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LARGE DEVIATIONS FOR A SUPER-HEAVY TAILED β-MIXING SEQUENCE

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Abstract. Let $\{X, X_n; n \geq 1\}$ be a β-mixing sequence of identical non-negative random variables with super-heavy tailed distributions and $S_n = X_1 + X_2 + \cdots + X_n$. For $\varepsilon > 0$, $b > 1$ and appropriate values of x, we obtain the logarithmic asymptotics behaviors for the tail probabilities $\mathbb{P}(S_n > e^{\varepsilon n^x})$ and $\mathbb{P}(S_n > e^{\varepsilon b^n})$. Moreover, our results are applied to the log-Pareto distribution and the distribution for the super-Petersburg game.

1. Introduction

We are concerned with large deviations for a super-heavy tailed β-mixing sequence. Our approach is based on techniques of transforming dependent sequence to independent sequence and partitioning (see Berbee [2] and Liu and Hu [13]). Nakata [20] studied that large deviations for sums of independent and identically distributed random variables with non-negative super-heavy tailed distributions. We extend the results in Nakata [20] to β-mixing sequence.

1.1. Heavy-tailed random variables

Hu and Nyrhinen [10] introduced the following two parameters for non-negative random variable X, namely,

$$\alpha^* = -\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(\log X > t) \in [0, \infty]$$

and

$$\alpha_* = -\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}(\log X > t) \in [0, \infty].$$

Clearly, $\alpha^* \leq \alpha_*$. The parameters are finite and equal with the common value α if and only if for every $\varepsilon > 0$ and large t,

$$\frac{1}{t^{\alpha+\varepsilon}} \leq \mathbb{P}(X > t) \leq \frac{1}{t^{\alpha-\varepsilon}}. \tag{1.1}$$

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We may have $\alpha^* < \alpha_*$. To see this, let $\mathbb{P}(\log X = e^n) = C\exp(-e^n)$ for $n = 1, 2, \ldots$, where C is a constant such that $\sum_{n=1}^{\infty} C\exp(-e^n) = 1$. For this random variable, we have $\alpha^* = 1$ and $\alpha_* = e$. A further useful fact is that

$$\alpha^* = \sup\{m \geq 0; \mathbb{E}(X^m) < \infty\}.$$

The proof can be found in Rolski et al. [21, p. 39]. If $\alpha^* < \infty$, then X is heavy tailed, namely, $\mathbb{E}e^{mX} = \infty$ for every $m > 0$.

Hu and Nyrhinen [10] established the large deviations for the partial sums of non-negative independent and identically distributed random variables with heavy tails, which answered the conjecture in Gantert [8]. Miao et al. [17] showed the logarithmic asymptotic behaviors for the cases of m-dependent sequence and negatively associated sequences. Miao et al. [16] studied the logarithmic asymptotic behaviors for the largest order statistics from a Pareto distribution. Stoica [23] obtained the large deviations for the player's gains in the independent St. Petersburg games. Li and Miao [11] established the large deviations for the partial sums of independent identically distributed **B**-valued random variables. Miao and Li [15] further extended the works in Li and Miao [11].

1.2. Super-heavy tailed random variables

If $\alpha = 0$ then it seems that (1.1) is not so effective. Therefore, Nakata [20] tried to introduce parameters η_* and η^* as follows:

$$\eta^* = -\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(\log\log X > t) \in [0, \infty] \tag{1.2}$$

and

$$\eta_* = -\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}(\log\log X > t) \in [0, \infty]. \tag{1.3}$$

If the parameters are finite and equal with the common value η, then for each $\varepsilon > 0$ and large x, we get

$$\frac{1}{(\log x)^{\eta+\varepsilon}} \leq \mathbb{P}(X > x) \leq \frac{1}{(\log x)^{\eta-\varepsilon}},$$

whose tail is super-heavy. The terminology "super-heavy" is used in Falk et al. [6]. A further useful fact is that

$$\eta^* = \sup\{d \geq 0; \mathbb{E}((\log X)^d) < \infty\}.$$

The proof is used by the same method as Rolski et al. [21, p. 39].

Nakata [20] studied the following large deviations for sums of independent and identically distributed random variables $\{X, X_n; n \geq 1\}$ with super-heavy tailed distributions.

Theorem 1.1. *Assume that $0 < \eta^* < \infty$. Then for any $\varepsilon > 0$, the following statements hold.*

(i) *We have*

$$\limsup_{n\to\infty} \frac{\log \mathbb{P}\left(S_n > e^{\varepsilon n^x}\right)}{\log n} = 1 - \eta^* x, \quad \textit{for} \quad x > \max\{1, 1/\eta^*\}.$$

In addition, if $\eta_* < \infty$ *then*

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(S_n > e^{\varepsilon n^x}\right)}{\log n} = 1 - \eta_* x, \quad \textit{for} \quad x > \max\{1, 1/\eta^*\}.$$

(ii) *We have*

$$\limsup_{n\to\infty} \frac{\log \mathbb{P}\left(S_n > e^{\varepsilon b^n}\right)}{n} = -\eta^* \log b, \quad \textit{for} \quad b > 1.$$

In addition, if $\eta_* < \infty$ *then*

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(S_n > e^{\varepsilon b^n}\right)}{n} = -\eta_* \log b, \quad \textit{for} \quad b > 1.$$

Li et al. [12] obtained a general large deviation result for the tail probabilities $\mathbb{P}(\|S_n\| > sg(n))$ for all $s > 0$ by giving the exact values for

$$\lim_{n\to\infty} \frac{\log \mathbb{P}(\|S_n\| > sg(n))}{\log n} \quad \text{and} \quad \lim_{n\to\infty} \frac{\log \mathbb{P}(\|S_n\| > sg(n))}{h(n)},$$

where $\{X, X_n; n \geq 1\}$ is a sequence of independent and identically distributed $\mathbf{B}$-valued random variables. In their paper, $(\mathbf{B}, \|\cdot\|)$ is a real separable Banach space equipped with its Borel σ-algebra, i.e., the σ-algebra generated by the class of open subsets of $\mathbf{B}$ determined by $\|\cdot\|$. $g(n)$ is a continuous and strictly increasing function and $h(n)$ is an increasing regularly varying function.

1.3. β-mixing sequence

Let us recall the definition of the β-mixing coefficient, and for the definitions of other mixing coefficients as well as for the relations between them, we refer to Bradley [3]. Let X and Z be two random variables, and denote the distribution of (X, Z) by $\mu_{(X,Z)}$ and the distributions of X and Z by μ_X and μ_Z. The β-mixing coefficient of X and Z is defined as

$$\beta(X, Z) = \frac{1}{2}\|\mu_{(X,Z)} - \mu_X \otimes \mu_Z\|,$$

where $\|\mu - \nu\|$ denotes the (total) variation norm of the signed measure $\mu - \nu$. Now for a sequence of random variables $\{Y_n; n \geq 1\}$, define

$$\beta(n) = \sup_{k\in N} \beta((Y_1, Y_2, \ldots, Y_k), (Y_{k+n}, Y_{k+n+1}, \ldots)).$$

The sequence is called β-*mixing* (or *absolutely regular*) if $\beta(n) \to 0$ for $n \to \infty$.

In time series, asymptotic independent conditions such as mixing conditions are usually proposed to replace independent case, among which β-mixing is an important dependent structure and has been connected with a large class of time series including autoregressive moving average (ARMA) models, generalized autoregressive conditional heteroskedasticity (GARCH) models and certain Markov processes.

Masuda [14] considered a multidimensional diffusion with jumps and provided sets of conditions under which the multidimensional diffusion is exponentially β-mixing (i.e., $\beta(n) = O(e^{-\gamma n})$ for some $\gamma > 0$) and fulfils the ergodic theorem for any initial distribution. Specially, Masuda [14] proved that a special Lévy-driven Ornstein-Uhlenbeck processes is (exponentially) β-mixing based on the super-heavy tailed condition. Let $\mathcal{Q} \in \mathbb{R}^{d \otimes d}$ whose eigenvalues have positive real parts, and let Z be a nontrivial d-dimensional Lévy process. Then let X be a d-dimensional Ornstein-Uhlenbeck process given by

$$dX_t = -\mathcal{Q}X_t dt + dZ_t \tag{1.4}$$

with $\mathcal{L}(X_0) = \eta$. We beforehand know that a unique invariant distribution π exists if and only if $\int_{|z|>1} \log|z|\nu(dz) < \infty$, where ν is a Lévy measure. Masuda [14] showed the following results: Let X be the Ornstein-Uhlenbeck process given by (1.4), then:

(i) If $\int_{|z|>1} \log|z|\nu(dz) < \infty$, then X fulfils the ergodic theorem for any η and is β-mixing for $\eta = \pi$;

(ii) If $\int_{|z|>1} |z|^q \nu(dz) < \infty$ and $\int |x|^q \eta(dz) < \infty$ for some $q > 0$, then X is exponentially β-mixing and $\int |x|^q \pi(dz) < \infty$.

Athreya and Pantula [1] considered an autoregressive process given by $Y_n = \rho Y_{n-1} + \varepsilon_n$, $n = 1, 2, \ldots$, where $|\rho| < 1$ and $\{\varepsilon_n\}$ are i.i.d. random variables independent of Y_0. Assume that there exists a finite constant C such that $|\varepsilon_1| \le C$ and

$$\mathbb{E}(\log^+ |\varepsilon_1|) < \infty.$$

In addition, for some $n_0 \ge 1$, $U_{n_0} = \sum_{j=1}^{n_0} \rho^j \varepsilon_j$ has a non-trivial absolutely continuous component, then for any initial distribution of Y_0 concentrated on a bounded set, $\{Y_n\}$ is β-mixing (in fact, $\{Y_n\}$ is uniform mixing).

Liu and Hu [13] studied the logarithmic asymptotics for a stationary sequence of non-negative β-mixing random variables with heavy tails. The novel work of Chen et al. [4] made the first attempt to develop the theory of Cramér-type moderate deviations for self-normalized sums of weakly dependent random variables satisfying the geometrically β-mixing condition or geometric moment contraction. Gao et al. [9] further improved the results in Chen et al. [4] by applying their new framework on the general self-normalized sum. Miao and Yin [18] proved the logarithmic asymptotic behavior and the weak law of large numbers for a stationary sequence of nonnegative β-mixing random variables with heavy-tailed distributions.

In the present paper, let $\{X, X_n; n \ge 1\}$ be a β-mixing sequence of identical non-negative random variables with super-heavy tailed distributions and $S_n = X_1 + X_2 + \cdots + X_n$. The parameters η^* and η_* are defined in (1.2) and (1.3).

Furthermore, we assume that

$$\mathbb{P}(X \geq e) = 1$$

and $\mathbb{E}((\log X)^d) \geq 1$ for any $d \geq 0$. In Section 2, we give some preliminary lemmas. In Section 3, for $\varepsilon > 0$, $b > 1$ and appropriate values of x, we obtain the logarithmic asymptotics behaviors for the tail probabilities $\mathbb{P}(S_n > e^{\varepsilon n^x})$ and $\mathbb{P}(S_n > e^{\varepsilon b^n})$. Moreover, we apply our results to the log-Pareto distribution and the distribution for the super-Petersburg game. In Section 4, we state the proofs of the main results. In Section 5, we study a generalization of Theorems 3.1 and 3.2.

2. Some preliminary lemmas

We begin with a series of lemmas which are needed in the sequel. The following decoupling lemma was obtained by Berbee [2] and Schwarz [22]. It will be used to decouple X_i and X_j when $|i - j|$ is big enough.

Lemma 2.1. (Berbee [2, Lemma 2.1]) *Let $\{X, X_n; n \geq 1\}$ be random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and for every $1 \leq k \leq n$, define*

$$\beta_k = \beta((X_1, X_2, \ldots, X_k), (X_{k+1}, X_{k+2}, \ldots, X_n)).$$

Then there exist independent random variables $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n$ on the same probability space such that $\tilde{X}_i$ and X_i have the same distribution and

$$\|\mu_{(X_1,X_2,\ldots,X_n)} - \mu_{(\tilde{X}_1,\tilde{X}_2,\ldots,\tilde{X}_n)}\| \leq \beta_1 + \beta_2 + \cdots + \beta_n. \tag{2.1}$$

Lemma 2.2. (Nakata [20, Lemma 3.2]) *For $\varepsilon > 0$ and $x > 0$, we have*

$$\limsup_{n\to\infty} \frac{\log \mathbb{P}\left(\log X > \varepsilon n^x\right)}{\log n} = -\eta^* x \tag{2.2}$$

and

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(\log X > \varepsilon n^x\right)}{\log n} = -\eta_* x. \tag{2.3}$$

Remark 2.1. It is easy to see that (2.2) implies that for any $\delta > 0$, there exists a positive constant n_0, such that for all $n \geq n_0$,

$$\mathbb{P}\left(X > e^{\varepsilon n^x}\right) \leq n^{-\eta^* x + \delta}$$

and there exists a subsequence $\{n_k, k \geq 1\}$, such that

$$\mathbb{P}\left(X > e^{\varepsilon n_k^x}\right) \geq n_k^{-\eta^* x - \delta}.$$

Similarly, (2.3) implies that for any $\delta > 0$, there exists a positive constant n_0, such that for all $n \geq n_0$,

$$\mathbb{P}\left(X > e^{\varepsilon n^x}\right) \geq n^{-\eta_* x - \delta}$$

and there exists a subsequence $\{n_k, k \geq 1\}$, such that

$$\mathbb{P}\left(X > e^{\varepsilon n_k^x}\right) \leq n_k^{-\eta_* x+\delta}.$$

Lemma 2.3. (Nakata [20, Lemma 3.2]) *For* $\varepsilon > 0$ *and* $b > 1$ *we have*

$$\limsup_{n\to\infty} \frac{\log \mathbb{P}\left(\log X > \varepsilon b^n\right)}{n} = -\eta^* \log b \tag{2.4}$$

and

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(\log X > \varepsilon b^n\right)}{n} = -\eta_* \log b. \tag{2.5}$$

Remark 2.2. It is easy to see that (2.4) implies that for any $\delta > 0$, there exists a positive constant n_0, such that for all $n \geq n_0$,

$$\mathbb{P}\left(X > e^{\varepsilon b^n}\right) \leq e^{\delta n} b^{-\eta^* n}$$

and there exists a subsequence $\{n_k, k \geq 1\}$, such that

$$\mathbb{P}\left(X > e^{\varepsilon b^{n_k}}\right) \geq e^{-\delta n_k} b^{-\eta^* n_k}.$$

Similarly, (2.5) implies that for any $\delta > 0$, there exists a positive constant n_0, such that for all $n \geq n_0$,

$$\mathbb{P}\left(X > e^{\varepsilon b^n}\right) \geq e^{-\delta n} b^{-\eta_* n}$$

and there exists a subsequence $\{n_k, k \geq 1\}$, such that

$$\mathbb{P}\left(X > e^{\varepsilon b^{n_k}}\right) \leq e^{\delta n_k} b^{-\eta_* n_k}.$$

Lemma 2.4. (Nakata [20, Lemma 3.3]) *Assume that* $\{X, X_n; n \geq 1\}$ *is a sequence of independent identically distributed non-negative random variables with* $\mathbb{E}((\log X)^d) < \infty$ *for some* $0 < d < \infty$. *Denote* $\lambda = \min\{d, 1\}$. *Then for* $t > 0$,

$$1 \leq u \leq \exp\left(t^{1/\lambda}\left(1 - 2^{-1/\lambda}\right)\right)$$

and for $n = 1, 2, \ldots,$ *we have*

$$\mathbb{P}\left(S_n > e^{t^{1/\lambda}}\right) \leq n\mathbb{P}\left(X > \frac{e^{t^{1/\lambda}}}{u}\right) + \left(\frac{2en\mathbb{E}((\log X)^\lambda)}{ut}\right)^u.$$

3. Main results and applications

3.1. The main results

In this subsection, we state the main results of the paper. Write $\bar{x} = \max\{1, 1/\eta^*\}$ if $\eta^* \in (0, \infty]$, where by convention, $1/\infty = 0$. Write also $\underline{x} = \max\{1, 1/\eta_*\}$ if $\eta_* \in (0, \infty]$.

Theorem 3.1. *Assume that $0 < \eta^* < \infty$ and the mixing coefficient $\beta(n)$ satisfies*

$$\frac{\log \beta(n)}{\log n} \to -\infty \quad as \quad n \to \infty,$$

then for $\varepsilon > 0$ and every $x > \bar{x}$,

$$\limsup_{n\to\infty} \frac{\log \mathbb{P}\left(S_n > e^{\varepsilon n^x}\right)}{\log n} = 1 - \eta^* x.$$

In addition, if $\eta_* < \infty$, then for $\varepsilon > 0$ and every $x > \bar{x}$,

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(S_n > e^{\varepsilon n^x}\right)}{\log n} = 1 - \eta_* x. \tag{3.1}$$

Theorem 3.2. *Assume that $0 < \eta^* < \infty$ and the mixing coefficient $\beta(n)$ satisfies*

$$\frac{\log \beta(n)}{n \log n} \to -\infty \quad as \quad n \to \infty,$$

then for $\varepsilon > 0$ and every $b > 1$,

$$\limsup_{n\to\infty} \frac{\log \mathbb{P}\left(S_n > e^{\varepsilon b^n}\right)}{n} = -\eta^* \log b.$$

In addition, if $\eta_* < \infty$, then for $\varepsilon > 0$ and every $b > 1$,

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(S_n > e^{\varepsilon b^n}\right)}{n} = -\eta_* \log b. \tag{3.2}$$

From the theoretical point of view, it is also interesting to consider the above tail probabilities in the extreme cases $\eta^* = 0$ and $\eta^* = \infty$. The following results are complementary to Theorem 3.1 and 3.2.

Theorem 3.3. *Assume that $\eta^* = 0$ and the mixing coefficient $\beta(n)$ satisfies*

$$\frac{\log \beta(n)}{\log n} \to -\infty \quad as \quad n \to \infty,$$

then for $\varepsilon > 0$ and every $x > 1$,

$$\limsup_{n\to\infty} \mathbb{P}\left(S_n > e^{\varepsilon n^x}\right) = 1. \tag{3.3}$$

In addition, if $\eta^* = \infty$, then for $\varepsilon > 0$ and every $x > 1$,

$$\lim_{n\to\infty} \frac{\log \mathbb{P}\left(S_n > e^{\varepsilon n^x}\right)}{\log n} = -\infty. \tag{3.4}$$

Theorem 3.4. *Assume that $\eta^* = \infty$ and the mixing coefficient $\beta(n)$ satisfies*

$$\frac{\log \beta(n)}{n \log n} \to -\infty \quad as \quad n \to \infty,$$

then for $\varepsilon > 0$ and every $b > 1$,

$$\lim_{n\to\infty} \frac{\log \mathbb{P}\left(S_n > e^{\varepsilon b^n}\right)}{n} = -\infty. \tag{3.5}$$

Many results indicate that the extremal behaviour of the partial sum S_n of the super-heavy tailed sequence is caused by a similar behaviour of the maximum

$$M_n = \max\{X_1, X_2, \ldots, X_n\}.$$

We refer to [5]. Therefore, we get the following results.

Corollary 3.1. *Under the conditions in* Theorem 3.1*, for* $\varepsilon > 0$ *and every* $x > \bar{x}$*, we have*

$$\limsup_{n\to\infty} \frac{\log \mathbb{P}\left(M_n > e^{\varepsilon n^x}\right)}{\log n} = 1 - \eta^* x \tag{3.6}$$

and

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(M_n > e^{\varepsilon n^x}\right)}{\log n} = 1 - \eta_* x. \tag{3.7}$$

Corollary 3.2. *Under the conditions in* Theorem 3.2*, for* $\varepsilon > 0$ *and every* $b > 1$*, we have*

$$\limsup_{n\to\infty} \frac{\log \mathbb{P}\left(M_n > e^{\varepsilon b^n}\right)}{n} = -\eta^* \log b \tag{3.8}$$

and

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(M_n > e^{\varepsilon b^n}\right)}{n} = -\eta_* \log b. \tag{3.9}$$

3.2. Applications

In the subsection, we state two examples.

Example 3.1. (log-Pareto distribution) Let $X, X_1, X_2, \ldots$ be non-negative β-mixing random variables with

$$\mathbb{P}(X > x) = \frac{1}{\log x} \quad \text{for} \quad x \geq e,$$

which is called the log-Pareto distribution in Galambos [7]. Let $S_n = \sum_{i=1}^n X_i$. It turns out that $\eta = \eta^* = \eta_* = 1$ by calculating (1.2) and (1.3).

Assume that the mixing coefficient $\beta(n)$ satisfies

$$\frac{\log \beta(n)}{\log n} \to -\infty \quad \text{as} \quad n \to \infty,$$

then for $\varepsilon > 0$ and every $x > 1$, we get

$$\lim_{n\to\infty} \frac{\log \mathbb{P}\left(S_n > e^{\varepsilon n^x}\right)}{\log n} = 1 - x.$$

In addition, assume that the mixing coefficient $\beta(n)$ satisfies

$$\frac{\log \beta(n)}{n \log n} \to -\infty \quad \text{as} \quad n \to \infty,$$

then for $\varepsilon > 0$ and $b > 1$, we get

$$\lim_{n\to\infty} \frac{\log \mathbb{P}\left(S_n > e^{\varepsilon b^n}\right)}{n} = -\log b.$$

Example 3.2. (The distribution of the super-Petersburg game) Let $X, X_1, X_2, \ldots$ be non-negative β-mixing random variables with

$$\mathbb{P}(X = 2^{2^k}) = 2^{-k} \quad \text{for} \quad k = 1, 2, \ldots,$$

where X is the payoff of the super-Petersburg game. Some historical discussion of the game was written in Nakata [19]. The tail probability is

$$\frac{1}{\lg x} \le \mathbb{P}(X > x) = 2^{-[\lg \lg x]} = \frac{2^{\{\lg \lg x\}}}{\lg x} < \frac{2}{\lg x} \quad \text{for} \quad x > 4,$$

where $\lg x = (\log x)/(\log 2)$, $[x]$ is defined as the largest integer not exceeding x and $\{x\}$ stand for the fractional part of x, i.e., $\{x\} = x - [x]$. Let $S_n = \sum_{i=1}^n X_i$. It turns out that $\eta = \eta^* = \eta_* = 1$ by calculating (1.2) and (1.3).

Assume that the mixing coefficient $\beta(n)$ satisfies

$$\frac{\log \beta(n)}{\log n} \to -\infty \quad \text{as} \quad n \to \infty,$$

then for $\varepsilon > 0$ and every $x > 1$, we get

$$\lim_{n\to\infty} \frac{\log \mathbb{P}\left(S_n > e^{\varepsilon n^x}\right)}{\log n} = 1 - x.$$

In addition, assume that the mixing coefficient $\beta(n)$ satisfies

$$\frac{\log \beta(n)}{n \log n} \to -\infty \quad \text{as} \quad n \to \infty,$$

then for $\varepsilon > 0$ and $b > 1$, we get

$$\lim_{n\to\infty} \frac{\log \mathbb{P}\left(S_n > e^{\varepsilon b^n}\right)}{n} = -\log b.$$

4. Proofs of the main results

Proof of Theorem 3.1. Let $\gamma \in (0, 1)$. Decompose the set $\{1, 2, \ldots, n\}$ into $l(n)$ blocks of a length $k(n)$ and a block of a length less than $k(n)$, where $k(n)$, $l(n)$ are integers with

$$\frac{k(n)}{n^\gamma} \to 1, \quad \frac{l(n)}{n^{1-\gamma}} \to 1 \quad \text{as} \quad n \to \infty. \tag{4.1}$$

According to the above formulas, for any $0 < \delta < 1$, if n is large enough, it is easy to see that

$$(1 - \delta)n^{1-\gamma} \le l(n) \le (1 + \delta)n^{1-\gamma}. \tag{4.2}$$

By using Lemma 2.1, we know that there exists a sequence of independent random variables $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n$ such that for every $1 \le i \le n$, $\tilde{X}_i$ and X_i have the same distribution.

Step 1. We shall prove the lower bound of the limit (3.1), namely,

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(S_n > e^{\varepsilon n^x}\right)}{\log n} \geq 1 - \eta_* x. \tag{4.3}$$

Combining (2.1) with (4.2), we get

$$\begin{aligned}
&\mathbb{P}\left(S_n > e^{\varepsilon n^x}\right)\\
&\geq \mathbb{P}\left(\sum_{j=1}^{l(n)} X_{(j-1)k(n)+1} > e^{\varepsilon n^x}\right)\\
&\geq \mathbb{P}\left(\max_{1\leq j\leq l(n)} X_{(j-1)k(n)+1} > e^{\varepsilon n^x}\right)\\
&\geq \mathbb{P}\left(\max_{1\leq j\leq l(n)} \tilde{X}_{(j-1)k(n)+1} > e^{\varepsilon n^x}\right) - l(n)\beta(k(n))\\
&= 1 - \left(1 - \mathbb{P}\left(X > e^{\varepsilon n^x}\right)\right)^{l(n)} - l(n)\beta(k(n))\\
&\geq 1 - \left(1 - \mathbb{P}\left(X > e^{\varepsilon n^x}\right)\right)^{(1-\delta)n^{1-\gamma}} - (1+\delta)n^{1-\gamma}\beta(k(n)).
\end{aligned} \tag{4.4}$$

Note that by Remark 2.1, we have

$$\mathbb{P}\left(X > e^{\varepsilon n^x}\right) \geq n^{-\eta_* x - \delta},$$

which together with (4.4) and the following inequality

$$1 - y \leq e^{-y} \quad \text{for} \quad y \geq 0,$$

we get

$$\begin{aligned}
&\mathbb{P}\left(S_n > e^{\varepsilon n^x}\right)\\
&\geq 1 - \left(1 - n^{-\eta_* x - \delta}\right)^{(1-\delta)n^{1-\gamma}} - (1+\delta)n^{1-\gamma}\beta(k(n))\\
&\geq 1 - e^{-(1-\delta)n^{-\eta_* x - \delta + 1 - \gamma}} - (1+\delta)n^{1-\gamma}\beta(k(n))\\
&\geq (1+o(1))(1-\delta)n^{-\eta_* x - \delta + 1 - \gamma} - (1+\delta)n^{1-\gamma}\beta(k(n)).
\end{aligned}$$

Substituting (4.2) and $\log \beta(n)/\log n \to -\infty$ $(n \to \infty)$ into the above inequality yields

$$\liminf_{n\to\infty} \frac{\log\left((1+\delta)n^{1-\gamma}\beta(k(n))\right)}{\log n} = -\infty.$$

Therefore, we obtain

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}(S_n > e^{\varepsilon n^x})}{\log n} \geq 1 - \eta_* x - \delta - \gamma.$$

This implies (4.3) by letting $\delta \downarrow 0$ and $\gamma \downarrow 0$.

From Remark 2.1, we know that for any $\delta > 0$, there exists a subsequence $\{n_k; k \geq 1\}$ such that

$$\mathbb{P}\left(X > e^{\varepsilon n_k^x}\right) \geq n_k^{-\eta^* x - \delta}.$$

Hence, analogously to (4.3), it is straightforward that

$$\limsup_{n\to\infty} \frac{\log \mathbb{P}\left(S_n > e^{\varepsilon n^x}\right)}{\log n} \geq 1 - \eta^* x.$$

Step 2. We shall prove the upper bound of the limit (3.1), namely,

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(S_n > e^{\varepsilon n^x}\right)}{\log n} \leq 1 - \eta_* x. \tag{4.5}$$

Applying Lemma 2.1 and the inequality (4.2), we have

$$\begin{aligned}
\mathbb{P}\left(S_n > e^{\varepsilon n^x}\right) &= \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i > \frac{e^{\varepsilon n^x}}{n}\right) \\
&\leq \mathbb{P}\left(\frac{1}{k(n)l(n)}\sum_{i=1}^{n} X_i > \frac{e^{\varepsilon n^x}}{n}\right) \\
&\leq \mathbb{P}\left(\frac{1}{k(n)}\sum_{j=1}^{k(n)} \frac{1}{l(n)}\sum_{i=1}^{l(n)+1} X_{(i-1)k(n)+j} > \frac{e^{\varepsilon n^x}}{n}\right) \\
&= \mathbb{P}\left(\sum_{j=1}^{k(n)} \frac{1}{l(n)}\sum_{i=1}^{l(n)+1} X_{(i-1)k(n)+j} > k(n)\frac{e^{\varepsilon n^x}}{n}\right) \\
&\leq k(n)\mathbb{P}\left(\frac{1}{l(n)}\sum_{i=1}^{l(n)+1} X_{(i-1)k(n)+1} > \frac{e^{\varepsilon n^x}}{n}\right) \\
&\leq k(n)\mathbb{P}\left(\sum_{i=1}^{l(n)+1} X_{(i-1)k(n)+1} > (1-\delta)n^{-\gamma}e^{\varepsilon n^x}\right) \\
&\leq k(n)\mathbb{P}\left(\sum_{i=1}^{l(n)+1} \tilde{X}_{(i-1)k(n)+1} > (1-\delta)n^{-\gamma}e^{\varepsilon n^x}\right) \\
&\quad + k(n)(l(n)+1)\beta(k(n)) \\
&\leq k(n)\mathbb{P}\left(\tilde{S}_{l(n)+1} > (1-\delta)n^{-\gamma}e^{\varepsilon n^x}\right) + 2n\beta(k(n)),
\end{aligned}$$

where

$$\tilde{S}_{l(n)+1} := \sum_{i=1}^{l(n)+1} \tilde{X}_{(i-1)k(n)+1}.$$

If $\bar{x} = \max\{1, 1/\eta^*\} = 1$, let

$$\frac{1}{1+\varepsilon} < \lambda < 1.$$

Moreover, if $\bar{x} = \max\{1, 1/\eta^*\} = 1/\eta^*$, let

$$\frac{\eta^*}{1+\varepsilon\eta^*} < \lambda < \eta^*.$$

Then applying Lemma 2.4 with

$$t = \left(\log\left((1-\delta)n^{-\gamma}\right) + \varepsilon n^x\right)^\lambda,$$

we have

$$1 \leq u \leq \exp\left(\left(\log\left((1-\delta)n^{-\gamma}\right) + \varepsilon n^x\right)\left(1 - 2^{-1/\lambda}\right)\right).$$

In particular, choosing

$$u = \max\left\{\frac{2(\eta_* x - 1)}{\lambda x - 1}, 1\right\}, \tag{4.6}$$

we have

$$\begin{aligned}
&k(n)\mathbb{P}\left(\tilde{S}_{l(n)+1} > (1-\delta)n^{-\gamma}e^{\varepsilon n^x}\right) \\
&\leq k(n)(l(n)+1)\mathbb{P}\left(X > \frac{(1-\delta)n^{-\gamma}e^{\varepsilon n^x}}{u}\right) \\
&\quad + k(n)\left(\frac{2en\mathbb{E}((\log X)^\lambda)}{u\left(\log((1-\delta)n^{-\gamma}) + \varepsilon n^x\right)^\lambda}\right)^u \\
&\leq 2n\mathbb{P}\left(X > \frac{(1-\delta)n^{-\gamma}e^{\varepsilon n^x}}{u}\right) + k(n)\left(\frac{2en\mathbb{E}((\log X)^\lambda)}{u\left(\log((1-\delta)n^{-\gamma}) + \varepsilon n^x\right)^\lambda}\right)^u.
\end{aligned}$$

Together with (4.1) and $\log\beta(n)/\log n \to -\infty$ $(n \to \infty)$, we know that

$$\lim_{n\to\infty} \frac{\log(2n\beta(k(n)))}{\log n} = -\infty. \tag{4.7}$$

It follows, from Lemma 2.2, that

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{\log n}\log\left(2n\mathbb{P}\left(X > \frac{(1-\delta)n^{-\gamma}e^{\varepsilon n^x}}{u}\right)\right) \\
&= \liminf_{n\to\infty} \frac{1}{\log n}\log\left(2n\mathbb{P}\left(\log X > \log\left((1-\delta)n^{-\gamma}\right) + \varepsilon n^x - \log u\right)\right) \\
&\leq \liminf_{n\to\infty} \frac{1}{\log n}\log\left(2n\mathbb{P}\left(\log X > \frac{\varepsilon n^x}{2}\right)\right) \\
&= 1 - \eta_* x.
\end{aligned}$$

Note that by (4.1), it implies that

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{\log n} \log \left[k(n) \left(\frac{2en\mathbb{E}((\log X)^\lambda)}{u\left(\log((1-\delta)n^{-\gamma}) + \varepsilon n^x\right)^\lambda} \right)^u \right] \\
&= \liminf_{n\to\infty} \frac{\log k(n) + u[\log(2en\mathbb{E}((\log X)^\lambda)) - \log(u(\log((1-\delta)n^{-\gamma}) + \varepsilon n^x)^\lambda)]}{\log n} \\
&\le \liminf_{n\to\infty} \frac{\log k(n) + u\left[\log\left(2en\mathbb{E}((\log X)^\lambda)\right) - \log u - \log\left(\frac{1}{2}\varepsilon n^x\right)^\lambda\right]}{\log n} \\
&= \gamma + u(1-\lambda x).
\end{aligned}$$

Letting $\gamma \downarrow 0$, it follows that

$$\liminf_{n\to\infty} \frac{1}{\log n} \log \left[k(n) \left(\frac{2en\mathbb{E}((\log X)^\lambda)}{u\left(\log((1-\delta)n^{-\gamma}) + \varepsilon n^x\right)^\lambda} \right)^u \right] = u(1-\lambda x).$$

Therefore, by using (4.6), we have

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(S_n > e^{\varepsilon n^x}\right)}{\log n} \le 1 - \eta_* x.$$

Similarly, we can get

$$\limsup_{n\to\infty} \frac{\log \mathbb{P}\left(S_n > e^{\varepsilon n^x}\right)}{\log n} \le 1 - \eta^* x.$$

□

Proof of Theorem 3.2. Decompose the set $\{1, 2, \ldots, n\}$ into $l(n)$ blocks of a length $k(n)$ and a block of a length less than $k(n)$, where $k(n)$, $l(n)$ are integers with

$$\frac{k(n)}{n/\log n} \to 1, \quad \frac{l(n)}{\log n} \to 1 \quad \text{as} \quad n \to \infty. \tag{4.8}$$

Note that by the above formula, for $0 < \delta < 1$, if n is large enough, we have

$$(1-\delta)\log n \le l(n) \le (1+\delta)\log n. \tag{4.9}$$

From Lemma 2.1, we know that there exists a sequence of independent random variables $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n$ such that for every $1 \le i \le n$, $\tilde{X}_i$ and X_i have the same distribution.

Step 1. We shall prove the lower bound of the limit (3.2), namely,

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(S_n > e^{\varepsilon b^n}\right)}{n} \ge -\eta_* \log b. \tag{4.10}$$

Combining with (2.1) and (4.9), we get

$$
\begin{aligned}
&\mathbb{P}\left(S_n > e^{\varepsilon b^n}\right)\\
&\geq \mathbb{P}\left(\sum_{j=1}^{l(n)} X_{(j-1)k(n)+1} > e^{\varepsilon b^n}\right)\\
&\geq \mathbb{P}\left(\max_{1\leq j\leq l(n)} X_{(j-1)k(n)+1} > e^{\varepsilon b^n}\right)\\
&\geq \mathbb{P}\left(\max_{1\leq j\leq l(n)} \tilde{X}_{(j-1)k(n)+1} > e^{\varepsilon b^n}\right) - l(n)\beta(k(n))\\
&= 1 - \left(1 - \mathbb{P}(X > e^{\varepsilon b^n})\right)^{l(n)} - l(n)\beta(k(n))\\
&\geq 1 - (1 - \mathbb{P}(X > e^{\varepsilon b^n}))^{(1-\delta)\log n} - (1+\delta)(\log n)\beta(k(n)).
\end{aligned}
\tag{4.11}
$$

From Remark 2.2, for $\varepsilon > 0$, we have

$$\mathbb{P}\left(X > e^{\varepsilon b^n}\right) > e^{-\delta n} b^{-\eta_* n},$$

which combine with (4.11) and the following inequality

$$1 - y \leq e^{-y} \quad \text{for} \quad y \geq 0,$$

we obtain

$$
\begin{aligned}
&\mathbb{P}\left(S_n > e^{\varepsilon b^n}\right)\\
&\geq 1 - \left(1 - e^{-\delta n} b^{-\eta_* n}\right)^{(1-\delta)\log n} - (1+\delta)(\log n)\beta(k(n))\\
&\geq 1 - e^{-(1-\delta)(\log n) e^{-\delta n} b^{-\eta_* n}} - (1+\delta)(\log n)\beta(k(n))\\
&\geq (1+o(1))(1-\delta)(\log n) e^{-\delta n} b^{-\eta_* n} - (1+\delta)(\log n)\beta(k(n)).
\end{aligned}
$$

Substituting (4.1) and $\log\beta(n)/n\log n \to -\infty$ $(n\to\infty)$ into the above inequality yields

$$\lim_{n\to\infty} \frac{\log((1+\delta)(\log n)\beta(k(n)))}{n} = -\infty.$$

It is straightforward that

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(S_n > e^{\varepsilon b^n}\right)}{n} \geq -\delta - \eta_* \log b.$$

Because of the arbitrariness of δ, (4.10) holds.

Note that by Remark 2.2, we know that for any $\delta > 0$, there exists a subsequence $\{n_k; k\geq 1\}$ such that

$$\mathbb{P}\left(X > e^{\varepsilon b^{n_k}}\right) \geq e^{-\delta n_k} b^{-\eta^* n_k}.$$

Hence, analogously to (4.10), it is straightforward that

$$\limsup_{n\to\infty}\frac{\log\mathbb{P}\left(S_n>e^{\varepsilon b^n}\right)}{n}\geq -\eta^*\log b.$$

Step 2. We shall prove the upper bound of the limit (3.2), namely,

$$\liminf_{n\to\infty}\frac{\log\mathbb{P}\left(S_n>e^{\varepsilon b^n}\right)}{n}\leq -\eta_*\log b. \tag{4.12}$$

Applying Lemma 2.1 and the inequality (4.9), we have

$$\begin{aligned}
\mathbb{P}\left(S_n>e^{\varepsilon b^n}\right) &= \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}X_i>\frac{e^{\varepsilon b^n}}{n}\right)\\
&\leq \mathbb{P}\left(\frac{1}{k(n)l(n)}\sum_{i=1}^{n}X_i>\frac{e^{\varepsilon b^n}}{n}\right)\\
&\leq \mathbb{P}\left(\frac{1}{k(n)}\sum_{j=1}^{k(n)}\frac{1}{l(n)}\sum_{i=1}^{l(n)+1}X_{(i-1)k(n)+j}>\frac{e^{\varepsilon b^n}}{n}\right)\\
&= \mathbb{P}\left(\sum_{j=1}^{k(n)}\frac{1}{l(n)}\sum_{i=1}^{l(n)+1}X_{(i-1)k(n)+j}>k(n)\frac{e^{\varepsilon b^n}}{n}\right)\\
&\leq k(n)\mathbb{P}\left(\frac{1}{l(n)}\sum_{i=1}^{l(n)+1}X_{(i-1)k(n)+1}>\frac{e^{\varepsilon b^n}}{n}\right)\\
&\leq k(n)\mathbb{P}\left(\sum_{i=1}^{l(n)+1}X_{(i-1)k(n)+1}>\frac{(1-\delta)(\log n)e^{\varepsilon b^n}}{n}\right)\\
&\leq k(n)\mathbb{P}\left(\sum_{i=1}^{l(n)+1}\tilde{X}_{(i-1)k(n)+1}>\frac{(1-\delta)(\log n)e^{\varepsilon b^n}}{n}\right)\\
&\quad + k(n)(l(n)+1)\beta(k(n))\\
&\leq k(n)\mathbb{P}\left(\tilde{S}_{l(n)+1}>\frac{(1-\delta)(\log n)e^{\varepsilon b^n}}{n}\right)+2n\beta(k(n)),
\end{aligned}$$

where

$$\tilde{S}_{l(n)+1}:=\sum_{i=1}^{l(n)+1}\tilde{X}_{(i-1)k(n)+1}.$$

Applying Lemma 2.4 with

$$t=\left(\log\left((1-\delta)\frac{\log n}{n}\right)+\varepsilon b^n\right)^{\lambda},$$

where $\lambda = 1/n$, we have

$$1 \le u \le \exp\left(\left(\log\left((1-\delta)\frac{\log n}{n}\right) + \varepsilon b^n\right)\left(1-2^{-n}\right)\right).$$

We can put $u = u'n$, where

$$u' = \max\left\{2e^2 b^{-1}, \eta_* \log b, 1\right\}. \tag{4.13}$$

It is not difficult to get that

$$\begin{aligned} & k(n)\mathbb{P}\left(\tilde{S}_{l(n)+1} > \frac{(1-\delta)(\log n)e^{\varepsilon b^n}}{n}\right) \\ & \le k(n)(l(n)+1)\mathbb{P}\left(X > \frac{(1-\delta)\frac{\log n}{n}e^{\varepsilon b^n}}{u}\right) \\ & \quad + k(n)\left(\frac{2en\mathbb{E}((\log X)^{1/n})}{u\left(\log\left((1-\delta)\frac{\log n}{n}\right)+\varepsilon b^n\right)^{1/n}}\right)^u \\ & \le 2n\mathbb{P}\left(X > \frac{(1-\delta)\frac{\log n}{n}e^{\varepsilon b^n}}{u'n}\right) \\ & \quad + k(n)\left(\frac{2e\mathbb{E}((\log X)^{1/n})}{u'\left(\log\left((1-\delta)\frac{\log n}{n}\right)+\varepsilon b^n\right)^{1/n}}\right)^{u'n}. \end{aligned}$$

Together with (4.8) and $\log\beta(n)/n\log n \to -\infty$ ($n \to \infty$), we notice that

$$\lim_{n\to\infty} \frac{\log(2n\beta(k(n)))}{n} = -\infty. \tag{4.14}$$

It follows, from Lemma 2.3, that

$$\begin{aligned} & \liminf_{n\to\infty} \frac{1}{n}\log\mathbb{P}\left(X > \frac{(1-\delta)\frac{\log n}{n}e^{\varepsilon b^n}}{u'n}\right) \\ & = \liminf_{n\to\infty} \frac{1}{n}\log\mathbb{P}\left(\log X > \log\left((1-\delta)\frac{\log n}{n}\right) + \varepsilon b^n - \log(u'n)\right) \\ & \le \liminf_{n\to\infty} \frac{1}{n}\log\mathbb{P}\left(\log X > \frac{\varepsilon b^n}{2}\right) \\ & = -\eta_*\log b. \end{aligned} \tag{4.15}$$

Note that by (4.8), we get

$$\liminf_{n\to\infty}\frac{1}{n}\log\left[k(n)\left(\frac{2e\mathbb{E}\left((\log X)^{1/n}\right)}{u'\left(\log\left((1-\delta)\frac{\log n}{n}\right)+\varepsilon b^n\right)^{1/n}}\right)^{u'n}\right]$$

$$
\begin{aligned}
&= \liminf_{n\to\infty}\left[\frac{\log k(n)}{n}+\right.\\
&\left.\frac{u'n\left[\log\left(2e\mathbb{E}\left((\log X)^{1/n}\right)\right)-\log\left(u'\left(\log\left((1-\delta)\frac{\log n}{n}\right)+\varepsilon b^n\right)^{1/n}\right)\right]}{n}\right]\\
&\le \liminf_{n\to\infty}\frac{\log k(n)+u'n\left[\log\left(2e\mathbb{E}\left((\log X)^{1/n}\right)\right)-\log u'-\log\left(\frac{1}{2}\varepsilon b^n\right)^{1/n}\right]}{n}\\
&= -u'\log\frac{u'b}{2e}.
\end{aligned}
$$

Therefore, from (4.13),(4.14), (4.15) and the above inequality, we get

$$\liminf_{n\to\infty}\frac{\log\mathbb{P}(S_n>e^{\varepsilon b^n})}{n}\le-\eta_*\log b.$$

Analogously to (4.12), it is straightforward that

$$\limsup_{n\to\infty}\frac{\log\mathbb{P}\left(S_n>e^{\varepsilon b^n}\right)}{n}\le-\eta^*\log b. \qquad \square$$

Proof of Theorem 3.3. Since $\mathbb{E}((\log X)^d)<\infty$ for any $d\ge 0$. Let $\eta^*=0$ and $\gamma,\delta\in(0,1)$ satisfy $1-\gamma-\delta>0$. By Remark 2.2, there exists a subsequence $\{n_k,k\ge 1\}$ such that

$$\mathbb{P}\left(S_{n_k}>e^{\varepsilon n_k^x}\right)>n_k^{-\delta}.$$

By using the same proof for lower bound in Theorem 3.1, we have

$$
\begin{aligned}
\mathbb{P}\left(S_{n_k}>e^{\varepsilon n_k^x}\right)&\ge 1-\left(1-\mathbb{P}\left(X>e^{\varepsilon n_k^x}\right)\right)^{(1-\delta)n_k^{1-\gamma}}-(1+\delta)n_k^{1-\gamma}\beta(k(n_k))\\
&\ge 1-\left(1-n_k^{-\delta}\right)^{(1-\delta)n_k^{1-\gamma}}-(1+\delta)n_k^{1-\gamma}n_k^{-((1+\delta)/\gamma-1)\gamma}\\
&\ge 1-e^{-(1-\delta)n_k^{1-\gamma-\delta}}-(1+\delta)n_k^{-\delta}.
\end{aligned}
$$

It is easy to get that

$$\lim_{k\to\infty}\mathbb{P}\left(S_{n_k}>e^{\varepsilon n_k^x}\right)=1.$$

Therefore, (3.3) holds.

Let $\eta^*=\infty$ and $x>1$. By using the same proof for upper bound in Theorem 3.1, we have

$$
\begin{aligned}
\mathbb{P}\left(S_n>e^{\varepsilon n^x}\right)&\le k(n)\mathbb{P}\left(\tilde{S}_{l(n)+1}>(1-\delta)n^{-\gamma}e^{\varepsilon n^x}\right)+2n\beta(k(n))\\
&=k(n)\mathbb{P}\left(\log\tilde{S}_{l(n)+1}>\log\left((1-\delta)n^{-\gamma}e^{\varepsilon n^x}\right)\right)+2n\beta(k(n))\\
&\le k(n)\mathbb{P}\left(\sum_{i=1}^n\log\tilde{X}_i>\log\left((1-\delta)n^{-\gamma}e^{\varepsilon n^x}\right)\right)+2n\beta(k(n)).
\end{aligned}
$$

Together with the Markov's inequality and c_r-inequality, we get

$$\begin{aligned}&\mathbb{P}\left(\sum_{i=1}^{n}\log\tilde{X}_i>\log\left((1-\delta)n^{-\gamma}e^{\varepsilon n^x}\right)\right)\\ \le\ &\left(\log\left((1-\delta)n^{-\gamma}\right)+\varepsilon n^x\right)^{-d}\mathbb{E}\left(\sum_{i=1}^{n}\log\tilde{X}_i\right)^d\\ \le\ &\left(\log\left((1-\delta)n^{-\gamma}\right)+\varepsilon n^x\right)^{-d}n^d\mathbb{E}\left((\log X)^d\right),\end{aligned}$$

which implies that

$$\frac{1}{\log n}\log\left(k(n)\mathbb{P}\left(\sum_{i=1}^{n}\log\tilde{X}_i>\log\left((1-\delta)n^{-\gamma}e^{\varepsilon n^x}\right)\right)\right)\le d(1-x).$$

Letting d tend to infinity, we have

$$\lim_{n\to\infty}\frac{1}{\log n}\log\left(k(n)\mathbb{P}\left(\sum_{i=1}^{n}\log\tilde{X}_i>\log\left((1-\delta)n^{-\gamma}e^{\varepsilon n^x}\right)\right)\right)=-\infty.$$

From the above equation and (4.7), (3.4) holds.

Proof of Theorem 3.4. Since $\mathbb{E}((\log X)^d)<\infty$ for any $d\ge 0$. Let $\eta^*=\infty$. For $b>1$, by using the same proof for upper bound in Theorem 3.2, we have

$$\begin{aligned}\mathbb{P}\left(S_n>e^{\varepsilon b^n}\right)&\le k(n)\mathbb{P}\left(\tilde{S}_{l(n)+1}>\frac{(1-\delta)\log ne^{\varepsilon b^n}}{n}\right)+2n\beta(k(n))\\&=k(n)\mathbb{P}\left(\log\tilde{S}_{l(n)+1}>\log\left(\frac{(1-\delta)\log ne^{\varepsilon b^n}}{n}\right)\right)+2n\beta(k(n))\\&\le k(n)\mathbb{P}\left(\sum_{i=1}^{n}\log\tilde{X}_i>\log\left(\frac{(1-\delta)\log ne^{\varepsilon b^n}}{n}\right)\right)+2n\beta(k(n)).\end{aligned}$$

Combining with the Markov's inequality and c_r-inequality, we get

$$\begin{aligned}&\mathbb{P}\left(\sum_{i=1}^{n}\log\tilde{X}_i>\log\left(\frac{(1-\delta)\log ne^{\varepsilon b^n}}{n}\right)\right)\\ \le\ &\left(\log\left(\frac{(1-\delta)(\log n)e^{\varepsilon b^n}}{n}\right)\right)^{-d}\mathbb{E}\left(\sum_{i=1}^{n}\log\tilde{X}_i\right)^d\\ \le\ &\left(\log\left(\frac{(1-\delta)(\log n)e^{\varepsilon b^n}}{n}\right)\right)^{-d}n^d\mathbb{E}\left((\log X)^d\right),\end{aligned}$$

which implies that

$$\frac{1}{n}\log\left(k(n)\mathbb{P}\left(\sum_{i=1}^{n}\log\tilde{X}_i>\log\left(\frac{(1-\delta)\log ne^{\varepsilon b^n}}{n}\right)\right)\right)\le -d\log b.$$

Letting d tend to infinity, we have

$$\lim_{n\to\infty}\frac{1}{n}\log\left(k(n)\mathbb{P}\left(\sum_{i=1}^{n}\log\tilde{X}_i>\log\left(\frac{(1-\delta)\log ne^{\varepsilon b^n}}{n}\right)\right)\right)=-\infty.$$

Under the above equation and (4.14), (3.5) holds. □

Proof of Corollary 3.1. We only show (3.7) by omitting the proof of (3.6). From the proof of lower bound and the inequality (4.4), we get

$$\liminf_{n\to\infty}\frac{\log\mathbb{P}\left(M_n>e^{\varepsilon n^x}\right)}{\log n}\geq 1-\eta_* x.$$

Note that $M_n\leq S_n$. By using (4.5), we obtain

$$\liminf_{n\to\infty}\frac{\log\mathbb{P}\left(M_n>e^{\varepsilon n^x}\right)}{\log n}\leq\liminf_{n\to\infty}\frac{\log\mathbb{P}\left(S_n>e^{\varepsilon n^x}\right)}{\log n}\leq 1-\eta_* x. \qquad \square$$

Proof of Corollary 3.2. We only show (3.9) by omitting the proof of (3.8). From the proof of lower bound and the inequality (4.11), we get

$$\liminf_{n\to\infty}\frac{\log\mathbb{P}\left(M_n>e^{\varepsilon b^n}\right)}{n}\geq -\eta_*\log b.$$

Note that $M_n\leq S_n$. By using (4.12), it implies that

$$\liminf_{n\to\infty}\frac{\log\mathbb{P}\left(M_n>e^{\varepsilon b^n}\right)}{n}\leq\liminf_{n\to\infty}\frac{\log\mathbb{P}\left(S_n>e^{\varepsilon b^n}\right)}{n}\leq -\eta_*\log b. \qquad \square$$

5. Generalization

We have assumed $0<\eta^*<\infty$ in Theorems 3.1 and 3.2. However, it is also possible that $\eta^*=0$ for heavy tailed distributions. A general framework is needed to handle this case. Let us introduce parameters $\eta^*(k)$ and $\eta_*(k)$ for $k=1,2,\ldots$ as follows.

$$\eta^*(k)=-\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(\log_k X>t\right)\in[0,\infty]$$

and

$$\eta_*(k)=-\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(\log_k X>t\right)\in[0,\infty],$$

where

$$\log_k x=\begin{cases}\log(\log_{k-1}x) & \text{if } k\geq 1\\ x & \text{if } k=0.\end{cases}$$

Note that

$$\eta_*(1)=\alpha_*,\ \eta^*(1)=\alpha^*,\ \eta_*(2)=\eta_*,\ \eta^*(2)=\eta^*.$$

Theorem 5.1. *Fix an integer $k \geq 1$. Assume that $0 < \eta^*(k) < \infty$ and the mixing coefficient $\beta(n)$ satisfies*

$$\frac{\log \beta(n)}{\log n} \to -\infty \quad \textit{as} \quad n \to \infty,$$

then for $\varepsilon > 0$ and every $x > \max\{1, 1/\eta^*(k)\}$,

$$\limsup_{n\to\infty} \frac{\log \mathbb{P}\left(\log_{k-1} S_n > \varepsilon n^x\right)}{\log n} = 1 - \eta^*(k)x.$$

In addition, if $\eta_*(k) < \infty$, then for $\varepsilon > 0$ and every $x > \max\{1, 1/\eta^*(k)\}$,

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(\log_{k-1} S_n > \varepsilon n^x\right)}{\log n} = 1 - \eta_*(k)x.$$

Theorem 5.2. *Fix an integer $k \geq 1$. Assume that $0 < \eta^*(k) < \infty$ and the mixing coefficient $\beta(n)$ satisfies*

$$\frac{\log \beta(n)}{n \log n} \to -\infty \quad \textit{as} \quad n \to \infty,$$

then for $\varepsilon > 0$ and every $b > 1$,

$$\limsup_{n\to\infty} \frac{\log \mathbb{P}\left(\log_{k-1} S_n > \varepsilon b^n\right)}{n} = -\eta^*(k) \log b.$$

In addition, if $\eta_*(k) < \infty$, then for $\varepsilon > 0$ and every $b > 1$,

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(\log_{k-1} S_n > \varepsilon b^n\right)}{n} = -\eta_*(k) \log b.$$

The proofs of Theorems 5.1 and 5.2 are the same as the proofs of Theorems 3.1 and 3.2, respectively.

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