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SOLUTIONS FOR QUADRATIC TRINOMIAL PARTIAL DIFFERENTIAL-DIFFERENCE EQUATIONS IN $\mathbb{C}^n$

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Abstract. In this paper, we utilize Nevanlinna theory to study the existence and forms of solutions for quadratic trinomial complex partial differential-difference equations of the form $aF^2 + 2\omega FG + bG^2 = \exp(g)$, where $ab \neq 0, \omega \in \mathbb{C}$ with $\omega^2 \neq 0, ab$ and g is a polynomial in $\mathbb{C}^n$. In order to achieve a comprehensive and thorough analysis, we study the characteristics of solutions in two specific cases: one when $\omega^2 \neq 0, ab$ and the other when $\omega = 0$. Because polynomials in several complex variables may exhibit periodic behavior, a property that differs from polynomials in single complex variables, our study of finding solutions of equations in $\mathbb{C}^n$ is significant. The main results of the paper improved several known results in $\mathbb{C}^n$ for $n \geq 2$. Additionally, the corollaries generalize results of Xu *et al.* [Rocky Mountain J. Math. **52**(6) (2022), 2169–2187] for trinomial equations with arbitrary coefficients in $\mathbb{C}^n$. Finally, we provide examples that endorse the validity of the conclusions drawn from the main results and their related remarks.

1. Introduction

Partial differential equations ($PDEs$) or partial differential-difference equations ($PDDEs$) are fundamental mathematical equations used to describe a wide range of phenomena in science and engineering. While various numerical and computational methods have been developed to obtain approximate solutions for these intricate equations, the exploration of true analytic solutions has been comparatively limited. This paper aims to bridge this gap by proposing a novel approach that harnesses the power of Nevanlinna theory for several complex variables to unveil the elusive realm of analytic solutions of $PDEs$ and $PDDEs$. Nevanlinna theory, rooted in complex analysis, offers a unique perspective on the behavior of meromorphic functions, which in turn provides a powerful tool for uncovering the intricacies of analytic solutions to complex $PDEs$ and $PDDEs$. Through the application of Nevanlinna theory, this study

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strives to shed light on unexplored analytical solutions, offering new insights into the underlying structure and properties of these equations.

Moreover, the Nevanlinna theory has emerged as a powerful and flexible tool in the investigation of functional equations, particularly in the context of Fermat-type equations that involve several complex variables. Researchers have extensively utilized this theory to explore the presence of solutions to these equations in the complex domain. By employing essential concepts like the logarithmic derivative lemma and its difference analogue, mathematicians have achieved significant advancements in understanding the intricate behavior of these functional equations in several complex variables. The adaptation and development of Nevanlinna theory to accommodate several complex variables have recently spurred a surge in research activities, with a specific focus on unraveling the solutions of Fermat-type functional equations in $\mathbb{C}^2$ or even higher-dimensional spaces like $\mathbb{C}^n$. This broader approach has opened up novel avenues of exploration, shedding light on the existence and characteristics of solutions for these equations in complex higher-dimensional settings, presenting a captivating and challenging frontier for mathematical inquiry.

In this paper, we consider solutions of certain functional equations in $\mathbb{C}^n$ related to Fermat varieties. Among the most basic functional equations are the circle functional equation $f^2 + g^2 = 1$, and the Fermat cubic $f^3 + g^3 = 1$. Generalizations of these power equations are called Fermat-type functional equations, which are associated with diagonal varieties, and have been the subject of interest in global complex analysis in connection with the extensions of Picard-type theorems and results on hyperbolic sub-manifolds of projective space (see for example [10, 19, 41]). Due to the development of the difference analogue lemma of logarithmic derivative lemma, in recent year an increasing amount of interests has been grown up for several properties of entire and meromorphic solutions of several difference functional equations both in one and several complex variables. Since non-constant polynomials in $\mathbb{C}^n$ (for $n \geq 2$) may be periodic, the nature of solutions of Fermat-type equations in $\mathbb{C}^n$ is completely different from that in $\mathbb{C}$. This is one of the reason why we consider Fermat-type functional equations in several complex variables in our study.

The study of Fermat-type functional equation has been an interesting subject in the field of complex analysis in connection with extensions of Nevanlinna's theory. For extensive research on the Fermat-type functional equations, we refer to the articles [1, 5, 6, 35, 36, 38, 40] and references therein. We will assume that the reader is familiar with basic elements of the Nevanlinna's theory of meromorphic function f in one or several complex variables (see e.g., [16, 17, 43]), such as the characteristic function $T(r, f)$, the counting function $N(r, f)$ for poles of f, reduce counting function $\overline{N}(r, f)$ of f, proximation function $m(r, f)$ in the value distribution theory, also known as Nevanlinna theory. We denote by $S(r, f)$, any function satisfying $S(r, f) = \circ\{T(r, f)\}$ as $r \to \infty$, possibly outside a set of finite measure. In addition, we use the notation $\rho(f)$ to denote the order of growth of the meromorphic function f in $\mathbb{C}^n$, and defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

It has always been a well-known and interesting problem to investigate the existence and form of solutions to Fermat-type functional equations of the form

$$f^n(z) + g^n(z) = 1 \tag{1.1}$$

regard as the Fermat diophantine equation $x^n + y^n = 1$ over functional fields, where $n \geq 2$ is an integer. The classical results on meromorphic solutions in $\mathbb{C}$ of (1.1) have been studied and forms of the solutions are obtained (see e.g. [2, 11, 28]). It is understood that (1.1) does not admit transcendental meromorphic (resp. entire) solutions when $n \geq 4$ (resp. $n \geq 3$). If $n = 3$, then equation (1.1) admits meromorphic solutions $f = (3 + \sqrt{3}\wp'(\beta))/6\wp(\beta)$ and $g = \eta(3 - \sqrt{3}\wp'(\beta))/6\wp(\beta)$, for some non-constant entire function β, where $\eta^3 = 1$ and $\wp$ denotes the Weierstrass $\wp$-function satisfying $(\wp')^2 \equiv 4\wp^3 - 1$ after appropriately choosing its periods. For $n = 2$, (1.1) has nontrivial (non-constant) entire solutions $f(z) = \cos(\psi(z))$ and $g(z) = \sin(\psi(z))$, where ψ is an entire function. For the study of meromorphic solutions to (1.1) in $\mathbb{C}^n$ and applications to complex partial differential equations, we refer the reader to [22–25] and references therein.

This article mainly concerns the analytic solutions for trinomial quadratic partial differential-difference equations (in short, $PDDE$) with arbitrary coefficients of the form $aF^2 + 2\omega FG + bG^2 = \exp(g)$, where a, b, ω are complex constants and g is a polynomial in $\mathbb{C}^n$. In particular, for $a = 1 = b$ and $g(z) = 2k\pi i$, k being an integer, there are number of results in $\mathbb{C}$ and $\mathbb{C}^2$. In fact, what could be the characterization of solutions of the trinomial in $\mathbb{C}^n$ is not explored yet and need to study. In general, one cannot expect the existence of analytic solutions, and even when global analytic or entire solutions exist, it is difficult to find such solutions in closed form in $\mathbb{C}^n$. The finite order solutions to the Fermat-type binomial and trinomial equations in $\mathbb{C}$ over some commonly studied function fields have been investigated by many authors, and there is an extensive literature on these equations and generalizations as well as connections to other problems (see e.g., [4, 7, 11–13, 28, 34, 41, 42]). Furthermore, it appears that the solutions of the system of Fermat-type binomial or trinomial equations in $\mathbb{C}^2$ has been recently studied in [35, 38]. However, no study has so far been done on the solutions of quadratic trinomial functional equations in $\mathbb{C}^n$. In this paper, our main aim is to describe transcendental solutions for quadratic trinomial $PDDEs$ in $\mathbb{C}^n$. Henceforth, throughout this paper, we assume that $z + c = (z_1 + c_1, \ldots, z_n + c_n)$, for any $z = (z_1, \ldots, z_n)$ and $c = (c_1, \ldots, c_n)$ are in $\mathbb{C}^n$. The difference operator $\Delta_c f$ of entire functions f in $\mathbb{C}^n$ is defined by $\Delta_c f(z) := f(z + c) - f(z)$.

Liu *et al.* [26] have investigated the Fermat-type difference equation $f^2(z)+f^2(z+c) = 1$ in $\mathbb{C}$ and obtained the finite order transcendental entire solutions satisfy $f(z) = \sin(Az + B)$, where B is a constant and $A = ((4k + 1)\pi)/(2c)$, where k is an integer. Later, Han and Lu [15] established the solution to the more general complex difference equation $f^n(z)+g^n(z) = e^{\alpha z+\beta}$. Moreover, Liu *et al.* [26] showed that the existence of solutions for the complex differential-difference equations $f'(z)^2 + f(z + c)^2 = 1$ and $f'(z)^2 + [\Delta_c f(z)]^2 = 1$ in $\mathbb{C}$.

Our aim is to analyze solutions of partial differential equations in $\mathbb{C}^n$. As is known to all, partial differential equations ($PDEs$) are occurring in various areas of applied mathematics, such as fluid mechanics, nonlinear acoustics, gas dynamics, and traffic flow (see [8, 9]). In general, it is difficult to find entire and meromorphic solutions for a nonlinear PDE. By employing Nevanlinna theory and the method of complex analysis, there were a number of literature focusing on the solutions of some $PDEs$ and theirs many variants (see [3, 5, 6, 14, 18, 21, 24, 27, 32, 39]).

1.1. Solutions of Fermat-type partial differential equations in $\mathbb{C}^2$:

The solutions of Fermat-type $PDEs$ were investigated by [3, 20, 31]. Most noticeably, in 1995, Khavinson [18] derived that any entire solution of the partial differential equation in $\mathbb{C}^2$,

$$\left(\frac{\partial u}{\partial z_1}\right)^2 + \left(\frac{\partial u}{\partial z_2}\right)^2 = 1$$

is necessarily linear, i.e., $u(z_1, z_2) = az_1+bz_2+c$, where $a, b, c \in \mathbb{C}$, and $a^2+b^2 = 1$. This PDE in the real variable case occurs in the study of characteristic surfaces and wave propagation theory, and it is the two-dimensional eiconal equation, one of the main equations of geometric optics (see [8]). Furthermore, Li [22, 24] have continued the research and discussed solutions of a series of $PDEs$ with more general forms including $(\frac{\partial f}{\partial z_1})^2+(\frac{\partial f}{\partial z_2})^2 = e^g$, $(\frac{\partial f}{\partial z_1})^2+(\frac{\partial f}{\partial z_2})^2 = p$, etc., where g, p are polynomials in $\mathbb{C}^2$. In 2020, Xu and Cao [37] investigated the entire and meromorphic solutions of the Fermat-type functional equations such as partial differential equation and obtained the following result

Theorem 1.1 ([37, Theorem 1.4])**.** *Any transcendental entire solution with finite order of Fermat-type partial differential equation*

$$f^2(z_1, z_2) + \left(\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^2 = 1 \tag{1.2}$$

has the form of $f(z_1, z_2) = \sin(z_1 + g(z_2))$*, where* $g(z_2)$ *is a polynomial in one variable* z_2*.*

Recently, Xu *et al.* [40] have established solutions of the following $PDDEs$

$$\left(\alpha\frac{\partial f(z)}{\partial z_1} + \beta\frac{\partial f(z)}{\partial z_2}\right)^2 + f(z + c)^2 = \exp(g(z)) \tag{1.3}$$

and

$$\left(\alpha\frac{\partial f(z)}{\partial z_1}+\beta\frac{\partial f(z)}{\partial z_2}\right)^2+[\Delta_c f(z)]^2=\exp(g(z)) \tag{1.4}$$

in $\mathbb{C}^2$, and they have obtained the precise form of the solution in $\mathbb{C}^2$.

1.2. Motivations and some questions:

Initially, several researchers considered Fermat-type functional equations on $\mathbb{C}$ and extensively explored the solutions. The main tools used in such equations are the Nevanlinna theory, especially the logarithmic derivative lemma and the difference analogue lemma of the logarithmic derivative lemma. However, with the development of these lemmas for several complex variables, there has been a recent surge in research focusing on exploring the solutions of Fermat-type functional equations in $\mathbb{C}^2$ or even in $\mathbb{C}^n$. While the results obtained for $\mathbb{C}^2$ have limited scope, similar equations can be considered in $\mathbb{C}^n$, making the study interesting to explore the solutions in that case. Consequently, through a detailed exploration of the results concerning Fermat-type equations in $\mathbb{C}^2$ and their proofs, it has become evident that these results can be extended to $\mathbb{C}^n$. Additionally, the scope for selecting combinations of partial derivatives can be made broad than what was considered in the earlier known results. The results presented above naturally prompt several questions to be raised.

Question 1.1. What can be said about the form of solutions in $\mathbb{C}^n$, if we extend the binomial equations (1.3) and (1.4) to a more general trinomial equation (2.1) with an arbitrary coefficient.

Question 1.2. Does there exists solutions of (1.2) in Theorem 1.1, if we take $L_k(f) := \sum_{t=1}^{k}\lambda_t\frac{\partial^t f(z)}{\partial z_i^t}$ in the place of $\frac{\partial f(z_1,z_2)}{\partial z_1}$ in $\mathbb{C}^n$ with arbitrary coefficients? If exists, then what would be the form of solutions, where λ_t are constants in $\mathbb{C}$, $t=1,2,\ldots,k$?

Motivated by the above question, our purpose of this article is to exploring the finite order transcendental entire solutions of the quadratic trinomial partial differential equations. To find precise solutions of trinomial quadratic functional equations we use with certain techniques. More precisely, Saleeby [32] initiates this type of study considering the quadratic trinomial equations of the form $f^2+2\alpha fg+g^2=1$, where $\alpha\in\mathbb{C}\setminus\{-1,1\}$, which is associated with the partial differential equations

$$u_x^2+2\alpha u_x u_y+u_y^2=1, \tag{1.5}$$

where $(x,y)\in\mathbb{C}^2$ and showed that the entire and meromorphic solutions of (1.5) have the form $u(x,y)=ax+by+c$, where $a^2+2\alpha ab+b^2=1$

The main tools used in this paper are the Nevanlinna theory and the characteristic equations for quasi-linear *PDEs* and linear *PDEs*. The paper is organized as follows. Our main results about the existence and the forms of entire solutions and their corollaries with examples will be exhibited in Section 2. The proofs of the main results will be given in Section 3.

2. Main results, corollaries, and examples

Motivated by method of proof of results in [3, 40], we explore the finite order transcendental entire solutions of quadratic trinomial partial differential equations in $\mathbb{C}^n$. Henceforth, throughout this paper, we assume that $z + c = (z_1 + c_1, \ldots, z_n + c_n)$, for any $z = (z_1, \ldots, z_n)$ and $c = (c_1, \ldots, c_n)$ are in $\mathbb{C}^n$. To serve the purpose, we define $\omega_1 := -\frac{\omega}{\sqrt{ab}} \pm \frac{\sqrt{\omega^2-ab}}{\sqrt{ab}}$ and $\omega_2 := -\frac{\omega}{\sqrt{ab}} \mp \frac{\sqrt{\omega^2-ab}}{\sqrt{ab}}$. Let $g(z) = \sum_{|I|=0}^{p} a_{\alpha_1,\ldots,\alpha_n} z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ be a polynomial in $\mathbb{C}^n$, where $I = (\alpha_1, \ldots, \alpha_n)$ are two multi-index with $|I| = \sum_{j=0}^{n} \alpha_j$ and α_j are non-negative integers.

We aim to investigate the solutions of the following trinomial equation

$$\begin{aligned} & a\left(\alpha\frac{\partial f(z)}{\partial z_i} + \beta\frac{\partial f(z)}{\partial z_j}\right)^2 \\ & + 2\omega\left(\alpha\frac{\partial f(z)}{\partial z_i} + \beta\frac{\partial f(z)}{\partial z_j}\right)(\gamma f(z+c) + \delta f(z)) \\ & + b\left(\gamma f(z+c) + \delta f(z)\right)^2 = \exp\left(g(z)\right) \end{aligned} \tag{2.1}$$

in $\mathbb{C}^n$ that corresponds to (1.3) and (1.4).

Our objective is to thoroughly investigate the explicit representation of solutions f for the equation (2.1) and the polynomial g. The resulting conclusion provides a comprehensive response to Question 1.1.

Theorem 2.1. *Let $c = (c_1, \ldots, c_n) \in \mathbb{C}^n \setminus \{(0, \ldots, 0)\}$, $ab \neq 0$, $\alpha, \beta, \gamma, \delta$ be constants in $\mathbb{C}$ that are not all zero and $\omega^2 \neq 0, ab$, and $1 \leq i < j \leq n$, and $(\alpha c_j - \beta c_i) \neq 0$, where $c_i, c_j \in \mathbb{C}$. If f is a finite order transcendental entire solution of the $PDDE$ (2.1), then f must assume one of the following forms*:

(i)

$$f(z) = \phi\left(z_j - \frac{\beta}{\alpha}z_i\right),$$

where ϕ is a finite order transcendental entire function satisfying

$$\gamma\phi\left(z_j - \frac{\beta}{\alpha}z_i + c_j - \frac{\beta}{\alpha}c_i\right) + \delta\phi\left(z_j - \frac{\beta}{\alpha}z_i\right) = \pm\frac{1}{\sqrt{b}}\exp\left(\frac{g(z)}{2}\right).$$

(ii)

$$f(z) = \pm\frac{1}{\alpha\sqrt{a}}\int_0^{z_i/\alpha} \exp\left(\frac{L(z) + H(s_1) + R}{2}\right) dz_i + \psi_1\left(z_j - \frac{\beta}{\alpha}z_i\right),$$

$g(z) = L(z) + H(s_1) + R$, where $L(z) = a_1 z_1 + \cdots + a_n z_n$ and $H(s_1)$ is a polynomial in $s_1 := d_1 z_1 + \cdots + d_n z_n$ with $d_1 c_1 + \cdots + d_n c_n = 0$ such that $H(z + c) = H(z)$, $R \in \mathbb{C}$, $a_1 c_1 + \cdots + a_n c_n = 2\ln(-\frac{\delta}{\gamma})$

and $a_i, d_i \in \mathbb{C}$ *for* $i = 1, \ldots, n$ *and* ψ_1 *is a finite order entire function satisfying*

$$\psi_1\left(z_j - \frac{\beta}{\alpha}z_i + c_j - \frac{\beta}{\alpha}c_i\right) = -\frac{\delta}{\gamma}\psi_1\left(z_j - \frac{\beta}{\alpha}z_i\right).$$

(iii) *if* $\alpha d_i + \beta d_j \neq 0$ *for* $d_i, d_j \in \{d_1, \ldots, d_n\} \subset \mathbb{C}$, *then*

$$f(z) = \frac{2(\omega_2\xi^2 - \omega_1)}{\xi\sqrt{a}(\omega_2 - \omega_1)(k_i\alpha + k_j\beta)}\exp\left(\frac{L(z) + R_2}{2}\right) + \phi_1\left(z_j - \frac{\beta}{\alpha}z_i\right),$$

$g(z) = L(z) + R_2$, *where* $L(z) = k_1z_1 + \cdots + k_nz_n$, $R_2 \in \mathbb{C}$, ϕ_1 *is a finite order entire function satisfying similar condition as* ψ_1 *in* (ii) *with*

$$\frac{\sqrt{a}(\xi^2 - 1)}{2\gamma\sqrt{b}(\omega_2\xi^2 - \omega_1)}(\alpha k_i + \beta k_j) - \frac{\delta}{\gamma} = \exp\left(\frac{k_1c_1 + \cdots + k_nc_n}{2}\right).$$

(iv) *if* $\alpha d_i + \beta d_j \neq 0$ *for* $d_i, d_j \in \{d_1, \ldots, d_n\} \subset \mathbb{C}$, *then*

$$f(z) = \frac{1}{\sqrt{a}(\omega_2 - \omega_1)}\left(\frac{\omega_2\exp(L_1(z) + R_3)}{(\alpha a_{1i} + \beta a_{1j})} - \frac{\omega_1\exp(L_2(z) + R_4)}{(\alpha a_{2i} + \beta a_{2j})}\right) + \phi_2\left(z_j - \frac{\beta}{\alpha}z_i\right),$$

$g(z) = L_1(z) + L_2(z) + R_3 + R_4$, $L_1(z) \neq L_2(z)$, *where* $L_l(z) = a_{l1}z_1 + \cdots + a_{ln}z_n$ *and* $R_3, R_4 \in \mathbb{C}$, ϕ_2 *is a finite order function with satisfying similar condition as* ψ_1 *in* (ii) *with*

$$\begin{cases} \dfrac{\sqrt{a}}{\gamma\omega_2\sqrt{b}}\left((\alpha a_{1i} + \beta a_{1j}) - \delta\sqrt{b}\omega_2\right)\exp(-L_1(c)) \equiv 1, \\ \dfrac{\sqrt{a}}{\gamma\omega_1\sqrt{b}}\left((\alpha a_{2i} + \beta a_{2j}) - \delta\sqrt{b}\omega_1\right)\exp(-L_2(c)) \equiv 1. \end{cases}$$

Remark 2.1. It is evident that Theorem 2.1 in $\mathbb{C}^n$ has been established in a manner that extends the scope of the study conducted by Xu *et al.* [40, Theorems 2.1, 2.2] and Xu *et al.* [36, Theorem 1.2].

The following examples are exhibited to validate the existence and precise form of the solutions of equations in Theorem 2.1.

Example 2.1. For $c = (2, 2, 3) \in \mathbb{C}^3$ and $a = \beta = 1, b = 3, \alpha = 2, \gamma = 2, \delta = -5$, the function

$$f(z_1, z_2, z_3) = \frac{(4 \mp 3\sqrt{13})}{\mp 4\sqrt{26}}\exp\left(\frac{3z_1 + \ln\left(\frac{22\mp 15\sqrt{13}}{2(4+\sqrt{13})}\right)z_2 - 2z_3 + \frac{\pi i}{7}}{2}\right) + \phi_1\left(z_3 - \frac{1}{2}z_1\right)$$

for $\phi_1\left(z_3-\frac{1}{2}z_1+2\right)=\frac{5}{2}\phi_1\left(z_3-\frac{1}{2}z_1\right)$; is a transcendental entire solution in $\mathbb{C}^3$ of the differential-difference equation

$$\left(2\frac{\partial f(z)}{\partial z_1}+\frac{\partial f(z)}{\partial z_3}\right)^2-8\omega\left(2\frac{\partial f(z)}{\partial z_1}+\frac{\partial f(z)}{\partial z_3}\right)(2f(z+c)-5f(z)) \\ +3\left(2f(z+c)-5f(z)\right)^2=\exp\left(g(z)\right),$$

where $g(z)=3z_1+\ln\left(\frac{22\mp15\sqrt{13}}{2(4+\sqrt{13})}\right)z_2-2z_3+\frac{\pi i}{7}$.

Example 2.2. For $c=(3,1,-4)\in\mathbb{C}^3$ and $a=\alpha=3, b=\gamma=1, \beta=2, \delta=-1$, the function

$$f(z_1,z_2,z_3)=\frac{(5\mp\sqrt{22})}{\mp36\sqrt{66}}\exp(g_1(z_1,z_2,z_3))-\frac{(5\pm\sqrt{22})}{\mp72\sqrt{66}}\exp(g_2(z_1,z_2,z_3)) \\ +\exp\left(\pi i\left(\frac{2z_1}{3}+z_3\right)\right),$$

where

$$\begin{cases} g_1(z_1,z_2,z_3)=4z_1+\ln\left(\dfrac{3(23\mp\sqrt{22})}{5\mp\sqrt{22}}\right)z_2+3z_3+\dfrac{(\pi i+\sqrt{3})}{\sqrt{7}} \\ g_2(z_1,z_2,z_3)=8z_1+\ln\left(\dfrac{\sqrt{3}(36\sqrt{3}+5\pm\sqrt{22})}{5\pm\sqrt{22}}\right)z_2+6z_3+\dfrac{(\pi i+\sqrt{5})}{\sqrt{7}} \end{cases}$$

are transcendental entire solutions in $\mathbb{C}^3$ of the differential-difference equation

$$3\left(3\frac{\partial f(z)}{\partial z_1}+2\frac{\partial f(z)}{\partial z_3}\right)^2 \\ -10\left(3\frac{\partial f(z)}{\partial z_1}+2\frac{\partial f(z)}{\partial z_3}\right)(f(z+c)-f(z)) \\ +(f(z+c)-f(z))^2=\exp(g(z)),$$

where

$$g(z)=12z_1+\left(\ln\left(\frac{3(23\mp\sqrt{22})}{5\mp\sqrt{22}}\right)+\ln\left(\frac{\sqrt{3}(36\sqrt{3}+5\pm\sqrt{22})}{5\pm\sqrt{22}}\right)\right)z_2 \\ +9z_3+\frac{(2\pi i+\sqrt{5}+\sqrt{3})}{\sqrt{7}}.$$

Hereafter, we explore our second principal finding, which involves deriving the precise solutions to a quadratic trinomial partial differential equation, providing a comprehensive response to Question 1.2. In this case, we consider the following trinomial PDE

$$af^2(z)+2\omega f(z)\left(\sum_{t=1}^{k}\lambda_t\frac{\partial^t f(z)}{\partial z_i^t}\right)+b\left(\sum_{t=1}^{k}\lambda_t\frac{\partial^t f(z)}{\partial z_i^t}\right)^2=1 \tag{2.2}$$

and obtained the following result.

Theorem 2.2. *Let $ab \neq 0$, and $1 \leq i \leq n$. The PDE* (2.2) *admits a transcendental entire solution of finite order in $\mathbb{C}^n$. Moreover,*

(I) *if $\omega^2 \neq 0, ab$, then f assumes the following form*

$$f(z) = \frac{\omega_2 \exp(F(z)) - \omega_1 \exp(-F(z))}{\sqrt{a}(\omega_2 - \omega_1)},$$

where $F(z) = \beta z_i + \phi(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$ with $\beta \in \mathbb{C} \setminus \{0\}$ and ϕ is an arbitrary polynomial in $z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n$ such that

$$\omega_2 \sqrt{b} \sum_{t=1}^{k} \lambda_t \beta^t = \sqrt{a} \ \text{ and } \ \omega_1 \sqrt{b} \sum_{t=1}^{k} (-1)^t \lambda_t \beta^t = \sqrt{a}.$$

(II) *if $\omega^2 = 0$, then f assumes the following form*

$$f(z) = \frac{1}{\sqrt{a}} \cosh\left(\eta z_i + \phi(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)\right),$$

where η is a non-zero constant in $\mathbb{C}$ and $\phi(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$ is an arbitrary polynomial in $z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n$ such that

$$i\sqrt{b} \sum_{t=1}^{k} \lambda_t \eta^t = \sqrt{a} \ \text{ and } \ i\sqrt{b} \sum_{t=1}^{k} (-1)^t \lambda_t \eta^t = -\sqrt{a}.$$

Remark 2.2. It is apparent that Theorem 2.2 in $\mathbb{C}^n$ extends the Fermat-type functional equations [3, Theorem 1.13] and [37, Theorem 1.4] to the quadratic trinomial equation with a more general setting.

As a consequence of Theorem 2.2, the following result can be derive easily which is an improved version of [3, Theorem 1.13] and [37, Theorem 1.4] in $\mathbb{C}^2$. The corollary, in essence, offers a more comprehensive and inclusive perspective, encompassing the entire scope of [3, Theorem 1.13] and [37, Theorem 1.4].

Corollary 2.1. *Suppose that $ab \neq 0$. The PDE*

$$af^2(z) + 2\omega f(z) \left(\sum_{t=1}^{k} \lambda_t \frac{\partial^t f(z)}{\partial z_1^t} \right) + b \left(\sum_{t=1}^{k} \lambda_t \frac{\partial^t f(z)}{\partial z_1^t} \right)^2 = 1. \tag{2.3}$$

in $\mathbb{C}^2$ admits transcendental entire solutions f of finite order. Moreover,

(I) *if $\omega^2 \neq 0, ab$, then f assumes the following form*

$$f(z) = \frac{\omega_2 \exp(\beta z_1 + \psi(z_2)) - \omega_1 \exp(-\beta z_1 - \psi(z_2))}{\sqrt{a}(\omega_2 - \omega_1)},$$

where β is a non-zero constant in $\mathbb{C}$ and $\psi(z_2)$ is an arbitrary polynomial in z_2 such that

$$\omega_2 \sqrt{b} \sum_{t=1}^{k} \lambda_t \beta^t = \sqrt{a} \ \text{ and } \ \omega_1 \sqrt{b} \sum_{t=1}^{k} (-1)^t \lambda_t \beta^t = \sqrt{a}.$$

(II) *if* $\omega^2 = 0$*, then* f *takes the following form*

$$f(z) = \frac{1}{\sqrt{a}} \cosh\left(\eta z_1 + \phi(z_2)\right),$$

where η *is a non-zero constant in* $\mathbb{C}$ *and* $\phi(z_2)$ *is an arbitrary polynomial in* z_2 *such that*

$$i\sqrt{b}\sum_{t=1}^{k}\lambda_t\eta^t = \sqrt{a} \;\; \text{and} \;\; i\sqrt{b}\sum_{t=1}^{k}(-1)^t\lambda_t\eta^t = -\sqrt{a}.$$

By the following examples, we show that the solution of equation (2.2) for (I) and (II) in Theorem 2.2 is precise.

Example 2.3. For $\omega = \sqrt{3}, a = 2$ and $b = 1$, the function

$$f(z) = \frac{1}{\sqrt{2}}\cosh\left(\sqrt{2}iz_1 + \phi(z_2, z_3)\right) + \frac{\sqrt{3}}{\sqrt{2}}\sinh\left(\sqrt{2}iz_1 + \phi(z_2, z_3)\right),$$

where ϕ is an arbitrary polynomial in z_2, z_3, is a transcendental entire solution in $\mathbb{C}^3$ of the partial differential equation

$$2f^2(z) + 2\sqrt{3}f(z)\left(\sqrt{2}\frac{\partial f(z)}{\partial z_1} + \frac{\sqrt{3}}{2}\frac{\partial^2 f(z)}{\partial z_1^2} + \frac{(1+2i)}{2\sqrt{2}i}\frac{\partial^3 f(z)}{\partial z_1^3}\right)$$
$$+\left(\sqrt{2}\frac{\partial f(z)}{\partial z_1} + \frac{\sqrt{3}}{2}\frac{\partial^2 f(z)}{\partial z_1^2} + \frac{(1+2i)}{2\sqrt{2}i}\frac{\partial^3 f(z)}{\partial z_1^3}\right)^2 = 1.$$

Example 2.4. For $a = 3$, $b = 2$ and $\eta = \sqrt{3}$, the function

$$f(z) = \frac{1}{\sqrt{3}}\cosh\left(\sqrt{3}z_1 + \psi(z_2, z_3)\right),$$

where ψ is an arbitrary polynomial in z_2, z_3, is a transcendental entire solution in $\mathbb{C}^3$ of the quadratic partial differential equation

$$3f^2(z) + \left(\sqrt{5}\frac{\partial f(z)}{\partial z_1} + \frac{(1-i\sqrt{10})}{3\sqrt{2}i}\frac{\partial^3 f(z)}{\partial z_1^3}\right)^2 = 1.$$

In the next section, we will introduce several key lemmas in Nevanlinna theory. In the detailed discussion of the proof for the main results, these lemmas play a key role.

3. Some key lemmas and proof of the main results

First, we present here some necessary lemmas which will play a key roles in proving the main results of this paper.

Lemma 3.1 ([30, 33])**.** *For any entire function* F *on* $\mathbb{C}^n$*,* $F(0) \neq 0$ *and put* $\rho(n_F) = \rho < \infty$*, where* $\rho(n_F)$ *denotes be the order of the counting function of zeros of* F*. Then there exist a canonical function* f_F *and a function* $g_F \in \mathbb{C}^n$

such that $F(z) = f_F(z)e^{g_F(z)}$. For the special case $n = 1$, f_F is the canonical product of Weierstrass.

Lemma 3.2 ([29]). *If g and h are entire functions on the complex plane $\mathbb{C}$ and $g(h)$ is an entire function of finite order, then there are only two possible cases: either*

(i) *the internal function h is a polynomial and the external function g is of finite order; or*

(ii) *the internal function h is not a polynomial but a function of finite order, and the external function g is of zero order.*

Lemma 3.3 ([17]). *Suppose that $a_0(z), a_1(z), \ldots, a_m(z)$ $(m \geq 1)$ are meromorphic functions on $\mathbb{C}^n$ and $g_0(z), g_1(z), \ldots, g_m(z)$ are entire functions on $\mathbb{C}^n$ such that $g_i(z) - g_j(z)$ are not constants for $0 \leq i < j \leq m$. If*

$$\sum_{i=0}^{m} a_i(z)e^{g_i(z)} \equiv 0$$

and $||T(r, a_i) = o(T(r)), i = 0, 1, \ldots, m$ hold, where $T(r) := \min_{0 \leq i < j \leq m} T(r, e^{g_i - g_j})$, then $a_i(z) \equiv 0$ for $i = 0, 1, \ldots, m$.

Lemma 3.4 ([17]). *Let $f_j (\not\equiv 0)$, $j = 1, 2, 3$, be meromorphic functions on $\mathbb{C}^n$ such that f_1 is non-constant and $f_1 + f_2 + f_3 = 1$ such that*

$$\sum_{j=1}^{3} \left\{ N_2\left(r, \frac{1}{f_j}\right) + 2\overline{N}(r, f_j) \right\} < \lambda T(r, f_1) + O(\log^+ T(r, f_1)),$$

for all r outside possibly a set with finite logarithmic measure, where $\lambda < 1$ is a positive number. Then either $f_2 = 1$ or $f_3 = 1$.

Remark 3.1. Here, note that $N_2(r, 1/f)$ is the counting function of the zeros of function f in open disk $|z| \leq r$, where the simple zero is counted once, and the multiple zero is counted twice.

3.1. Proof of Theorems 2.1 and 2.2

For the convenience of the reader, we will present our proofs of Theorems 2.1 and 2.2 in detail.

Proof of Theorem 2.1. Suppose that $f(z)$ is a finite order transcendental entire solution of (2.1). The equation (2.1) can be written as

$$(\sqrt{a}F - \omega_1\sqrt{b}G)(\sqrt{a}F - \omega_2\sqrt{b}G) = 1, \tag{3.1}$$

where F and G are defined by

$$F := \frac{\alpha\dfrac{\partial f(z)}{\partial z_i} + \beta\dfrac{\partial f(z)}{\partial z_j}}{\exp\left(\frac{g(z)}{2}\right)} \quad \text{and} \quad G := \frac{\gamma f(z+c) + \delta f(z)}{\exp\left(\frac{g(z)}{2}\right)}. \tag{3.2}$$

Since f is a finite order transcendental entire function and g is a polynomial, by Lemmas 3.1 and 3.2, there exists a polynomial p in $\mathbb{C}^n$ such that

$$\sqrt{a}F - \omega_1\sqrt{b}G = \exp(p) \quad \text{and} \quad \sqrt{a}F - \omega_2\sqrt{b}G = \exp(-p)\,. \tag{3.3}$$

A simple computation using (3.2) and (3.3) gives us

$$\alpha\frac{\partial f(z)}{\partial z_i} + \beta\frac{\partial f(z)}{\partial z_j} = \frac{\omega_2\exp(h_1(z)) - \omega_1\exp(h_2(z))}{\sqrt{a}(\omega_2-\omega_1)} \tag{3.4}$$

and

$$\gamma f(z+c) + \delta f(z) = \frac{\exp(h_1(z)) - \exp(h_2(z))}{\sqrt{b}(\omega_2-\omega_1)}, \tag{3.5}$$

where

$$h_1(z) = \frac{g(z)}{2} + p(z) \quad \text{and} \quad h_2(z) = \frac{g(z)}{2} - p(z). \tag{3.6}$$

Thus, it follows from (3.4) and (3.5) that

$$\begin{aligned} &H_{21}(z)\exp(h_1(z)-h_1(z+c)) - H_{22}(z)\exp(h_2(z)-h_1(z+c)) \\ &+ K_2\exp(h_2(z+c)-h_1(z+c)) \equiv 1, \end{aligned} \tag{3.7}$$

where

$$\begin{cases} H_{21}(z) = \dfrac{\sqrt{a}\left(\alpha\dfrac{\partial h_1(z)}{\partial z_i} + \beta\dfrac{\partial h_1(z)}{\partial z_j}\right) - \delta\sqrt{b}\omega_2}{\gamma\omega_2\sqrt{b}}, \\ H_{22}(z) = \dfrac{\sqrt{a}\left(\alpha\dfrac{\partial h_2(z)}{\partial z_i} + \beta\dfrac{\partial h_2(z)}{\partial z_j}\right) - \delta\sqrt{b}\omega_1}{\gamma\omega_2\sqrt{b}} \end{cases} \text{and } K_2 = \frac{\omega_1}{\omega_2}.$$

The equation (3.7) can be written as $g_{21}+g_{22}+g_{23}=1$. It is easy to see that $g_{23}\neq 0$, where

$$\begin{cases} g_{21} = H_{21}(z)\exp(h_1(z)-h_1(z+c)), \\ g_{22} = -H_{22}(z)\exp(h_2(z)-h_1(z+c)), \\ g_{23} = K_2\exp(h_2(z+c)-h_1(z+c)). \end{cases}$$

Case A: If $\exp(h_2(z+c)-h_1(z+c))$ is a constant, then $h_2(z+c)-h_1(z+c) = K$, where $K\in\mathbb{C}$ is a constant. From (3.6), it is easy to see that $p(z)=-K$ is a constant. Suppose that $\xi = \exp(p(z))$. Now, the equations (3.4) and (3.5) become

$$\begin{cases} \alpha\dfrac{\partial f(z)}{\partial z_i} + \beta\dfrac{\partial f(z)}{\partial z_j} = M_1\exp\left(\frac{g(z)}{2}\right); \\ \gamma f(z+c) + \delta f(z) = M_2\exp\left(\frac{g(z)}{2}\right), \end{cases} \tag{3.8}$$

where

$$M_1 = \frac{\omega_2\xi - \omega_1\xi^{-1}}{\sqrt{a}(\omega_2 - \omega_1)} \text{ and } M_2 = \frac{\xi - \xi^{-1}}{\sqrt{b}(\omega_2 - \omega_1)}$$

satisfying

$$M_1^2 + M_2^2 = \frac{b(\omega_2\xi^2 - \omega_1)^2 + a(\xi^2 - 1)^2}{ab\xi^2(\omega_2 - \omega_1)^2}. \tag{3.9}$$

Now, we discuss the following three sub-cases.

Sub-case A1: If $M_1 = 0$, then we obtain $\xi^2 = \omega_1/\omega_2$. In view of (3.9) and using $\omega_1\omega_2 = 1$, it is easy to see that $M_2 = \pm(1/\sqrt{b})$. The equation (3.8) can be written as

$$\alpha\frac{\partial f(z)}{\partial z_i} + \beta\frac{\partial f(z)}{\partial z_j} = 0 \text{ and} \tag{3.10}$$

$$\gamma f(z + c) + \delta f(z) = M_2 \exp\left(\frac{g(z)}{2}\right). \tag{3.11}$$

Solving (3.10), we obtain that

$$f(z) = \phi\left(z_j - \frac{\beta}{\alpha}z_i\right),$$

where $\phi(z_j - \frac{\beta}{\alpha}z_i)$ is a finite order transcendental entire function satisfying

$$\gamma\phi\left(z_j - \frac{\beta}{\alpha}z_i + c_j - \frac{\beta}{\alpha}c_i\right) + \delta\phi\left(z_j - \frac{\beta}{\alpha}z_i\right) = \pm\frac{1}{\sqrt{b}}\exp\left(\frac{g(z)}{2}\right).$$

Sub-case A2: If $M_2 = 0$, then we see that $\xi^2 = 1$. Using (3.9), a simple computation shows that $M_1 = \pm 1/\sqrt{a}$. Therefore, from (3.8) it follows that

$$\alpha\frac{\partial f(z)}{\partial z_i} + \beta\frac{\partial f(z)}{\partial z_j} = \pm\frac{1}{\sqrt{a}}\exp\left(\frac{g(z)}{2}\right) \text{ and } \gamma f(z + c) + \delta f(z) = 0. \tag{3.12}$$

We see from second equation of (3.12) that

$$\gamma\left(\alpha\frac{\partial f(z + c)}{\partial z_i} + \beta\frac{\partial f(z + c)}{\partial z_j}\right) + \delta\left(\alpha\frac{\partial f(z)}{\partial z_i} + \beta\frac{\partial f(z)}{\partial z_j}\right) = 0,$$

which implies that $\exp\left((g(z + c) - g(z))/2\right) = -(\delta/\gamma)$. It is evident that $g(z + c) - g(z)$ must be a constant. Consequently, we have $g(z) = L(z) + H(s_1) + R$, where $L(z) = a_1z_1 + \cdots + a_nz_n$ and $H(s_1)$ is a polynomial in $s_1 := d_1z_1 + \cdots + d_nz_n$ with $d_1c_1 + \cdots + d_nc_n = 0$ such that $H(z + c) = H(z)$, $d_1, \ldots, d_n, R \in \mathbb{C}$, and $a_1c_1 + \cdots + a_nc_n = 2\ln\left(-(\delta/\gamma)\right)$. The characteristic equations for the first equation of (3.12) are

$$\frac{dz_i}{dt} = \alpha, \quad \frac{dz_j}{dt} = \beta, \quad \frac{df}{dt} = \pm\frac{1}{\sqrt{a}}\exp\left(\frac{g(z)}{2}\right).$$

Using the initial conditions: $z_i = 0$, $z_j = s$ and $f = f(0,s) := \psi_1(s)$, with a parameter s, we obtain the following parametric representation for the solutions of the characteristic equations: $z_i = \alpha t$, $z_j = \beta t + s$,

$$f(s,t) = \pm\frac{1}{\sqrt{a}}\int_0^t \exp\left(\frac{g(z)}{2}\right) dt + \psi_1(s)$$

or,

$$f(z) = \pm\frac{1}{\alpha\sqrt{a}}\int_0^{z_i/\alpha} \exp\left(\frac{a_1z_1 + \cdots + a_nz_n + H(s_1) + R}{2}\right) dz_i + \psi_1\left(z_j - \frac{\beta}{\alpha}z_i\right),$$

where ψ_1 is a finite order entire function. Substituting $f(z)$ into the second equation of (3.12) and comparing both side, we obtain

$$\psi_1\left(z_j - \frac{\beta}{\alpha}z_i + c_j - \frac{\beta}{\alpha}c_i\right) = -\frac{\delta}{\gamma}\psi_1\left(z_j - \frac{\beta}{\alpha}z_i\right).$$

Sub-case A3: Suppose that $M_1 \neq 0$ and $M_2 \neq 0$. Then, a simple computation using (3.8) shows that

$$\frac{M_2}{2\gamma M_1}\left(\alpha\frac{\partial g(z)}{\partial z_i} + \beta\frac{\partial g(z)}{\partial z_j}\right) - \frac{\delta}{\gamma} = \exp\left(\frac{g(z+c) - g(z)}{2}\right). \tag{3.13}$$

As $g(z)$ is a polynomial, from (3.13) it follows that $g(z+c) - g(z) = \eta$, where η is a constant in $\mathbb{C}$. It yields that $g(z) = L_1(z) + H(s_1) + R_1$, where $L_1(z) = a_{11}z_1 + \cdots + a_{1n}z_n$ and $H(s_1)$ is a polynomial in $s_1 := d_1z_1 + \cdots + d_nz_n$ with $d_1c_1 + \cdots + d_nc_n = 0$ such that $H(z+c) = H(z)$, $R_1 \in \mathbb{C}$. Thus, from (3.13) we see that

$$\alpha\frac{\partial L_1(z)}{\partial z_i} + \beta\frac{\partial L_1(z)}{\partial z_j} + \alpha\frac{\partial H(z)}{\partial z_i} + \beta\frac{\partial H(z)}{\partial z_j} \equiv M_5$$

or

$$\alpha\frac{\partial H(s)}{\partial z_i} + \beta\frac{\partial H(s)}{\partial z_j} \equiv (\alpha d_i + \beta d_j)H' \equiv M_6,$$

where

$$M_5 = (2\gamma M_1/M_2)\left(\exp(\eta/2) + (\delta/\gamma)\right) \text{ and } M_6 = M_5 - (\alpha a_{1i} + \beta a_{1j}).$$

Since $\alpha d_i + \beta d_j \neq 0$, hence H' must be a constant. Thus, it follows that $H(s) = A_3s + A_4 = A_3(d_1z_1 + \cdots + d_nz_n) + A_4$, where $A_3 = M_6/(\alpha d_i + \beta d_j)$, $A_4 \in \mathbb{C}$. Therefore, we obtain

$$g(z) = L_1(z) + H(s_1) + R_1 = L(z) + R_2 = k_1z_1 + \cdots + k_nz_n + R_2, \tag{3.14}$$

where $k_1 = (A_3d_1 + a_{11}), \ldots, k_n = (A_3d_n + a_{1n})$ and $R_2 = A_4 + R_1$. In view of (3.13) and (3.14), we see that

$$\frac{M_2}{2\gamma M_1}(\alpha k_i + \beta k_j) - \frac{\delta}{\gamma} = \exp\left(\frac{k_1c_1 + \cdots + k_nc_n}{2}\right).$$

Also, the first equation of (3.8) can be written as

$$\alpha\frac{\partial f(z)}{\partial z_i} + \beta\frac{\partial f(z)}{\partial z_j} = M_1\exp\left(\frac{L(z)+R_2}{2}\right). \tag{3.15}$$

Solving the PDE (3.15), we obtain

(3.16)

$$f(z) = \frac{2(\omega_2\xi^2 - \omega_1)}{\xi\sqrt{a}(\omega_2-\omega_1)(k_i\alpha + k_j\beta)}\exp\left(\frac{L(z)+R_2}{2}\right) + \phi_1\left(z_j - \frac{\beta}{\alpha}z_i\right).$$

Moreover, substituting (3.16) into the second equation of (3.8) and comparing both sides, we obtain

$$\phi_1\left(z_j - \frac{\beta}{\alpha}z_i + c_j - \frac{\beta}{\alpha}c_i\right) = -\frac{\delta}{\gamma}\phi_1\left(z_j - \frac{\beta}{\alpha}z_i\right).$$

Case B: If $\exp(h_2(z+c) - h_1(z+c))$ is non-constant, then obviously, both $H_{21}(z) \equiv 0$ and $H_{22}(z) \equiv 0$ cannot hold simultaneously. Otherwise, from (3.7) we see that $K_2\exp(h_2(z+c) - h_1(z+c)) \equiv 1$, which is a contradiction.

If $H_{21}(z) \equiv 0$ and $H_{22}(z) \not\equiv 0$, then in view of (3.7), we see that

$$-H_{22}(z)\exp(h_2(z) - h_1(z+c)) + K_2\exp(h_2(z+c) - h_1(z+c)) \equiv 1. \tag{3.17}$$

Because $\exp(h_2(z+c) - h_1(z+c))$ is non-constant, in view of (3.17), we conclude that $\exp(h_2(z) - h_1(z+c))$ is also non-constant. Consequently, it can be shown that $\exp(h_2(z+c) - h_2(z))$ is non-constant. Otherwise, if $h_2(z+c) - h_2(z) = \eta_1$, where $\eta_1 \in \mathbb{C}$, then, from (3.17) we see that $(-H_{22}(z)e^{-\eta_1} + K_2)\exp(h_2(z+c) - h_1(z+c)) \equiv 1$, which is a contraction as

$$\exp(h_2(z+c) - h_1(z+c))$$

is non-constant. Therefore, the equation (3.17) can be written as

$$-H_{22}(z)\exp(h_2(z)) + K_2\exp(h_2(z+c)) - \exp(h_1(z+c)) \equiv 0. \tag{3.18}$$

In view of Lemma 3.3, from (3.18), we arrive at a contradiction.

By the similar argument, we get a contradiction if $H_{21}(z) \not\equiv 0$ and $H_{22}(z) \equiv 0$. Therefore, we obtain that $H_{21}(z) \not\equiv 0$ and $H_{22}(z) \not\equiv 0$. Since $H_{21}(z) \not\equiv 0$ and $H_{22}(z) \not\equiv 0$, it is clear that the entire functions g_{21} and g_{22} both have no zeros and poles. As h_1 and h_2 are polynomials and $K_1\exp(h_2(z+c) - h_1(z+c))$ is non-constant, it is easy to see that the following condition in Lemma 3.4

$$\sum_{j=1}^{3}\left\{N_2\left(r, \frac{1}{g_{2j}}\right) + 2\overline{N}(r, g_{2j})\right\} = 0 < \lambda T(r, g_{23}) + O(\log^+ T(r, g_{23}))$$

is satisfied for all r outside possibly a set with finite logarithmic measure, where $\lambda < 1$ is a positive number. Thus, in view of Lemma 3.4, we have

$$H_{21}(z)\exp(h_1(z)-h_1(z+c)) \equiv 1 \text{ or } -H_{22}(z)\exp(h_2(z)-h_1(z+c)) \equiv 1.$$

Sub-case B1: Assume that $H_{21}(z)\exp(h_1(z)-h_1(z+c)) \equiv 1$. Then from (3.7), it is easy to see that $\frac{H_{22}(z)}{K_2}\exp(h_2(z)-h_2(z+c)) \equiv 1$. Since h_1, h_2 are polynomials, it follows that $h_1(z)-h_1(z+c)=\eta_2$ and $h_2(z)-h_2(z+c)=\eta_3$, where $\eta_2, \eta_3 \in \mathbb{C}$. Thus we have $h_1(z) = L_1(z)+H_1(s_1)+R_3$ and $h_2(z) = L_2(z)+H_2(s_1)+R_4$, where $L_l(z) = a_{l1}z_1+\cdots+a_{ln}z_n$ and $H_l(s_1)$ for $l=1,2$ are polynomial in $s_1 := d_1z_1+\cdots+d_nz_n$ with $d_1c_1+\cdots+d_nc_n=0$ such that $H_l(z+c)=H_l(z)$ for $l=1,2$, and $R_3, R_4 \in \mathbb{C}$. Since $\alpha d_i+\beta d_j \neq 0$, by the similar argument as in Case A, we see that $H_l(s_1)$ is a linear polynomial in s. Therefore, it is easy to see that $L_l(z)+H_l(s_1)$ $(l=1,2)$ composed of linear functions. For convenience, we always refer to $h_1(z)=L_1(z)+R_3$ and $h_2(z)=L_2(z)+R_4$. Obviously, $L_1(z) \neq L_2(z)$, otherwise, $h_2(z+c)-h_1(z+c)$ will be a constant, which will turn out that $\exp(h_2(z+c)-h_1(z+c))$ is a constant, a contradiction. Substituting $h_1(z)$ and $h_2(z)$ into $H_{21}(z)\exp(h_1(z)-h_1(z+c)) \equiv 1$ and $\frac{H_{22}(z)}{K_2}\exp(h_2(z)-h_2(z+c)) \equiv 1$, we obtain

$$\begin{cases} \dfrac{\sqrt{a}}{\gamma\omega_2\sqrt{b}}\Big((\alpha a_{1i}+\beta a_{1j})-\delta\sqrt{b}\omega_2\Big)\exp(-L_1(c)) \equiv 1, \\ \dfrac{\sqrt{a}}{\gamma\omega_1\sqrt{b}}\Big((\alpha a_{2i}+\beta a_{2j})-\delta\sqrt{b}\omega_1\Big)\exp(-L_2(c)) \equiv 1. \end{cases}$$

The equation (3.4) can be written as

$$\alpha\frac{\partial f(z)}{\partial z_i}+\beta\frac{\partial f(z)}{\partial z_j} = \frac{\omega_2\exp(L_1(z)+R_3)-\omega_1\exp(L_2(z)+R_4)}{\sqrt{a}(\omega_2-\omega_1)}. \tag{3.19}$$

Solving the PDE (3.19), we obtain

$$\begin{aligned} f(z) = {} & \frac{\omega_2\exp(L_1(z)+R_3)}{\sqrt{a}(\omega_2-\omega_1)(\alpha a_{1i}+\beta a_{1j})} - \frac{\omega_1\exp(L_2(z)+R_4)}{\sqrt{a}(\omega_2-\omega_1)(\alpha a_{2i}+\beta a_{2j})} \\ & + \phi_2\left(z_j-\frac{\beta}{\alpha}z_i\right). \end{aligned} \tag{3.20}$$

Furthermore, substituting (3.20) into the second equation of (3.5) and comparing both sides, we obtain

$$\phi_2\left(z_j-\frac{\beta}{\alpha}z_i+c_j-\frac{\beta}{\alpha}c_i\right) = -\frac{\delta}{\gamma}\phi_2\left(z_j-\frac{\beta}{\alpha}z_i\right).$$

From (3.6), it follows that

$$g(z) = h_1(z)+h_2(z) = L(z)+R_5,$$

where $L(z)=L_1(z)+L_2(z)$ and $R_5=R_3+R_4$.

Sub-case B2: Suppose that $-H_{22}(z)\exp(h_2(z)-h_1(z+c))\equiv 1$. Then, from (3.7) we see that $-\frac{H_{21}(z)}{K_2}\exp(h_1(z)-h_2(z+c))\equiv 1$. Since h_1, h_2 are polynomials, it follows that $h_2(z)-h_1(z+c)=\eta_4$ and $h_1(z)-h_2(z+c)=\eta_5$, where $\eta_4,\eta_5\in\mathbb{C}$. A simple computation shows that $h_1(z+2c)-h_1(z)=-\eta_4-\eta_5$ and $h_2(z+2c)-h_2(z)=-\eta_4-\eta_5$. Therefore, we conclude that $h_1(z)=L(z)+H(s_1)+R_6$ and $h_2(z)=L(z)+H(s_1)+R_7$, where $L(z)=a_1z_1+\cdots+a_nz_n$ and $H(s_1)$ is a polynomial in $s_1:=d_1z_1+\cdots+d_nz_n$ with $d_1c_1+\cdots+d_nc_n=0$ such that $H(z+c)=H(z)$, and $R_6,R_7\in\mathbb{C}$. Now, we see that $h_2(z+c)-h_1(z+c)=R_7-R_6$, which shows that $\exp(h_2(z+c)-h_1(z+c))$ is a constant, a contradiction. This completes the proof. □

Proof of Theorem 2.2. For a better clarity in our presentation, we divide the proof into two cases:

Case I: Assume that $\omega^2\neq 0, ab$ and f is a finite order transcendental entire solution of (2.2). The equation (2.2) can be written as

$$(\sqrt{a}f-\omega_1\sqrt{b}L_k(f))(\sqrt{a}f-\omega_2\sqrt{b}L_k(f))=1. \tag{3.21}$$

By the similar argument being used in the proof of the Theorem 2.1, there exists a non-constant polynomial p in $\mathbb{C}^n$ such that

$$\sqrt{a}f-\omega_1\sqrt{b}L_k(f)=\exp(p) \quad \text{and} \quad \sqrt{a}f-\omega_2\sqrt{b}L_k(f)=\exp(-p). \tag{3.22}$$

From (3.22), it is easy to se that

$$f(z)=\frac{\omega_2\exp(p(z))-\omega_1\exp(-p(z))}{\sqrt{a}(\omega_2-\omega_1)} \tag{3.23}$$

and

$$L_k(f)=\frac{\exp(p(z))-\exp(-p(z))}{\sqrt{b}(\omega_2-\omega_1)}. \tag{3.24}$$

Differentiating (3.23) t-times partially with respect to z_i, we obtain

$$\frac{\partial^t f(z)}{\partial z_i^t}=\frac{\omega_2 h_{1t}(z)\exp(p(z))-\omega_1 h_{2t}(z)\exp(-p(z))}{\sqrt{a}(\omega_2-\omega_1)}, \tag{3.25}$$

where

$$\begin{cases} h_{1t}(z)=\left(\dfrac{\partial p}{\partial z_i}\right)^t+H_{1t}\left(\dfrac{\partial^t p}{\partial z_i^t},\ldots,\dfrac{\partial p}{\partial z_i}\right) \\ h_{2t}(z)=(-1)^t\left(\dfrac{\partial p}{\partial z_i}\right)^t+H_{2t}\left(\dfrac{\partial^t p}{\partial z_i^t},\ldots,\dfrac{\partial p}{\partial z_i}\right), \end{cases} \tag{3.26}$$

H_{1t} and H_{2t} are polynomials of partial derivatives of $p(z)$ of degree less than t, $t=1,2,\ldots,k$. In view of (3.23), (3.24) and (3.25), a simple computation shows that

$$\left(\omega_2\sqrt{b}\sum_{t=1}^{k}\lambda_t h_{1t}(z)-\sqrt{a}\right)\exp(2p(z))=\left(\omega_1\sqrt{b}\sum_{t=1}^{k}\lambda_t h_{2t}(z)-\sqrt{a}\right). \tag{3.27}$$

Since p is a non-constant polynomial, it follows from (3.27) that

$$\omega_2\sqrt{b}\sum_{t=1}^{k}\lambda_t h_{1t}(z)=\sqrt{a} \text{ and } \omega_1\sqrt{b}\sum_{t=1}^{k}\lambda_t h_{2t}(z)=\sqrt{a}. \tag{3.28}$$

From (3.26) and (3.28), it is easy to observe that $\frac{\partial p}{\partial z_i}$ must be a non-zero constant in $\mathbb{C}$, say β. Consequently, we have

$$p(z)=\beta z_i+\phi(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_n),$$

where $\phi(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_n)$ is a polynomial in $z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_n$ in $\mathbb{C}^n$. Clearly, it follows from (3.28) that

$$\omega_2\sqrt{b}\sum_{t=1}^{k}\lambda_t\beta^t=\sqrt{a} \text{ and } \omega_1\sqrt{b}\sum_{t=1}^{k}(-1)^t\lambda_t\beta^t=\sqrt{a}.$$

Hence, from (3.23) we obtain

$$f(z)=\frac{\omega_2\exp\left(F(z)\right)-\omega_1\exp\left(-F(z)\right)}{\sqrt{a}(\omega_2-\omega_1)},$$

where $F(z)=\beta z_i+\phi(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_n)$. This completes the proof of (I).

Case II: Let $\omega^2=0$ and f be a finite order transcendental entire solution of (2.2). The equation (2.2) can be written as

$$\left(\sqrt{a}f(z)+i\sqrt{b}\sum_{t=1}^{k}\lambda_t\frac{\partial^t f(z)}{\partial z_i^t}\right)\left(\sqrt{a}f(z)-i\sqrt{b}\sum_{t=1}^{k}\lambda_t\frac{\partial^t f(z)}{\partial z_i^t}\right)=1.$$

By the similar argument being used in the proof of the Theorem 2.1, there exists a non-constant polynomial p in $\mathbb{C}^n$ such that

$$f(z)=\frac{\exp\left(p(z)\right)+\exp\left(-p(z)\right)}{2\sqrt{a}} \text{ and } \tag{3.29}$$

$$\sum_{t=1}^{k}\lambda_t\frac{\partial^t f(z)}{\partial z_i^t}=\frac{\exp\left(p(z)\right)-\exp\left(-p(z)\right)}{2i\sqrt{b}}.$$

Similarly, as in Case I, form (3.29), we obtain

$$\left(i\sqrt{b}\sum_{t=1}^{k}\lambda_t h_{1t}(z)-\sqrt{a}\right)\exp\left(2p(z)\right)=-\left(i\sqrt{b}\sum_{t=1}^{k}\lambda_t h_{2t}(z)+\sqrt{a}\right). \tag{3.30}$$

As p is a non-constant polynomial, from (3.30) we observe that

$$i\sqrt{b}\sum_{t=1}^{k}\lambda_t h_{1t}(z)=\sqrt{a} \text{ and } i\sqrt{b}\sum_{t=1}^{k}\lambda_t h_{2t}(z)=-\sqrt{a}. \tag{3.31}$$

In view of (3.26) and (3.31), it is easy to see that $\frac{\partial p}{\partial z_i}$ must be a non-zero constant in $\mathbb{C}$, say η. Thus, we see that

$$p(z) = \eta z_i + \psi(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n),$$

where $\psi(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$ is a polynomial in $z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n$ in $\mathbb{C}^n$. Therefore, from (3.31), we see that

$$i\sqrt{b}\sum_{t=1}^{k} \lambda_t \eta^t = \sqrt{a} \text{ and } i\sqrt{b}\sum_{t=1}^{k} \lambda_t \eta^t = -\sqrt{a}.$$

Hence, f takes the following form

$$f(z) = \frac{1}{\sqrt{a}} \cosh\left(\eta z_i + \psi(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)\right).$$

This completes the proof of (II). □

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