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CASCADES OF TORIC LOG DEL PEZZO SURFACES OF PICARD NUMBER ONE

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Abstract. We classify toric log del Pezzo surfaces of Picard number one by introducing the notion, cascades. As an application, we show that if such a surface admits a Kähler–Einstein metric, then it should admit a special cascade and it satisfies the equality of the orbifold Bogomolov–Miyaoka–Yau inequality, i.e., $K^2 = 3e_{orb}$. Moreover, we provide an algorithm to compute a toric log del Pezzo surfaces of Picard number one for a given input of singularity types.

1. Introduction

A normal projective surface S with quotient singularities is called a *log del Pezzo surface* if its anticanonical divisor $-K_S$ is an ample $\mathbb{Q}$-Cartier divisor. A lot of work has been devoted to classify log del Pezzo surfaces. In particular, they are classified up to index 3. Here the *index* is defined to be the smallest integer n such that $-nK_S$ becomes a Cartier divisor. See [14], [1], [20], [28], and [11]. For the Picard number one case, see also [32] and [23].

In addition, if we further assume that S is toric, a lot of things have been investigated thanks to the one-to-one correspondence between toric log del Pezzo surfaces and certain convex lattice polygons, called *LDP-polygons*, due to Dais and Nill ([8]). In particular, toric log del Pezzo surfaces are completely classified up to index 17 ([6], [21]). See [12] for the list. Moreover, all toric log del Pezzo surfaces with 1 singular point are completely classified in [7], and those with 2 singular points are completely classified in [31] which also contains a partial classification of those with 3 singular points.

In this paper, we shall classify toric log del Pezzo surfaces of Picard number one by using the notion, a cascade, which was also introduced in [15] for a larger class of rational $\mathbb{Q}$-homology projective planes. In fact, even though there exist infinitely many toric log del Pezzo surfaces of Picard number one, one might think that classifying them is not a very difficult task at least in

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the sense that it is easy to describe their corresponding *LDP triangles*, i.e., LDP-polygons with 3 vertices. But it does not give us any geometric intuition and thus sometimes it is not easy to derive geometric consequences. By describing the classification in terms of cascades, which we shall soon define, one can understand the underlying geometry more clearly. See Section 4 for the applications. For example, one can easily determine whether a toric log del Pezzo surface of Picard number one with given singularity types exists or not. See Theorem 4.1, Corollary 4.2 and Algorithm 4.6.

Definition. Let S be a toric log del Pezzo surface of Picard number one. We say that S *admits a cascade* if there exists a diagram as follows:

$$\begin{tikzcd} S' = S'_t \arrow[r, "\phi_t"] \arrow[d, "\pi_t"'] & S'_{t-1} \arrow[r, "\phi_{t-1}"] \arrow[d, "\pi_{t-1}"'] & \dots \arrow[r, "\phi_1"] & S'_0 \arrow[d, "\pi_0"'] \\ S_t := S & S_{t-1} & \dots & S_0 \end{tikzcd}$$

where for each k

(1) ϕ_k is a toric blow-down,
(2) π_k is the minimal resolution,
(3) S_k is a toric log del Pezzo surface of Picard number one, and
(4) S_0 is *basic*, i.e., for every torus-invariant curve D intersecting a (-1)-curve C in S', $D^2 \geq -2$.

In this case, we also say that S *admits a cascade to* S_0, and S_0 is the *basic surface* of S.

The first main result of the present paper is to show the existence of a cascade for every toric log del Pezzo surface of Picard number one.

Theorem 1.1. *Every toric log del Pezzo surface of Picard number one admits a cascade.*

The condition that the Picard number is one is crucial. See [16, Subsection 3.4] for the counterexample and further discussions when the Picard number is greater than one.

The proof uses the standard theory of $\mathbb{P}^1$-fibrations. By looking at the dual graph of the torus-invariant divisors, one can immediately extract the information of $\mathbb{P}^1$-fibration structure on the corresponding smooth toric surface. See Notation 1 for dual graphs.

Moreover, it is easy to classify basic toric log del Pezzo surfaces of Picard number one.

Proposition 1.2. *If S_0 is a basic toric del Pezzo surface of Picard number one, then S_0 belongs to one of the following five types of surfaces*:

(1) $\mathbb{P}^2$.
(2) $\mathbb{P}(1, 1, n)$ *for* $n \geq 2$.

(3) *The log del Pezzo surface of Picard number one with two singular points of type* $A_1 + \frac{1}{2n-1}(1,2)$*, denoted by* $S_n(0,2)$.
(4) *The log del Pezzo surface of Picard number one with three singular points of type* $2A_1 + \frac{1}{4n-4}(1,2n-1)$*, denoted by* $S_n(2,2)$.
(5) *The log del Pezzo surface of Picard number one with* 3 *singular points of type* A_2*, denoted by* $S_2(2,4)$.

Conversely, by inverting the cascade process, one can obtain every toric log del Pezzo surface of Picard number one from the basic ones.

Theorem 1.3. *The minimal resolution of every toric log del Pezzo surface of Picard number one that is not basic is obtained from one of the three basic toric surfaces* $S_n(0,2)$, $S_n(2,2)$ *and* $S_2(2,4)$ *by a sequence of toric blowups at the intersection point of a* (-1)*-curve and a torus-invariant curve with self-intersection number at most* -2.

Since the cascade and its inverse process preserve the number of singular points of S, we can describe all toric log del Pezzo surfaces of Picard number one with respect to the given number of singular points.

Theorem 1.4. *Let* S *be a toric log del Pezzo surface of Picard number one. If* S *is not basic, it admits a cascade to one of the three basic surfaces*: $S_n(0,2)$, $S_n(2,2)$ *and* $S_2(2,4)$. *In particular, we have the following.*

(1) *If* $|Sing(S)| = 0$*, then* $S \cong \mathbb{P}^2$.
(2) *If* $|Sing(S)| = 1$*, then* $S \cong \mathbb{P}(1,1,n)$*, where* $n \geq 2$.
(3) *If* $|Sing(S)| = 2$*, then* $S \cong \mathbb{P}(1,p,q)$ *and it admits a cascade to* $S_n(0,2)$.
(4) *If* $|Sing(S)| = 3$*, then* S *admits a cascade to either* $S_n(2,2)$ *or* $S_2(2,4)$.

In particular, this reproves the theorems in [7] and [31] for the Picard number one case.

As an application, we shall consider the orbifold Bogomolov–Miyaoka–Yau inequality together with the Kähler–Einstein property. We first recall what is known about toric Fano varieties admitting a Kähler–Einstein metric.

Theorem 1.5 ([30], [2])**.** *A toric Fano variety admits a Kähler–Einstein metric if and only if the barycenter of its moment polytope is the origin.*

Motivated by the above theorem, the following definition is widely used in the literature.

Definition. A toric log del Pezzo surface is said to be *Kähler–Einstein* if the barycenter of its moment polytope is the origin.

Now we recall that the orbifold Bogomolov–Miyaoka–Yau inequality does not hold for Fano manifolds *in general*. However, Chan and Leung proposed a Miyaoka–Yau type inequality for Kähler–Einstein toric Fano manifolds.

Theorem 1.6 ([3, Theorem 1.2])**.** *Let* X *be a Kähler–Einstein toric Fano manifold of dimension* n*. Then, for any nef class* H*, we have*

$$c_1^2(X)H^{n-2} \leq 3c_2(X)H^{n-2}$$

if either $n = 2, 3, 4$, or each facet of the corresponding dual polytope of the Fano polytope of X contains a lattice point in its interior.

One might ask whether the above inequality can be generalized in singular setting. In the surface case, the inequality has the same form as the orbifold Bogomolov–Miyaoka–Yau inequality, which holds under the opposite condition of the positivity of the canonical divisor.

Theorem 1.7 ([24–26])**.** (Simpler form) *Let S be a normal projective surface with quotient singularities. If K_S is nef, then we have the following inequality.*

$$K^2(S) \leq 3e_{orb}(S).$$

Unfortunately, the inequality does not hold for Kähler-Einstein toric del Pezzo surfaces in general, as illustrated in the following example informed to the author by Yeonsu Kim.

Example 1.8. The Miyaoka-Yau inequality does not hold for a Kähler-Einstein toric log del Pezzo surface of Picard number two. Let S be a toric log del Pezzo surface corresponding to the LDP polygon

$$P = conv\{(0,1), (-m,-1), (0,-1), (m,1)\},$$

where $m > 2$. It is easy to see that S is Kähler-Einstein by Theorem 1.5. However, $K^2 - 3e_{orb} = \frac{2m-4}{m} > 0$.

However, when the Picard number is one, we have the equality of the above inequality.

Theorem 1.9. *Let S be a toric log del Pezzo surface of Picard number one. If S is Kähler–Einstein, then we have the following properties.*

(1) $K_S^2 = 3e_{orb}$.
(2) *S is either isomorphic to $\mathbb{P}^2$ or S has exactly 3 singular points.*
(3) *If S is not isomorphic to $\mathbb{P}^2$, it admits a cascade to $S_2(2,4)$, i.e., a cubic surface with 3 singular points of type A_2, not to $S_n(2,2)$.*

We emphasize that the Kähler-Einstein property is closely related with the property of cascades. In particular, the condition of being Kähler-Einstein forces a singular toric log del Pezzo surface of Picard number one to admits a cascade to a *special* basic surface, which is the unique singular cubic surface with 3 singular points of type A_2, not to the basic surface with 3 singular points of type $2A_1 + \frac{1}{4n-4}(1, 2n-1)$.

As a final application, we give a simple observation that every finite cyclic group is a Brauer group of a toric log del Pezzo surface of Picard number one. See Theorem 4.9.

We work over an algebraically closed field of arbitrary characteristic.

2. Basic toric log del Pezzo surfaces of Picard number one

Throughout this section, let S be a toric log del Pezzo surface of Picard number one and $f\colon S' \to S$ be its minimal resolution. Note that if S is singular, i.e., the Picard number of S' is greater than one, the torus-invariant divisors form two sections and two fibers of a suitable $\mathbb{P}^1$-fibration $\Phi : S' \to \mathbb{P}^1$. For generalities about $\mathbb{P}^1$-fibrations on rational surfaces, see [27] or [13].

Notation 2.1. We denote by $[[s_1^2, F_1, s_2^2, F_2]]$ the smooth toric surface S' equipped with a $\mathbb{P}^1$-fibration $\Phi : S' \to \mathbb{P}^1$, where s_1 and s_2 are the two torus-invariant sections of Φ and; F_1 and F_2 are the two torus-invariant fibers of Φ.

Definition. Let F be a singular fiber of a $\mathbb{P}^1$-fibration on S'.

(1) F is said to be *of type* I_0 if its dual graph is of the form $\overset{-2}{\circ} - \overset{-1}{\circ} - \overset{-2}{\circ}$.
(2) F is said to be *of type* I if it can be contracted to a fiber of type I_0.
(3) F is said to be *of type* II_0 if its dual graph is of the form $\overset{-1}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-1}{\circ}$.
(4) F is said to be *of type* II if it can be contracted to a fiber of type II_0.

Notation 2.2. Let Φ be a $\mathbb{P}^1$-fibration.

(1) A smooth fiber is denoted by F_0.
(2) A singular fiber of type I_0 is denoted by F_1^0.
(3) A singular fiber of type I is denoted by F_1.
(4) A singular fiber of type II_0 is denoted by F_2^0.
(5) A singular fiber of type II is denoted by F_2.

The below lemma immediately follows from the standard theory of smooth projective rational surfaces.

Lemma 2.3. *Let S be a toric log del Pezzo surface of Picard number one and S' be its minimal resolution. Denote by n the number of torus-invariant curves on S' and by N the sum of all self-intersection numbers of the torus-invariant curves. Then we have $N = 12 - 3n$.*

The following notion is essential in the description of the cascades.

Definition. A smooth rational surface S' is said to be *basic* if $D^2 \geq -2$ for every torus-invariant curve D on S' intersecting C, where C is any (-1)-curve on S'. A toric log del Pezzo surface S is said to be *basic* if its minimal resolution S' is basic.

For later use, we introduce the following notation.

Notation 2.4. We will use the following notion.

(1) $S(\mathbb{P}^2) = \mathbb{P}^2$.
(2) $S'_n(0,0) := [[-n, F_0, n, F_0]]$.
(3) $S'_n(0,2) := [[-n, F_0, n-1, F_1^0]]$.
(4) $S'_n(2,2) := [[-n, F_1^0, n-2, F_1^0]]$.

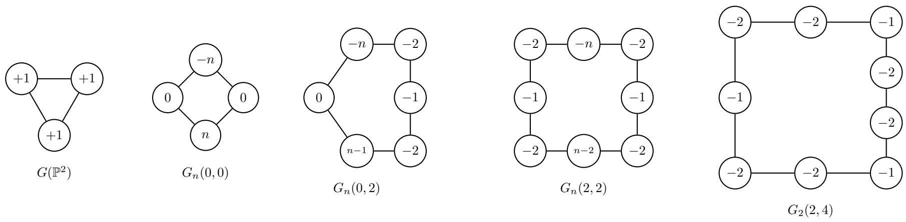


FIGURE 1. Basic dual graphs

(5) $S'_2(2,4) := [[-2, F_1^0, -2, F_2^0]]$.
(6) $S_n(a,b)$ denotes the anticanonical model of $S'_n(a,b)$, i.e., the surface obtained from $S'_n(a,b)$ by contracting all $(-n)$-curves with $n \geq 2$.

Figure 1 describes the *basic dual graphs*, i.e., the dual graphs of the torus-invariant curves on the above five surfaces.

Now we are ready to determine basic toric log del Pezzo surfaces of Picard number one.

Proposition 2.5 (=Proposition 1.2)**.** *If S is basic, then S belongs to one of the following five types of surfaces*: $S(\mathbb{P}^2)$, $S_n(0,0)$, $S_n(0,2)$, $S_n(2,2)$, *and* $S_2(2,4)$.

Proof. Assume that S is not isomorphic to $\mathbb{P}^2$. Then, S is singular and the Picard number of its minimal resolution S' is greater than one. Thus, S' admits a $\mathbb{P}^1$-fibration $\pi : S' \to \mathbb{P}^1$, where the cycle of torus-invariant curves forms two singular fibers and two sections of π.

If π is relatively minimal, then S' is isomorphic to the Hirzebruch surface $\mathbb{F}_n = S'_n(0,0)$ with $n \neq 1$. In this case, S is isomorphic to $\mathbb{P}(1,1,n)$.

From now on, we assume that π is not relatively minimal. In particular, there exists a (-1)-curve on S'. Moreover, since S is singular, there exists a torus-invariant curve with self-intersection number at most -2.

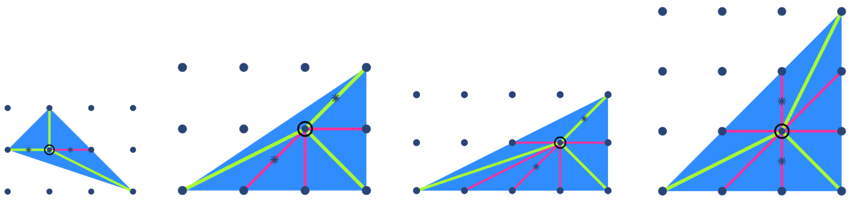
Note that there exists a (-1)-curve E meeting one of the exceptional curves of f. Let $D_1, D_2, \ldots, D_k$ be a chain of torus-invariant curves which contracts to one of the singular points of S such that E intersects D_1. Let C be the other torus-invariant curve intersecting E. Since S is basic, $D_1^2 \geq -2$ and $C^2 \geq -2$. We first consider the case $D_1^2 = C^2 = -2$. Since $D_1 + 2E + C$ induces a $\mathbb{P}^1$-fibration structure on S' on which it forms a singular fiber, there exists another torus-invariant fiber F. Since S is basic, it is easy to see that F belongs to one of the following four cases:

$$\overset{0}{\circ},\ \overset{-1}{\circ} - \overset{-1}{\circ},\ \overset{-2}{\circ} - \overset{-1}{\circ} - \overset{-2}{\circ},\ \text{and}\ \overset{-1}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \cdots - \overset{-2}{\circ} - \overset{-1}{\circ}.$$

In the first and third case, by Lemma 2.3, we see that the corresponding $\mathbb{P}^1$-fibration structures are $S'_n(0,2)$ and $S'_n(2,2)$, respectively, where $n \geq 2$. In the second case, one can show that S is of Picard number 2 or 3, a contradiction. In the final case, since S is basic and $\rho(S) = 1$, the torus-invariant sections have self-intersection number -2, so S has only rational double points as singular

TABLE 1. Vertices of the basic Fano triangles

$P(\mathbb{P}^2)$	$\{(0,1),(-1,-1),(1,0)\}$
$P_n(0,0), n \geq 2$	$\{(0,1),(-1,0),(n,-1)\}$
$P_n(0,2), n \geq 2$	$\{(1,1),(-n,-n+1),(1,-1)\}$
$P_n(2,2), n \geq 2$	$\{(-1,-1),(3,1),(2n-5,2n-3)\}$
$P_2(2,4)$	$\{(1,2),(-2,1),(1,-1)\}$



$P_2(0,0)$ $P_2(0,2)$ $P_2(2,2)$ $P_2(2,4)$

FIGURE 2. Reflexive singular basic Fano polygons

points. Thus, by Lemma 2.3, one can see that F should be of type (II_0). Hence, the corresponding surface is $S_2'(2,4)$.

Now we consider the case $C^2 = -1$ and assume that there is no (-1)-curve such that its adjacent torus-invariant curves have self-intersection number at most -2. Then, since $E+C$ induces a $\mathbb{P}^1$-fibration on which it forms a complete fiber, there exists another torus-invariant fiber F. By assumption, we see that the fiber F is one of the following:

$$\overset{0}{\circ},\ \overset{-1}{\circ} - \overset{-1}{\circ},\ \text{and}\ \ \overset{-1}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \cdots - \overset{-2}{\circ} - \overset{-1}{\circ}.$$

One can see that $\rho(S) > 1$ in all of the above cases, which is a contradiction.

Finally, we may assume that, for every (-1)-curve E intersecting an exceptional curve D_1 of f, the other torus-invariant curve C intersecting E have $C^2 \geq 0$. Then, by the similar analysis as above, one can see that C is a section of a $\mathbb{P}^1$-fibration Φ, D_1 is part of a fiber of type II and the other fiber is either of type II or of the form $\overset{0}{\circ}$. In any case, we have $\rho(S) > 1$, a contradiction. $\square$

Remark 2.6. The proof of Proposition 2.5 shows that $D^2 \geq -2$ can be replaced by $D^2 = -2$ in the definition of a basic surface.

Every toric log del Pezzo surface of Picard number one corresponds to a Fano triangle. See [8], [5] and [22] for LDP polygons or Fano polytopes in general. For each basic surface S in Notation 2.4, we denote by P the corresponding Fano triangle, which we call a *basic Fano triangle*. See Table 1 for some fixed choices for the explicit coordinates for the vertices of the basic Fano triangles.

See also Figure 2 for the drawings of reflexive singular basic Fano triangles.

3. Cascades of toric log del Pezzo surfaces of Picard number one

In this section we prove the main theorems of this paper and prove some properties preserved by cascades. For convenience we introduce the following notion.

Definition. Let S be a toric log del Pezzo surface of Picard number one. We say that S *admits a one-step cascade* if there exists a diagram as follows:

$$\begin{array}{ccc} S' & \xrightarrow{\phi} & \bar{S}' \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle \bar{\pi}}\downarrow \\ S & & \bar{S} \end{array}$$

where

(1) ϕ is a blow-down of a (-1)-curve,
(2) π and $\bar{\pi}$ are minimal resolutions, and
(3) $\bar{S}$ is a toric log del Pezzo surface of Picard number one.

Remark 3.1. In order to satisfy (3), the blowdown in (1) should be special.

Now we prove the main theorems of this paper.

Proof of Theorem 1.1. If S is basic, we are done. Assume that S is not basic. Then, there exists a (-1)-curve E that intersects a torus-invariant curve C with $C^2 \leq -3$. Let D be the other torus-invariant curve intersecting E. We claim that $D^2 = -2$. By [32, Lemma 1.4], $D^2 \geq -2$. If $D^2 \geq -1$, then, by contracting E and then contracting all torus-invariant curves with self-intersection number at most -2, we get a projective surface of Picard number zero, which is a contradiction. Thus, we have $D^2 = -2$. In fact, this can also be derived from [32, Lemma 4.2]. Now, contracting E induces a one-step cascade. Note that the new surface $\bar{S}$ in the diagram in Definition 3 is a log del Pezzo surface since the corresponding Fano polygon is a triangle, hence automatically convex. □

Proof of Theorem 1.3. In the process of each one-step cascade ϕ, the blowing-up locus of ϕ, as in the above diagram, is exactly the intersection point of two torus-invariant curves: one of them is a (-1)-curve and the other one has self-intersection number at most -2. Since there are exactly three basic surfaces $S'_n(0,2)$, $S'_n(2,2)$ and $S'_2(2,4)$ containing a torus-invariant (-1)-curve, the result follows. □

To describe the applications of Theorem 1.1 and Theorem 1.3, we introduce the notion of a trace of S.

Definition. The absolute value of the sum of self-intersection numbers of all irreducible components of exceptional curves of f is called the *trace* $tr(S)$ of S. In other words,

$$tr(s) = -\sum D_i^2,$$

where the sum runs over all exceptional curves D_i over f.

Now, the proof of Theorem 1.3 immediately yields the following.

Corollary 3.2. *The number of singular points of S and the number $tr(S) - 3L$ are invariant under a cascade, where L denotes the number of exceptional curves of the minimal resolution.*

Proof. It is enough to consider only a one-step cascade. It is clear that $tr(S) - 3L$ is invariant under a one-step cascade. This also follows from Lemma 2.3. Note that a one-step cascade does not increase the number of singular points. Assume that a one-step cascade decreases the number of singular points of S. Then, there is a chain of torus-invariant curves whose dual graph is of the form

$$\overset{-n}{\circ} - \overset{-1}{\circ} - \overset{-2}{\circ} - \overset{-m}{\circ},$$

where $-m \geq -1$ and $-n \leq -3$ since S is not basic. Let E be the (-1)-curve intersecting the $(-n)$-curve in the dual graph. By blowing down E, we can see that $n = 3$ by [32, Lemma 4.2], hence we get the following dual graph:

$$\overset{-2}{\circ} - \overset{-1}{\circ} - \overset{-m}{\circ}.$$

This cannot be possible since the Picard number is one. □

Remark 3.3. By Corollary 3.2, we can easily compute the trace of toric log del Pezzo surface of Picard number one once we know its basic surface.

S	$S(\mathbb{P}^2)$	$S_n(0,0)$	$S_n(0,2)$	$S_n(2,2)$	$S_2(2,4)$
tr(S)	-3	n	$3L-5+n(\geq 3L-3)$	$3L-7+n(\geq 3L-5)$	$3L-6$

The above table shows that the number of singular points and the trace of S determine uniquely the original surface S and its basic surface and vice versa. This will be used in Algorithm 4.6.

4. Applications

We completely classify toric log del Pezzo surfaces of Picard number one and the dual graphs of their singularities.

4.1. Classification

Theorem 4.1. *Let S be a toric log del Pezzo surface of Picard number one and S' be its minimal resolution. Then,*

1. *Either $S \cong \mathbb{P}(1,1,n)$ with $n \geq 1$ or S admits a cascade to one of the following: $S_n(0,2)$, $S_n(2,2)$, and $S_2(2,4)$ where $n \geq 2$.*
2. *Let T be the basic surface of S. Then, we have the following:*
   - (a) *if $T = S_n(0,2)$, then $S' = [[-n, F_1, n-1, F_0]]$,*
   - (b) *if $T = S_n(2,2)$, then $S' = [[-n, F_1, n-2, F_1']]$,*
   - (c) *if $T = S_2(2,4)$, then $S' = [[-n, F_1, -m, F_2]]$,*

   where F_1' is a fiber of type I possibly different from the fiber F_1.

Proof. Since $S_n(0,0) \cong \mathbb{P}(1,1,n)$, (1) immediately follows from Theorem 1.1 and Theorem 1.3.

We may assume that S is not basic. Then, by taking a finite number of one-step cascades, we can always find three torus-invariant curves whose dual graph is of the form $\overset{-2}{\circ} - \overset{-1}{\circ} - \overset{-2}{\circ}$. Note that they induce a $\mathbb{P}^1$-fibration Φ on the minimal resolution S' of S, on which they form a singular fiber F of type (I_0).

Consider the case $T = S_n(0,2)$. Since the inverting process only changes the singular fiber F of the $\mathbb{P}^1$-fibration, we see that $G(S) = [[-n, F_1, n-1, 0]]$ for some integer $n \geq 2$ with the unique singular fiber F_1 of type I.

Consider the case $T = S_n(2,2)$. Then only the two torus-invariant sections of Φ are invariant under the inverse process among all torus-invariant curves. Thus, we have $G(S) = [[-n, F_1, n-2, F_1']]$ for some integer $n \geq 2$, where both F_1 and F_1' are of type I.

Consider the case $T = S_2(2,4)$. Since no torus-invariant curve is invariant under the process of cascades in general, the result follows. □

Now we classify toric log del Pezzo surfaces of Picard number one according to the number of singular points.

Corollary 4.2. *Let S be a toric log del Pezzo surface of Picard number one. Then we have the following.*

(1) *If $|Sing(S)| \leq 1$, then $S \cong \mathbb{P}(1,1,n)$ for some n in $\mathbb{Z}^+$.*
(2) *If $|Sing(S)| = 2$, then $S \cong \mathbb{P}(1,q,(n-1)q+q_1)$ for some positive integers q, q_1, n, where $q > q_1$ and $gcd(q,q_1) = 1$.*
(3) *If $|Sing(S)| = 3$, then S is obtained by inverting a cascade from $S_2(2,4)$ or $S_n(2,2)$.*

In particular, if $|Sing(S)| \leq 2$, then S is a weighted projective plane.

To prove Corollary 4.2, we recall the Hirzebruch-Jung continued fraction.

Definition. For integers n_1, n_2, …, n_l, we set the following notation,

$$[n_1, n_2, \ldots, n_l] := n_1 - \cfrac{1}{n_2 - \cfrac{1}{\ddots - \cfrac{1}{n_l}}}.$$

If $n_i \geq 2$ for each i, then it is called a *Hirzebruch-Jung continued fraction.*

Proof of Corollary 4.2. For (1) and (3), the result follows from Theorem 4.1 and Corollary 3.2. Assume that $|Sing(S)| = 2$. By Theorem 4.1 and Corollary 3.2, S is obtained by inverting the cascade from $S_n(0,2)$.

Let F be a singular fiber of a $\mathbb{P}^1$-fibration of the form

$$\underset{F_1}{\square} - \underset{-1}{\circ} - \underset{F_2}{\square},$$

where F_1 is the dual graph corresponding to the Hirzebruch-Jung continued fraction $[n_1, \ldots, n_l]$ and F_2 corresponds to $[m_1, \ldots, m_t]$. Since $F^2 = 0$,

$$[n_1, \ldots, n_l, 1, m_1, \ldots, m_t] = 0.$$

Write $\frac{q}{q_1} = [m_1, \ldots, m_t]$. By Lemma 4.5 below,

$$Q = [n, n_1, \ldots, n_l] = n - \frac{q - q_1}{q} = \frac{(n-1)q + q_1}{q}.$$

Thus, S and $\mathbb{P}(1, q, (n-1)q + q_1)$ have the same singularity types. This completes the proof since the singularity type uniquely determines the surface when $|Sing(S)| = 2$. $\square$

Remark 4.3. It is well known that a weighted projective plane is a toric log del Pezzo surface of Picard number one. One can easily construct infinitely many toric surfaces of Picard number one which is not a weighted projective plane by inverting the cascade from $S_2(2,4)$. See the construction in the proof of Theorem 4.9.

Remark 4.4. Corollary 4.2 reproves the results in [7] and [31] for the Picard number one case.

Lemma 4.5. *Let $[n_1, \ldots, n_l]$ and $[m_1, \ldots, m_t]$ be Hirzebrugh-Jung continued fractions such that $[n_1, \ldots, n_l, 1, m_1, \ldots, m_t] = 0$. If $[m_1, \ldots, m_t] = \frac{q}{q_1}$, then $[n_l, \ldots, n_1] = \frac{q}{q-q_1}$.*

Proof. This lemma is well-known and easy to prove. See [29, Example 1] for the algorithm to compute $[n_1, \ldots, n_l]$ for a given $[m_1, \ldots, m_t]$. $\square$

Algorithm 4.6. By Corollary 4.2, we can determine whether there exists a toric log del Pezzo surface S of Picard number one with given singularity types.

<u>INPUT</u>: a k-tuple $(\frac{n_1}{m_1}, \ldots, \frac{n_k}{m_k})$ of positive rational numbers greater than 1, where k denotes the number of singular points of S and each rational number describes the singularity type.

<u>OUTPUT</u>: False if there exists no toric log del Pezzo surface of Picard number one having the given singularity type in INPUT. If it exists, we return S if S is basic, or S together with its basic surface if otherwise.

<u>PROCEDURE</u>: (using notation in Remark 3.3)

(1) If the input is empty, i.e., $k = 0$, then $S = \mathbb{P}^2$.
(2) If $k = 1$ and $m_1 = 1$, then $S = \mathbb{P}(1, 1, n_1)$.
(3) If $k \geq 2$, then reorder the k-tuple so that $i \geq j$ if and only if either $n_i > n_j$, or $n_i = n_j$ and $m_i \geq m_j$.

(4) If $k=2$, let m'_i be the integer such that $\frac{n_i}{m'_i}=[n_t,n_{t-1},\dots,n_1]$, where $\frac{n_i}{m_i}=[n_1,n_2,\dots,n_l]$ for $i=1,2$. If $n_2=m_1,m'_1$ and $n_1=m_2,m'_2$ modulo n_2, then $S\cong\mathbb{P}(1,n_2,n_1)$.
(5) If $k=3$ and $tr=3L-6$, then consider the three dual graphs of the singularities corresponding to the triple in INPUT. Form a cycle G by adding one vertex of weight -1 between any two of the three dual graphs. Note that there are four possible ways for forming the cycle. If the graph is $G_2(2,4)$ after a finite number of "blowing-down" of the graph, then S is the toric log del Pezzo surface of Picard number one whose dual graph of the torus-invariant divisors is G.
(6) If $k=3$ and $tr\geq 3L-5$, then consider the three dual graphs G_1, G_2, G_3 of the singularities corresponding to $\frac{n_1}{m_1},\frac{n_2}{m_2},\frac{n_3}{m_3}$. Form a tree G by adding one vertex of weight -1 between G_1 and G_2; and between G_1 and G_3. Note that there are three possible ways for forming the tree. If the graph is $G_n(2,2)$ after a finite number of "blowing-down" of the graph, then S is the toric log del Pezzo surface of Picard number one whose dual graph of the torus-invariant divisors is G.
(7) Return False.

4.2. Kähler–Einstein toric log del Pezzo surfaces of Picard number one

We shall show that the Kähler–Einstein property is closely related with the existence of a cascade to a particular basic surface. The main result in this direction is Theorem 1.9.

Proof of Theorem 1.9. Since $\mathbb{P}^2$ is Kähler–Einstein, (2) follows from [18, Corollary 4.7].

Let S be a Kähler–Einstein log del Pezzo surface of Picard number one. It is enough to assume that S is singular. Consider the minimal resolution $f\colon S'\to S$. Let $D_1,\dots,D_L$ be the all irreducible components of the reduced part $\mathcal{D}$ of the f-exceptional divisor. By [18, Corollary 4.7], S has 3 singular points, each of which has local fundamental group of order a, i.e., the finite groups inducing the three singular points have the same order. Then, by [17, Section 3 and Lemma 3.6],

$$K_S^2=tr-3L+6+\frac{9}{a},$$

where $tr=-\sum_{k=1}^{L}D_k^2$. Since $3e_{orb}=\frac{9}{a}$, we see that $K_S^2=3e_{orb}$ if and only if $tr=3L-6$ if and only if S admits a cascade to $S_2(2,4)$. The last equivalence follows from Corollary 4.2 and Lemma 3.3. Thus, it remains to show that S admits a cascade to $S_2(2,4)$. Now the below lemma completes the proof by Corollary 4.2. □

Lemma 4.7. *Let S be a log del Pezzo surface of Picard number one. If S admits a cascade to $S_n(2,2)$, then S is not Kähler–Einstein.*

Proof. Let P be the Fano polygon corresponding to S. It is enough to show that the barycenter of P is not the origin by [2, Theorem 1.2] together with [19, Proposition 3.1]. Since S admits a cascade to $S_n(2,2)$, P admits a cascade to $P_n(2,2)$. By Table 1, $P_n(2,2) = conv\{(-1,-1),(3,1),(2n-5,2n-3)\}$, so its barycenter lies below the y-axis. Since the y-coordinate of the barycenter is not increasing during the inverting process of the cascade, the barycenter of P cannot be the origin. □

4.3. Brauer groups

The Brauer–Grothendieck group $B(X)$, in short, the Brauer group, of a scheme X is defined by $B(X) = H^2_{\acute{e}t}(X, \mathbb{G}_m)$, where $\mathbb{G}_m$ denotes the multiplicative group. See [4] for an extensive discussion.

Ford showed that every finite abelian group is the Brauer group of a ring ([10]). In the course of the proof, he showed that every finite cyclic group is the Brauer group of a certain affine singular 3-fold. On the other hand, the Brauer group of a projective toric surface is always finite cyclic and can easily be computed by the following theorem.

Theorem 4.8 ([9, Corollary 2.9])**.** *Let X be a toric surface, Δ be the corresponding complete fan on $\mathbb{R}^2$ and $\Delta(1) = \{\rho_1, \ldots, \rho_n\}$. If $N' = \langle \rho_1 \cap N, \ldots, \rho_n \cap N\rangle$, then $B(X) \cong N/N'$.*

By using the above theorem, we show that every finite cyclic group is a Brauer group of a toric log del Pezzo surface of Picard number one. In fact, for each finite cyclic group G, we construct a toric log del Pezzo surface of Picard number one whose Brauer group is G by inverting a cascade.

Theorem 4.9. *For each positive integer n, there exists a toric log del Pezzo surface S of Picard number one with $Br(S) \cong \mathbb{Z}/n\mathbb{Z}$.*

Proof. First, we observe that $Br(\mathbb{P}^2)$ is a trivial group and $Br(S_2(2,2)) \cong \mathbb{Z}/2$. For each integer $n \geq 3$, we shall explicitly construct a toric log del Pezzo surface S of Picard number one with $Br(S) \cong \mathbb{Z}/n\mathbb{Z}$ by inverting the cascade from $S_2(2,4)$. Let $S_0 = S_2(2,4)$ and $f\colon S_0' \to S_0$ be its minimal resolution. Choose a chain of two (-2)-curves C_1 and C_2. Let E_i be a (-1)-curve intersecting C_i for each $i = 1, 2$. Blow up the intersection point of C_1 and E_1, and then blow up the intersection point of C_2 and E_2. Let S_1' be the resulting surface and S_1 be its anticanonical model. Note that there exists a (-1)-curve E_i' intersecting the proper transform C_i' of C_i for $i = 1, 2$. Blow up the intersection point of C_1' and E_1', and then blow up the intersection point of C_2' and E_2'. Let S_2' be resulting surface and S_2 be its anticanonical model. One can continue this inverting process. Note that S_n is a toric log del Pezzo surface of Picard number one with 3 singular points of type $2A_{n+2} + [n+2, n+2]$ for every $n \geq 0$. Now it is easy to see that $Br(S_n) \cong \mathbb{Z}/(n+3)\mathbb{Z}$ by Theorem 4.8. □

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