and $\mathfrak{b}$ of the theorem, so those ideals are prime as claimed.

The values c_j in Table 5 are such that $f_{73} = -39\mathfrak{f} + \sum_{j=1}^{8} c_j G[j]$ in Proposition 3.5. Thus

$$f_{73} - g_7(a) = -39\mathfrak{f} + \sum_{j=1}^{8}(c_j - d_{7,j}(a))G[j].$$

Let

$$w = \tfrac{134943}{2} - \tfrac{1626701}{8}a + \tfrac{6689239}{32}a^2 - \tfrac{2996715}{32}a^3 + \tfrac{80271}{4}a^4 - \tfrac{64895}{32}a^5 + \tfrac{2479}{32}a^6.$$

(How we found this w will be explained immediately after the proof.) This element of $\mathcal{O}_K$ has norm

$$\mathrm{N}(w) = -2^7 \cdot 3 \cdot 3919 \cdot 1941571 \cdot 8583739212883,$$

and so by the first bullet at the beginning of this section, the ideal $\langle 3, w\rangle$ is one of the norm-3 $\mathcal{O}_K$-ideals, $\langle 3, w\rangle = \mathfrak{a} = \langle 3, a\rangle$ or $\langle 3, w\rangle = \langle 3, a+2\rangle$. Further, $32w$ does not lie in $\langle 3, a+2\rangle$, as one can see by replacing a by 1 in $32w$ and then reducing modulo 3. Therefore $\langle 3, w\rangle = \mathfrak{a}$. Now set $\ell = 130627630879749647154734$. For $j = 1, \dots, 8$, compute $\ell(c_j + 2d_{7,j}(a)) \in \langle w\rangle$. The second bullet from the beginning of this section, with $k = 3$, ℓ as given, $\mathfrak{u} = \langle w\rangle$, and $c = c_j + 2d_{7,j}(a)$ gives $c_j + 2d_{7,j}(a) \in \langle 3, w\rangle = \mathfrak{a}$, and it follows that $c_j - d_{7,j}(a) \in \mathfrak{a}$. The Fourier coefficients of $-39\mathfrak{f}$ lie in $\mathfrak{a}$ as well. Thus $f_{73} \equiv g_7(a) \bmod \mathfrak{a}$ at the level of Fourier coefficients. Turning to the $\mathfrak{b} = \langle 13, a+6\rangle$ congruence, we evaluate the norm

$$\mathrm{N}(a+6) = 9270300 = 2^2 \cdot 3 \cdot 5^2 \cdot 13 \cdot 2377.$$

Set $\ell = 356550$. For $j = 1, \ldots, 8$, compute $\ell(c_j + 12d_{7,j}(a)) \in \langle a+6 \rangle$. The second bullet from the beginning of this section, with $k = 13$, ℓ as given, $\mathfrak{u} = \langle a+6 \rangle$, and $c = c_j + 12d_{7,j}(a)$ gives $c_j + 12d_{7,j}(a) \in \langle 13, a+6 \rangle = \mathfrak{b}$. Thus, similarly to just above, $f_{73} \equiv g_7(a) \bmod \mathfrak{b}$ at the level of Fourier coefficients. As in the proof of Theorem 4.3, this proves that all p-Euler polynomials of f_{73} and $g_7(a)$ are congruent modulo $\mathfrak{a}$ and modulo $\mathfrak{b}$.

The norms of possible prime-power congruence ideals are limited to 3 or 13 by computing the norms from $\mathcal{O}_K$ to $\mathbb{Z}$ of two eigenvalue differences and of their sum,

$$\begin{aligned} &\mathrm{N}\left(\lambda_{g_7(a)}(\mathrm{T}(2)) - \lambda_{f_{73}}(\mathrm{T}(2))\right) \\ &= \mathrm{N}\,(a+6) \\ &= 2^2 \cdot 3 \cdot 5^2 \cdot 13 \cdot 2377, \end{aligned}$$

$$\begin{aligned} &\mathrm{N}\left(\lambda_{g_7(a)}(\mathrm{T}(3)) - \lambda_{f_{73}}(\mathrm{T}(3))\right) \\ &= \mathrm{N}\left(\tfrac{241}{4} - \tfrac{3171}{16}a + \tfrac{13965}{64}a^2 - \tfrac{6297}{64}a^3 + \tfrac{167}{8}a^4 - \tfrac{133}{64}a^5 + \tfrac{5}{64}a^6 + 2\right) \\ &= 2^2 \cdot 3 \cdot 13 \cdot 195809, \end{aligned}$$

$$\begin{aligned} &\mathrm{N}\left(\lambda_{g_7(a)}(\mathrm{T}(2)) - \lambda_{f_{73}}(\mathrm{T}(2)) + \lambda_{g_7(a)}(\mathrm{T}(3)) - \lambda_{f_{73}}(\mathrm{T}(3))\right) \\ &= 3^6 \cdot 13 \cdot 61 \cdot 4793, \end{aligned}$$

because these norms have greatest common divisor $3 \cdot 13$. If the congruence ideal has norm 3 then because it contains the eigenvalue difference $a+6$ it contains $\langle 3, a+6 \rangle = \langle 3, a \rangle = \mathfrak{a}$, so it is $\mathfrak{a}$. Similarly, if the congruence ideal has norm 13 then it is $\langle 13, a+6 \rangle = \mathfrak{b}$. □

The integers $d_{1,j}$ and the polynomials $d_{7,j}$ in Table 5 were determined similarly to the polynomials d_j in Table 4. The value w in the proof was found by testing the set $\{c_j + nd_{7,j}(a)\}$ for various integers n until one element of the set divided all the others in $\mathcal{O}_K$; w came from $n = 2$ and then $j = 3$.

We remark that the Gritsenko lift with T(2)-eigenvalue given by q_1 arises from the elliptic newform 73.4.a.a at the database LMFDB [22, Modular Form 73.4.a.a] and similarly for q_7 and 73.4.a.b [22, Modular Form 73.4.a.b] .

Theorem 4.5. *Let $\mathcal{G} \subseteq \mathcal{S}_3(\mathrm{K}(79))$ be the subspace of Gritsenko lifts. The characteristic polynomial q of $\mathrm{T}(2)$ on $\mathcal{G}$ factors over $\mathbb{Q}$ as $q = q_2 q_5$, where its irreducible factors q_2 and q_5 are*

$$\begin{aligned} q_2(x) &= x^2 - 11x + 26, \\ q_5(x) &= x^5 - 27x^4 + 261x^3 - 1077x^2 + 1766x - 964. \end{aligned}$$

Let a be a root of q_2 and $K = \mathbb{Q}(a)$. With reference to the elements $d_{2,j}(a)$ of K in Table 6 and to the Gritsenko lifts $G[j]$ from Proposition 3.6, consider *an element of* $\mathcal{G}(K)$,

$$g_2(a) = \sum_{j=1}^{7} d_{2,j}(a) G[j].$$

Then $g_2(a)$ *lies in* $\mathcal{G}(\mathcal{O}_K)$, *and it is a* $\mathrm{T}(2)$*-eigenform with eigenvalue* a, *and it is an eigenform of* $\mathrm{T}(p)$ *and* $\mathrm{T}_{1,0}(p^2)$ *for all primes* p. *The* $\mathcal{O}_K$*-ideal*

$$\mathfrak{a} = \langle 2, a+1 \rangle$$

is prime, and the Fourier coefficients and the Euler polynomials of f_{79} *and* $g_2(a)$ *are congruent modulo* $\mathfrak{a}$. *The only (proper) ideal that gives a congruence between the Euler polynomials of* f_{79} *and* $g_2(a)$ *is* $\mathfrak{a}$.

Let b *be a root of* q_5 *and* $L = \mathbb{Q}(b)$. *With reference to the elements* $d_{5,j}(b)$ *of* L *in Table 6 and to the Gritsenko lifts* $G[j]$ *from Proposition 3.6, consider an element of* $\mathcal{G}(L)$,

$$g_5(b) = \sum_{j=1}^{7} d_{5,j}(b) G[j].$$

Then $g_5(b)$ *lies in* $\mathcal{G}(\mathcal{O}_L)$, *and it has a* $\mathrm{T}(2)$*-eigenform with eigenvalue* b, *and it is an eigenform of* $\mathrm{T}(p)$ *and* $\mathrm{T}_{1,0}(p^2)$ *for all primes* p. *Let* $w = \frac{35}{4} - \frac{435}{8}b + \frac{565}{16}b^2 - \frac{25}{4}b^3 + \frac{5}{16}b^4$, *an element of* $\mathcal{O}_L$. *The* $\mathcal{O}_L$*-ideal*

$$\mathfrak{b} = \langle 8, w \rangle$$

is the cube of a prime $\mathcal{O}_L$*-ideal over* 2 *of norm* 4. *The Fourier coefficients of* f_{79} *and* $5g_5(b)$ *are congruent modulo* $\mathfrak{b}$, *and the Euler polynomials of* f_{79} *and* $g_5(b)$ *are congruent modulo* $\mathfrak{b}$. *Every (proper) ideal that gives a congruence between the Euler polynomials of* f_{79} *and* $g_5(b)$ *divides* $\mathfrak{b}$.

TABLE 6. Coefficients of Gritsenko lifts for $N = 79$

j	c_j	$d_{2,j}(a)$	$d_{5,j}(b)$
1	4	$5-a$	$\frac{103}{2} - \frac{257}{4}b + \frac{185}{8}b^2 - 3b^3 + \frac{1}{8}b^4$
2	13	$-8+a$	$\frac{199}{4} - \frac{435}{8}b + \frac{257}{16}b^2 - \frac{7}{4}b^3 + \frac{1}{16}b^4$
3	-15	$6-a$	$-\frac{19}{4} + \frac{87}{8}b - \frac{113}{16}b^2 + \frac{5}{4}b^3 - \frac{1}{16}b^4$
4	8	$3-a$	$3-b$
5	0	$9-a$	$10-9b+b^2$
6	5	$4-a$	$4-b$
7	-11	$-18+3a$	$-\frac{203}{4} + \frac{443}{8}b - \frac{257}{16}b^2 + \frac{7}{4}b^3 - \frac{1}{16}b^4$

Proof. Similarly to the proof of Theorem 4.3, the matrix M of $\mathrm{T}(2)$ on $\mathcal{G}$ for the basis $\{G[j] : j = 1, \ldots, 7\}$ is

$$M = \begin{pmatrix} 0 & -2 & -2 & -2 & -8 & -2 & -6 \\ -4 & 2 & 0 & -3 & -7 & -2 & -5 \\ 3 & 0 & 9 & 1 & 2 & 2 & 4 \\ 0 & -1 & 0 & 11 & -1 & -6 & -1 \\ 3 & 0 & 1 & 2 & 9 & 2 & 5 \\ 0 & -1 & 0 & 7 & -1 & -3 & -1 \\ 5 & 2 & 1 & -3 & 8 & 6 & 10 \end{pmatrix}$$

The characteristic polynomial $q(x) = \det(xI - M) = q_2(x)q_5(x)$ is as stated.

Let a be a root of q_2 and $K = \mathbb{Q}(a)$. Each $d_{2,j}(a)$ lies in $\mathcal{O}_K$ and each $G[j]$ lies in $\mathcal{G}(\mathbb{Z})$, so $g_2(a)$ lies in $\mathcal{G}(\mathcal{O}_K)$. The vector $(d_{2,1}(a), \ldots, d_{2,7}(a))$ lies in $\mathrm{null}(aI - M)$, and so $g_2(a)$ is a $\mathrm{T}(2)$-eigenform in $\mathcal{G}(\mathcal{O}_K)$ with eigenvalue a. The characteristic polynomial q of $\mathrm{T}(2)$ is separable because its discriminant $\mathrm{disc}(q) = 2^{26} \cdot 17^3 \cdot 59^2 \cdot 4787257$ is nonzero, and so $g_2(a)$ is an eigenform of all the stated Hecke operators as in the proof of Theorem 4.3.

The reduction of $q_2(x)$ modulo 2 is $q_2(x) \equiv x(x+1) \bmod 2$, and because 2 does not divide $\mathrm{disc}(q_2) = 17$, The Kummer–Dedekind theorem says that the $\mathcal{O}_K$-ideal $\mathfrak{a} = \langle 2, a+1 \rangle$ is prime.

The values c_j in Table 6 are such that $f_{79} = -32\mathfrak{f} + \sum_{j=1}^{7} c_j G[j]$ in Proposition 3.6. Thus

$$f_{79} - g_2(a) = -32\mathfrak{f} + \sum_{j=1}^{7} (c_j - d_{2,j}(a)) G[j].$$

For $j = 1, \ldots, 7$ we compute that $53(c_j - d_{2,j}(a)) \in \langle a+5 \rangle$. The second bullet from the beginning of this section, with $k = 2$, $\ell = 53$, $\mathfrak{u} = \langle a+5 \rangle$, and $c = c_j - d_{2,j}(a)$ gives $c_j - d_{2,j}(a) \in \langle 2, a+5 \rangle = \mathfrak{a}$. The Fourier coefficients of $-32\mathfrak{f}$ lie in $\mathfrak{a}$ as well. Thus $f_{79} \equiv g_2(a) \bmod \mathfrak{a}$ at the level of Fourier coefficients. As in the proof of Theorem 4.3, this proves that all p-Euler polynomials of f_{79} and $g_2(a)$ are congruent modulo $\mathfrak{a}$.

The possible congruence ideals for f_{79} and $g_2(a)$ are limited to norm-2 ideals by computing the norm of an eigenvalue differences and then a second eigenvalue difference that is already a rational integer,

$$\begin{aligned} \mathrm{N}\left(\lambda_{g_2(a)}(\mathrm{T}(2)) - \lambda_{f_{79}}(\mathrm{T}(2))\right) &= \mathrm{N}(a+5) = 2 \cdot 53, \\ \lambda_{g_2(a)}(\mathrm{T}(3)) - \lambda_{f_{79}}(\mathrm{T}(3)) &= 11 - (-5) = 16 = 2^4, \end{aligned}$$

because the greatest common divisor of these values is 2. Any such congruence ideal also contains the eigenvalue difference $a+5$, so it contains $\langle 2, a+5 \rangle = \mathfrak{a}$, and so it is $\mathfrak{a}$.

Let b be a root of q_5 and $L = \mathbb{Q}(b)$. Each $d_{5,j}(b)$ lies in $\mathcal{O}_L$ and each $G[j]$ lies in $\mathcal{G}(\mathbb{Z})$, so $g_5(a)$ lies in $\mathcal{G}(\mathcal{O}_L)$. The vector $(d_{5,1}(b), \ldots, d_{5,7}(b))$ lies in $\mathrm{null}(bI - M)$., and so $g_5(b)$ is a $\mathrm{T}(2)$-eigenform in $\mathcal{G}(\mathcal{O}_L)$ with eigenvalue b.

Because the roots of q_5 are distinct, $g_5(b)$ is an eigenform of all the stated Hecke operators.

Similarly to just above,

$$f_{79} - 5g_5(b) = -32\mathfrak{f} + \sum_{j=1}^{7}(c_j - 5d_{5,j}(b))G[j].$$

Recall the element w of $\mathcal{O}_L$ and the ideal $\mathfrak{b} = \langle 8, w\rangle$ from the statement of the theorem. Computer software says that $\mathfrak{b}$ is the cube of the prime, norm 4 ideal $\langle 2, v\rangle$ over 2, where $v = \frac{1}{16}b^4 - \frac{5}{4}b^3 + \frac{129}{16}b^2 - \frac{151}{8}b + \frac{51}{4}$, and so $\mathfrak{b}$ has norm 64 as also can be confirmed directly. For $j = 1, \dots, 7$, we compute $635(c_j - 5d_{5,j}(b)) \in \langle w\rangle$. The second bullet from the beginning of this section, with $k = 8$, $\ell = 635$, $\mathfrak{u} = \langle w\rangle$, and $c = c_j - 5d_{5,j}(b)$ give $c_j - 5d_{5,j}(b) \in \langle 8, w\rangle = \mathfrak{b}$. The Fourier coefficients of $-32\mathfrak{f}$ lie in $\mathfrak{b}$ as well. Thus $f_{79} \equiv 5g_5(b) \bmod \mathfrak{b}$ at the level of Fourier coefficients. As in the proof of Theorem 4.3, and noting that scaling $g_5(b)$ by 5 has no effect on its Hecke eigenvalues or Euler polynomials, this proves that all p-Euler polynomials of f_{79} and $g_5(a)$ are congruent modulo $\mathfrak{b}$.

The possible congruence ideals for f_{79} and $g_5(a)$ are limited to divisors of $\mathfrak{b}$ by computing the norms of two eigenvalue differences,

$$\begin{aligned} &\mathrm{N}\left(\lambda_{g_5(b)}(\mathrm{T}(2)) - \lambda_{f_{79}}(\mathrm{T}(2))\right) \\ &= \mathrm{N}\,(b+5) \\ &= 2^8 \cdot 349, \end{aligned}$$

$$\begin{aligned} &\mathrm{N}\left(\lambda_{g_5(b)}(\mathrm{T}(5)) - \lambda_{f_{79}}(\mathrm{T}(5))\right) \\ &= \mathrm{N}\left(\tfrac{1884}{16} - \tfrac{1582}{16}b + \tfrac{401}{16}b^2 - \tfrac{36}{16}b^3 + \tfrac{1}{16}b^4 - 3\right) \\ &= 2^6 \cdot 5 \cdot 7^2 \cdot 67, \end{aligned}$$

because the greatest common divisor of these norms is 64, and so the congruence ideal has norm dividing 64. Letting $\mathfrak{c}$ denote the congruence ideal, also the least common multiple of $\mathfrak{b}$ and $\mathfrak{c}$ is a congruence ideal, so its norm divides $64 = \mathrm{N}(\mathfrak{b})$, so the least common multiple is $\mathfrak{b}$. That is, $\mathfrak{c} \mid \mathfrak{b}$. ☐

The w in this proof was found similarly to the $N = 73$ case, this time with $n = 5$ and $j = 3$. The Gritsenko lifts with T(2)-eigenvalues given by q_2 and q_5 arise from the LMFDB elliptic newforms 79.4.a.a [22, Modular Form 79.4.a.a] and 79.4.a.b [22, Modular Form 79.4.a.a].

5. Computation of eigenvalues

Throughout this section N is a positive integer, f an element of $\mathcal{M}_k(\mathrm{K}(N))$, and p a prime. Further, a, b, c are integers such that a matrix and two of its $\mathrm{SL}_2(\mathbb{Q}(\sqrt{p}))$-equivalents are positive,

$$s = \begin{pmatrix} a & b \\ b & c/N \end{pmatrix}, \qquad \check{s} = \begin{pmatrix} a/p & b \\ b & pc/N \end{pmatrix}, \qquad \hat{s} = \begin{pmatrix} pa & b \\ b & c/(pN) \end{pmatrix}.$$

Let $\phi_s(\tau) = \tau s$ for $\tau \in \mathcal{H}$, and let ϕ_s^* denote its pullback. Proposition 5.2 to follow gives initial formulas for the restrictions $\phi_s^*(f|_k\mathrm{T}(p))$ and $\phi_s^*(f|_k\mathrm{T}_1(p^2))$ when $p \nmid N$, and similarly Proposition 5.5 for $\phi_s^*(f|_k\mathrm{T}(p))$ and $\phi_s^*(f|_k\mathrm{T}_{1,0}(p^2))$ when $p\|N$. Each of these initial formulas consists of finitely many finite sums, but some of the sums are computationally intractable. Propositions 5.3 and 5.6 show that various sums in the initial formulas can be replaced by sums over smaller index sets, making them tractable after all. Some ideas from this section were introduced in [3] but complete details were not given there. Specifically, [3] states the first half of Proposition 5.2, the part about the Hecke operator $\mathrm{T}(p)$, but only alludes briefly to the second half, the part about $\mathrm{T}_1(p^2)$. The parts of Proposition 5.3 that speed up the second half of Proposition 5.2 are new, specifically the third formula in part (a), the second in part (b), and the second in part (d); in particular we prove the second formula in part (d) to illustrate ideas not present in [3]. Propositions 5.4 and 5.5, which give results for bad primes, are also new. The completeness of these propositions is needed to reproduce the computational results of [3] and of this paper.

Lemma 5.1. *With reference to the matrix s just above, define a map from the complex upper half plane to the 2-dimensional Siegel upper half space,*

$$\phi_s : \mathcal{H} \longrightarrow \mathcal{H}_2, \qquad \phi_s(\tau) = \tau s.$$

Let $R \subseteq \mathbb{C}$ be a subring. Then the pullback of ϕ_s is a ring homomorphism from the graded ring of Siegel paramodular forms of level N with coefficients in R to the graded ring of elliptic modular forms of level $\det(s)N$ with coefficients in R,

$$\phi_s^* : \mathcal{M}(\mathrm{K}(N))(R) \longrightarrow \mathcal{M}(\Gamma_0(\det(s)N))(R)$$

given by

$$(\phi_s^* f)(\tau) = f(\tau s).$$

The map ϕ_s^* multiplies weights by 2 and takes cusp forms to cusp forms.

The elliptic modular form $\phi_s^* f$ is the **restriction** of f to the curve $\phi_s(\mathcal{H})$, also called the restriction of f under s.

Proof. The proof follows from a straightforward modification of a result of Poor–Yuen [24, Proposition 5.4]. □

Let the paramodular form $f \in \mathcal{M}_k(\mathrm{K}(N))$ have Fourier expansion (in which $\langle t, \Omega\rangle = \mathrm{tr}(t\Omega)$)

$$f(\Omega) = \sum_{t \in \mathcal{X}_2(N)^{\mathrm{semi}}} a(t; f)\, \mathrm{e}(\langle t, \Omega\rangle).$$

Its restriction $\phi_s^* f \in \mathcal{M}_{2k}(\Gamma_0(\det(s)N))$ has Fourier expansion (in which $q = \mathrm{e}(\tau)$)

$$(\phi_s^* f)(\tau) = \sum_{n=0}^{\infty} \bigg(\sum_{t:\, \langle s,t\rangle = n} a(t; f) \bigg) q^n.$$

Furthermore, if f is slashed with a block upper triangular matrix $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in \mathrm{GSp}_4^+(\mathbb{Q})$ with similitude $\mu = \det(AD)^{1/2}$ then the restriction of the resulting function is

$$\begin{aligned}
&\phi_s^*(f|_k\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right))(\tau) = (f|_k\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right))(s\tau) \\
&= \det(AD)^{k-3/2}\det(D)^{-k} f(AsD^{-1}\tau + BD^{-1}) \\
&= \det(A)^k \det(AD)^{-3/2} \sum_{n\in\mathbb{Q}_{\geq 0}} \Bigg(\sum_{t:\,\langle AsD^{-1},t\rangle = n} a(t;f)\mathrm{e}\left(\langle BD^{-1},t\rangle\right) \Bigg) q^n.
\end{aligned} \tag{1}$$

In order to compute eigenvalues by the technique of restriction to a modular curve, we apply a restriction map ϕ_s^* to the eigenvalue equation $\lambda_f(T)f = f|T = \sum_{j=1}^m f|t_j$. We assume $T = \mathrm{K}(N)\,\mathrm{diag}(a,b,c,d)\mathrm{K}(N) = \bigsqcup_{j=1}^m \mathrm{K}(N)t_j$ where the t_j are upper block triangular and $m = \deg T$ is the number of cosets in $\mathrm{K}(N)\backslash T$. Using equation (1) for each term $\phi_s^*(f|t_j)$, the restricted eigenvalue equation

$$\lambda_f(T)\phi_s^*(f) = \sum_{j=1}^m \phi_s^*(f|t_j) \tag{2}$$

uniquely determines the eigenvalue $\lambda_f(T)$ as long as the elliptic modular form $\phi_s^*(f)$ does not vanish identically. Indeed, each successive power of $q = \mathrm{e}(\tau)$ in equation (2) provides an independent evaluation of the eigenvalue $\lambda_f(T)$ and is thus useful for checking computational infrastructure. For efficiency, the coefficients of equation (2) are evaluated over a finite field $\mathbb{F}_\ell$ rather than over $\overline{\mathbb{Q}}$.

Let T have a similitude μ that is a p-power. The factors $\mathrm{e}\left(\langle BD^{-1},t\rangle\right)$ in equation (1) are μ-th roots of unity. For simplicity assume $\lambda_f(T) \in \mathbb{Z}$. Choose an auxiliary prime ℓ that splits completely in the cyclotomic field $K = \mathbb{Q}(\mathrm{e}(1/\mu))$. By the Kummer–Dedekind theorem, the μ-th cyclotomic polynomial Φ_μ splits in $\mathbb{F}_\ell$. Let $r \in \mathbb{Z}$ give a root of Φ_μ in $\mathbb{F}_\ell$. In K we know that $\mathrm{N}\,(r - \mathrm{e}(1/\mu)) = \Phi_\mu(r) \equiv 0 \bmod \ell$, so that there is a prime ideal $\mathfrak{m}$ in $\mathcal{O}_K$ above $\langle r - \mathrm{e}(1/\mu), \ell\rangle$ by the first bullet at the beginning of the previous section. We evaluate the coefficients of equation (2) over the finite field $\mathcal{O}_K/\mathfrak{m} \cong \mathbb{F}_\ell$, using the congruence $\mathrm{e}(1/\mu) \equiv r \bmod \mathfrak{m}$ to reduce the computation to integers, and obtain $\lambda_f(T) \bmod \ell$. For sufficiently large ℓ, the general bound $|\lambda_f(T)| \leq \mu^{k-3}\deg T$, compare Proposition 6.7.1 in [3], determines $\lambda_f(T) \in \mathbb{Z}$.

The main computational advantage of restricting to modular curves, as opposed to using the formulae of Lemma 4.1, is that many terms in equation (2) may be omitted if we project onto integral powers of q after a partial summation. The speed-ups in Propositions 5.3 and 5.6 prove that we may partially sum over index sets that are roughly a factor of p smaller than $\deg T$ and still preserve equality for integral powers of q in equation (2).

A secondary benefit is that we may not need to compute Fourier coefficients of the eigenform f. For example, in section 3 each eigenform was given as a rational function of Gritsenko lifts $G[i]$. The specializations $\phi_s^*(G[i]|t_j)$ are computed from the Fourier coefficients of these Gritsenko lifts, which may be reduced to computing Fourier coefficients of Jacobi forms. It is also computationally beneficial to use single rather than multiple variable power series. The s actually used to restrict f can be selected from a number of candidates for speed and to make the q-order of $\phi_s^*(f)$ small. For $N = 61$ and good primes p, we used $s = \left(\begin{smallmatrix} 122 & 11 \\ 11 & 1 \end{smallmatrix}\right)$, which gave q-order 2; for $p = 61$, we used $s = \left(\begin{smallmatrix} 13 & 2 \\ 2 & 19/61 \end{smallmatrix}\right)$ with q-order 1. For $N = 73, 79$, in all cases we used $s = \left(\begin{smallmatrix} 146 & 17 \\ 17 & 2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 158 & 47 \\ 47 & 14 \end{smallmatrix}\right)$, respectively, each with q-order 3.

We now write speed-up theorems for computing the restrictions of $f|_k\mathrm{T}(p)$ and $f|_k\mathrm{T}_1(p^2)$ for $p \nmid N$ and of $f|_k\mathrm{T}(p)$ and $f|_k\mathrm{T}_{1,0}(p^2)$ for $p \| N$. Recall that these Hecke operators are defined as slashes by double cosets,

$$\mathrm{T}(p) = \mathrm{K}(N)\,\mathrm{diag}(1,1,p,p)\mathrm{K}(N)$$
$$\mathrm{T}_1(p^2) = \mathrm{K}(N)\,\mathrm{diag}(1,p,p^2,p)\mathrm{K}(N)$$
$$\mathrm{T}_{1,0}(p^2) = \mathrm{K}(N)\,\mathrm{diag}(p,p^2,p,1)\mathrm{K}(N).$$

See [3, 32] for the decompositions of these double cosets into right cosets.

For the case $p \nmid N$, we use the single coset decomposition from [3] and the following result after applying (1).

Proposition 5.2. *Let $N, f, p, s, \check{s}, \hat{s}$ be as at the beginning of this section. Let $p \nmid N$. For any integers i, j, k let $t_{i,j,k} = \left(\begin{smallmatrix} i/p & j/p \\ j/p & k/p \end{smallmatrix}\right)$, $u_{i,j,k} = \left(\begin{smallmatrix} i/p^2 & j/p \\ j/p & k/p^2 \end{smallmatrix}\right)$, and $v_i = pu_{0,ia,i(ia+2b)}$. The restrictions of $f|_k\mathrm{T}(p)$ and $f|_k\mathrm{T}_1(p^2)$ under s are*

$$\begin{aligned} &\phi_s^*(f|_k\mathrm{T}(p))(\tau) \\ &= p^{2k-3}f(ps\tau) + p^{k-3}\sum_{i \bmod p} f(\check{s}\tau + t_{i,0,0}) \\ &\quad + p^{k-3}\sum_{i,k \bmod p} f((\hat{s}+v_i)\tau + t_{0,0,k}) + p^{-3}\sum_{i,j,k \bmod p} f(s\tau/p + t_{i,j,k}) \end{aligned}$$

and

$$\begin{aligned} &\phi_s^*(f|_k\mathrm{T}_1(p^2))(\tau) \\ &= p^{3k-6}f(p\check{s}\tau) + p^{3k-6}\sum_{i \bmod p} f(p(\hat{s}+v_i)\tau) \\ &\quad + p^{2k-6}\sum_{i \not\equiv 0 \bmod p} f(s\tau + t_{i,0,0}) + p^{2k-6}\sum_{\substack{i \bmod p,\\ j \not\equiv 0 \bmod p}} f(s\tau + jt_{i^2,i,1}) \\ &\quad + p^{k-6}\sum_{\substack{i \bmod p^2,\\ j \bmod p}} f(\check{s}\tau/p + u_{i,j,0}) + p^{k-6}\sum_{\substack{i,j \bmod p,\\ k \bmod p^2}} f((\hat{s}+v_i)\tau/p + u_{0,j,k}) \end{aligned}$$

Upon expanding in Puiseux q-series, there is cancellation within the sums of restrictions in Proposition 5.2. The following proposition, which repeats Proposition 6.3.8 of [3] but also includes some additional formulas, shows that partial summation gives new restrictions whose sum *over smaller index sets* equals the original sum for integral powers of q. The proposition is subtle in that its simpler coefficients necessarily match the original ones only at integral powers. For a Puiseux series $f \in \mathbb{C}[[q^{1/\infty}]]$ and $e \in \mathbb{Q}_{\geq 0}$, let $\operatorname{coeff}_e f$ denote the coefficient of q^e in f, a complex number.

Proposition 5.3. *Let $N, f, p, s, \check{s}, \widehat{s}$ be as at the beginning of this section. Let $p \nmid N$. For any integers i, j, k let $t_{i,j,k} = \left(\begin{smallmatrix} i/p & j/p \\ j/p & k/p \end{smallmatrix}\right)$, $u_{i,j,k} = \left(\begin{smallmatrix} i/p^2 & j/p \\ j/p & k/p^2 \end{smallmatrix}\right)$, and $v_i = pu_{0,ia,i(ia+2b)}$. The following statements hold for all $e \in \mathbb{N}_0$.*

(a) *If $p \nmid a$ then*

$$\operatorname{coeff}_e \sum_{i \bmod p} f(\check{s}\tau + t_{i,0,0}) = p \operatorname{coeff}_e f(\check{s}\tau)$$

and

$$\operatorname{coeff}_e \sum_{i,j,k \bmod p} f(s\tau/p + t_{i,j,k}) = p \operatorname{coeff}_e \sum_{j,k \bmod p} f(s\tau/p + t_{0,j,k})$$

and

$$\operatorname{coeff}_e \sum_{\substack{i \bmod p^2 \\ j \bmod p}} f(\check{s}\tau/p + u_{i,j,0}) = p^2 \operatorname{coeff}_e \sum_{j \bmod p} f(\check{s}\tau/p + u_{0,j,0}).$$

(b) *If $p \nmid b$ then*

$$\operatorname{coeff}_e \sum_{i,j,k \bmod p} f(s\tau/p + t_{i,j,k}) = p \operatorname{coeff}_e \sum_{i,k \bmod p} f(s\tau/p + t_{i,0,k})$$

and

$$\operatorname{coeff}_e \sum_{\substack{i \bmod p^2 \\ j \bmod p}} f(\check{s}\tau/p + u_{i,j,0}) = p \operatorname{coeff}_e \sum_{i \bmod p^2} f(\check{s}\tau/p + u_{i,0,0}).$$

(c) *If $p \nmid c$ then*

$$\operatorname{coeff}_e \sum_{i,j,k \bmod p} f(s\tau/p + t_{i,j,k}) = p \operatorname{coeff}_e \sum_{i,j \bmod p} f(s\tau/p + t_{i,j,0}).$$

(d) *For $i \in \mathbb{Z}$, if $p \nmid c + i(ia+2b)N$ then*

$$\operatorname{coeff}_e \sum_{k \bmod p} f((\widehat{s} + v_i)\tau + t_{0,0,k}) = p \operatorname{coeff}_e f((\widehat{s} + v_i)\tau)$$

and

$$\operatorname{coeff}_e \sum_{\substack{j \bmod p \\ k \bmod p^2}} f((\widehat{s} + v_i)\tau/p + u_{0,j,k}) = p^2 \operatorname{coeff}_e \sum_{j \bmod p} f((\widehat{s} + v_i)\tau/p + u_{0,j,0}).$$

Proof. We prove the second part of (d), the others being similar; (c) is proved in [3]. Let $p \nmid c + i(ia + 2b)N$. Let $e \in \mathbb{N}_0$. With $t = \left(\begin{smallmatrix} n & r/2 \\ r/2 & mN \end{smallmatrix}\right)$, the coefficient of q^e on the left side is

$$\sum_{\substack{j \bmod p,\ k \bmod p^2 \\ n,r,m:\, an+(ia+b)r/p \\ +(c+i(ia+2b)N)mN/p^2=e}} a(t; f)\mathrm{e}(jr/p + kmN/p^2).$$

Because $\sum_{j \bmod p} \mathrm{e}(jr/p) = 0$ if $p \nmid r$, this sum is

$$\sum_{\substack{n,r,m:\, an+(ia+b)r/p \\ +(c+i(ia+2b)N)mN/p^2=e \\ p|r}} \sum_{j \bmod p,\ k \bmod p^2} a(t; f)\mathrm{e}(jr/p + kmN/p^2).$$

Because $p \nmid c + i(ia + 2b)N$, if $p|r$ then $p^2 \mid m$ inside the summation (since $p \nmid N$). Thus the above sum becomes

$$\begin{aligned}
&\sum_{\substack{n,r,m:\, an+(ia+b)r/p \\ +(c+i(ia+2b)N)mN/p^2=e \\ p|r}} \sum_{j \bmod p,\ k \bmod p^2} a(t; f)\mathrm{e}(jr/p + 0) \\
&= \sum_{\substack{n,r,m:\, an+(ia+b)r/p \\ +(c+i(ia+2b)N)mN/p^2=e \\ p|r}} \sum_{j \bmod p} p^2 a(t; f)\mathrm{e}(jr/p) \\
&= \sum_{\substack{n,r,m:\, an+(ia+b)r/p \\ +(c+i(ia+2b)N)mN/p^2=e}} \sum_{j \bmod p} p^2 a(t; f)\mathrm{e}(jr/p) \\
&= p^2 \sum_{j \bmod p} \sum_{\substack{n,r,m:\, an+(ia+b)r/p \\ +(c+i(ia+2b)N)mN/p^2=e}} a(t; f)\mathrm{e}(jr/p) \\
&= p^2 \sum_{j \bmod p} \mathrm{coeff}_e\, f((\widehat{s} + v_i)\tau/p + u_{0,j,0}).
\end{aligned}$$

□

We now give similar speed-up theorems for the case when $p\|N$. Having only one power of p divide N is needed to have all upper triangular coset representatives.

Proposition 5.4. *Let $p\|N$. Fix $\hat{p}, \hat{N} \in \mathbb{Z}$ such that $\hat{p}p + \hat{N}N/p = 1$. We have the following right coset decompositions.*

$$\begin{aligned}
&\mathrm{K}(N)\,\mathrm{diag}(p, p, 1, 1)\mathrm{K}(N) \\
&= \sum_{i,j,k \bmod p} \mathrm{K}(N) \begin{pmatrix} 1 & 0 & i & j \\ 0 & 1 & j & k/p \\ & & p & 0 \\ & & 0 & p \end{pmatrix} + \sum_{i,j \bmod p} \mathrm{K}(N) \begin{pmatrix} p & 0 & 0 & 0 \\ i & 1 & 0 & j/p \\ & & 1 & -i \\ & & 0 & p \end{pmatrix}
\end{aligned}$$

$$+ \sum_{i,j \bmod p} \mathrm{K}(N) \begin{pmatrix} 1 & -Nj & i & 0 \\ 0 & p & 0 & 0 \\ & & p & 0 \\ & & Nj & 1 \end{pmatrix} + \sum_{j \bmod p} \mathrm{K}(N) \begin{pmatrix} p & -N & jN/p & j \\ \hat{N} & \hat{p}p & \hat{p}j & -\hat{N}j/p \\ & & \hat{p}p & -\hat{N} \\ & & N & p \end{pmatrix}$$

$$+ \mathrm{K}(N) \begin{pmatrix} p & 0 & & \\ 0 & p & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix},$$

$$\mathrm{K}(N) \operatorname{diag}(p, p^2, p, 1) \mathrm{K}(N)$$

$$= \sum_{\substack{i,j \bmod p \\ k \bmod p^2}} \mathrm{K}(N) \begin{pmatrix} p & 0 & 0 & jp \\ i & 1 & j & -ij+k/p \\ & & p & -ip \\ & & 0 & p^2 \end{pmatrix} + \sum_{j \bmod p} \mathrm{K}(N) \begin{pmatrix} p & -Njp & & \\ 0 & p^2 & & \\ & & p & 0 \\ & & Nj & 1 \end{pmatrix}$$

$$+ \sum_{\substack{i,j,k \bmod p \\ k \neq 0 \bmod p}} \mathrm{K}(N) \begin{pmatrix} p & -jNp & 0 & 0 \\ -i & ijN+p & jNk/p & k/p \\ & & ijN+p & i \\ & & jNp & p \end{pmatrix}$$

$$+ \sum_{\substack{i,j \bmod p \\ i \neq 0 \bmod p}} \mathrm{K}(N) \begin{pmatrix} p & -Npj & -ijN & -i \\ \hat{N} & p-\hat{N}Nj & i(\hat{N}Nj/p-1) & i\hat{N}/p \\ & & p-\hat{N}Nj & -\hat{N} \\ & & jNp & p \end{pmatrix}.$$

Proof. The coset representatives of the decomposition of $\mathrm{K}(N) \operatorname{diag}(p, p, 1, 1) \mathrm{K}(N)$ are precisely the right coset representatives given in Proposition 2.10 of [32], except that we have replaced the last representative in Proposition 2.10 of [32] as follows: for $p \nmid i$, replace

$$\begin{pmatrix} p & 0 & & \\ 0 & p & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ 0 & i\hat{N}N & 1 & 0 \\ i\hat{N}N & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} p & -N & \hat{\imath}N/p & \hat{\imath} \\ \hat{N} & \hat{p}p & \hat{p}\hat{\imath} & -\hat{N}\hat{\imath}/p \\ & & \hat{p}p & -\hat{N} \\ & & N & p \end{pmatrix},$$

where $\hat{\imath}$ is such that $i\hat{\imath} \equiv 1 \bmod p$. It is a straightforward calculation that the left-hand representative multiplied on the right by the inverse of the right-hand representative is

$$\begin{pmatrix} (i\hat{\imath}\hat{N}N + p)/p & (i\hat{\imath}\hat{N}N^2 - Np)/p^2 & -\hat{\imath}N/p & -\hat{\imath} \\ (\hat{N}p - i\hat{\imath}\hat{N}^2 N)/p^2 & ((i\hat{\imath}\hat{N}N + p)\hat{p})/p & -\hat{\imath}\hat{p} & \hat{\imath}\hat{N}/p \\ (i\hat{N}^2 N)/p & -i\hat{N}N\hat{p} & p\hat{p} & -\hat{N} \\ -i\hat{N}N & -(i\hat{N}N^2)/p & N & p \end{pmatrix}.$$

Using the fact that $1 - i\hat{\imath}\hat{N}N/p$ is a multiple of p, it is straightforward to show that the entries satisfy the conditions for the matrix to be in $\mathrm{K}(N)$. Summing over $i \bmod p$, $i \neq 0$ is the same as summing over $\hat{\imath} \bmod p$, $\hat{\imath} \neq 0$, and so we replace $\hat{\imath}$ with j. Thus we may replace the representative as stated. The proof of the decomposition of $\mathrm{K}(N) \operatorname{diag}(p, p^2, p, 1) \mathrm{K}(N)$ is similar. $\square$

In the next two propositions, $t_{i,j,k}$ and $u_{i,j,k}$ are defined differently than they were in Propositions 5.2 and 5.3.

Proposition 5.5. *Let $N, f, p, s, \check{s}, \widehat{s}$ be as at the beginning of this section. Let $p \| N$. For any integers i, j, k let $t_{i,j,k} = \left(\begin{smallmatrix} i/p & j/p \\ j/p & k/p^2 \end{smallmatrix}\right)$, $u_{i,j,k} = \left(\begin{smallmatrix} i/p & j/p \\ j/p & k/p^3 \end{smallmatrix}\right)$, $v_i = pt_{0,ia,i(ia+2b)}$, and $w_j = j\left(\begin{smallmatrix} (-2b+jc)N/p & -c \\ -c & 0 \end{smallmatrix}\right)$. Fix $\hat{p}, \hat{N} \in \mathbb{Z}$ such that $\hat{p}p + \hat{N}N/p = 1$. The restrictions of $f|_k\mathrm{T}(p)$ and $f|_k\mathrm{T}_{1,0}(p^2)$ are*

$$
\begin{aligned}
&\phi_s^*(f|_k\mathrm{T}(p))(\tau) \\
&= p^{-3} \sum_{i,j,k \bmod p} f(s\tau/p + t_{i,j,k}) + p^{k-3} \sum_{i,k \bmod p} f((\widehat{s} + v_i)\tau + t_{0,0,k}) \\
&\quad + p^{k-3} \sum_{i,j \bmod p} f((\check{s} + w_j)\tau + t_{i,0,0}) \\
&\quad + p^{k-3} \sum_{j \not\equiv 0 \bmod p} f\Big(\begin{pmatrix} -2bN+cN/p+ap & b-\hat{p}c+a\hat{N}-2b\hat{N}N/p \\ b-\hat{p}c+a\hat{N}-2b\hat{N}N/p & \frac{-\hat{p}c\hat{N}N+a\hat{N}^2N+\hat{p}cp+2\hat{p}b\hat{N}Np}{Np} \end{pmatrix}\tau + t_{0,j,0}\Big) \\
&\quad + p^{2k-3} f(ps\tau),
\end{aligned}
$$

$$
\begin{aligned}
&\phi_s^*(f|_k\mathrm{T}_{1,0}(p^2))(\tau) \\
&= p^{k-6} \sum_{\substack{i,j \bmod p, \\ k \bmod p^2}} f((\widehat{s} + v_i)\tau/p + u_{0,j,k}) \\
&\quad + p^{2k-6} \sum_{\substack{i,j \bmod p, \\ i \neq 0}} f\Big(\Big(s + \begin{pmatrix} j(cj-2b)N & \frac{a\hat{N}-2b\hat{N}jN+c\hat{N}j^2N-cjp}{p} \\ \frac{\{a\hat{N}-2b\hat{N}jN+ c\hat{N}j^2N-cjp\}}{p} & \frac{\{a\hat{N}^2-2b\hat{N}^2jN+ c\hat{N}^2j^2N+2b\hat{N}p-2c\hat{N}jp\}}{p^2} \end{pmatrix}\Big)\tau + t_{0,i,0}\Big) \\
&\quad + p^{3k-6} \sum_{j \bmod p} f(p(\check{s} + w_j)\tau) \\
&\quad + p^{2k-6} \sum_{\substack{i,j \bmod p, \\ k \not\equiv 0 \bmod p}} f\Big(\Big(s + \begin{pmatrix} j(-2b+cj)N & \frac{-ai+2bijN-cij^2N-cjp}{p} \\ \frac{\{-ai+2bijN -cij^2N-cjp\}}{p} & \frac{\{ai^2-2bi^2jN+ci^2j^2N -2bip+2cijp\}}{p^2} \end{pmatrix}\Big)\tau + t_{0,0,k}\Big)
\end{aligned}
$$

Proof. Apply (1) to Proposition 5.4. □

We have the following available speed-ups.

Proposition 5.6. *Let $N, f, p, s, \check{s}, \widehat{s}$ be as at the beginning of this section. Let $p \| N$. For any integers i, j, k let $t_{i,j,k} = \left(\begin{smallmatrix} i/p & j/p \\ j/p & k/p^2 \end{smallmatrix}\right)$, $u_{i,j,k} = \left(\begin{smallmatrix} i/p & j/p \\ j/p & k/p^3 \end{smallmatrix}\right)$, $v_i = pt_{0,ia,i(ia+2b)}$, and $w_j = j\left(\begin{smallmatrix} (-2b+jc)N/p & -c \\ -c & 0 \end{smallmatrix}\right)$. Then the following statements hold for all $e \in \mathbb{Z}_{\geq 0}$.*

(a) *If $p \nmid a$, then*

$$
\operatorname{coeff}_e \sum_{i,j,k \bmod p} f(s\tau/p + t_{i,j,k}) = p \operatorname{coeff}_e \sum_{j,k \bmod p} f(s\tau/p + t_{0,j,k})
$$

and

$$\mathrm{coeff}_e \sum_{i,j \bmod p} f((\check{s}+w_j)\tau + t_{i,0,0}) = p\,\mathrm{coeff}_e \sum_{j \bmod p} f((\check{s}+w_j)\tau)$$

(b) *If $p \nmid b$ then*

$$\mathrm{coeff}_e \sum_{i,j,k \bmod p} f(s\tau/p + t_{i,j,k}) = p\,\mathrm{coeff}_e \sum_{i,k \bmod p} f(s\tau/p + t_{i,0,k}).$$

(c) *If $p \nmid c$ then*

$$\mathrm{coeff}_e \sum_{i,j,k \bmod p} f(s\tau/p + t_{i,j,k}) = p\,\mathrm{coeff}_e \sum_{i,j \bmod p} f(s\tau/p + t_{i,j,0})$$

and

$$\mathrm{coeff}_e \sum_{i,k \bmod p} f((\hat{s}+v_i)\tau + t_{0,0,k}) = p\,\mathrm{coeff}_e \sum_{i \bmod p} f((\hat{s}+v_i)\tau)$$

and

$$\mathrm{coeff}_e \sum_{\substack{i,j \bmod p,\\ k \bmod p^2}} f((\hat{s}+v_i)\tau/p + u_{0,j,k})$$
$$= p^2\,\mathrm{coeff}_e \sum_{i,j \bmod p} f((\hat{s}+v_i)\tau/p + u_{0,j,0}).$$

(d) *For fixed i, if $p \nmid ia+b$ then*

$$\mathrm{coeff}_e \sum_{\substack{j \bmod p,\\ k \bmod p^2}} f((\hat{s}+v_i)\tau/p + u_{0,j,k}) = p\,\mathrm{coeff}_e \sum_{k \bmod p^2} f((\hat{s}+v_i)\tau/p + u_{0,0,k}).$$

Proof. The proofs are similar to those of Proposition 5.3 □

Another speed-up is that for $X, Y \in \mathrm{M}_2^{\mathrm{sym}}$, if the set $\{t \in \mathcal{X}_2 : \mathrm{Tr}(Xt) = e\}$ is empty then $\mathrm{coeff}_e\, f(X\tau + Y) = 0$. Here the set is independent of Y and the conclusion holds for all Y, and so checking whether the set is empty can save significant computation time. Further, this result can be crucial when the denominator of a particular formula for f might restrict to zero for some X and Y, because we simply skip this X.

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