Bull. Korean Math. Soc. **61** (2024), No. 4, pp. 1121–1136
https://doi.org/10.4134/BKMS.b230654
pISSN: 1015-8634 / eISSN: 2234-3016

# THE TILTED CARATHÉODORY FUNCTION CLASS AND ITS PRACTICAL APPLICATIONS

Nak Eun Cho*, Inhwa Kim, and Young Jae Sim

Abstract. In this paper, by using a technique of the first-order differential subordination, we find several sufficient conditions for the tilted Carathéodory function of order $\beta$ and angle $\alpha$ ($\alpha \in (-\pi/2, \pi/2)$ and $\beta \in [0, \cos\alpha)$), which maps the unit disk $\mathbb{D}$ into the region $\{w \in \mathbb{C} : \mathrm{Re}\{e^{i\alpha}w\} > \beta\}$. Using these conditions, we also derive conditions for an analytic function that maps $\mathbb{D}$ into a sector defined by $\{w \in \mathbb{C} : |\arg(w - \gamma)| < (\pi/2)\delta\}$, where $\gamma \in [0, 1)$ and $\delta \in (0, 1]$. The results obtained here will be applied to find some conditions for spirallike functions and strongly starlike functions in $\mathbb{D}$.

## 1. Introduction

Let $\mathcal{H}_1$ be the class of functions $p$ analytic in $\mathbb{D}$ and satisfy $p(0) = 1$. Let us define two subfamilies $\mathcal{P}_\beta(\alpha)$ and $\mathcal{Q}_\gamma(\delta)$ of $\mathcal{H}_1$ by

$$\mathcal{P}_\beta(\alpha) = \{p \in \mathcal{H}_1 : \mathrm{Re}\{e^{i\alpha}p(z)\} > \beta \text{ for all } z \in \mathbb{D}\}$$

and

$$\mathcal{Q}_\gamma(\delta) = \{p \in \mathcal{H}_1 : |\arg(p(z) - \gamma)| < \frac{\pi}{2}\delta \text{ for all } z \in \mathbb{D}\},$$

where $-\pi/2 < \alpha < \pi/2$, $0 \le \beta < \cos\alpha$, $0 \le \gamma < 1$ and $0 < \delta \le 1$. Functions in $\mathcal{P}_\beta(0) \equiv \mathcal{Q}_\beta(1) := \mathcal{P}(\beta)$ are called Carathéodory functions of order $\beta$, and functions in $\mathcal{P}_0(0) \equiv \mathcal{Q}_0(1) := \mathcal{P}$ are called functions with positive real part or Carathéodory functions (refer to [1, Chapter 7] and [11, Section 3.1]), and they have an important role of studying Geometric Function Theory. For example, see [6–9]. Also, functions in $\mathcal{P}_0(\alpha)$ were named by tilted Carathéodory functions by angle $\alpha$ [12]. For that reason, we call $\mathcal{P}_\beta(\alpha)$ by the class of tilted Carathéodory functions of order $\beta$ and angle $\alpha$.

Received November 7, 2023; Accepted January 23, 2024.
2020 *Mathematics Subject Classification.* 30C45, 30C80.
*Key words and phrases.* Differential subordination, spirallike functions, strongly starlike functions.
* This work was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2019R1I1A3A01050861).

Let $\mathcal{A}$ denote the class of functions normalized by the condition $f(0) = f'(0) - 1 = 0$ which are analytic in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Also, let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{D}$. For $-\pi/2 < \alpha < \pi/2$ and $0 \leq \beta < \cos\alpha$, a function $f \in \mathcal{A}$ is called an $\alpha$-spirallike function of order $\beta$ [1, Vol. II, p. 89] (see also [4,10]) if and only if $f$ satisfies

$$\mathrm{Re}\left\{ e^{i\alpha} \frac{zf'(z)}{f(z)} \right\} > \beta, \quad z \in \mathbb{D}. \tag{1.1}$$

And, for $0 \leq \gamma < 1$ and $0 < \delta \leq 1$, $f \in \mathcal{A}$ is called a strongly starlike function of order $\delta$ and type $\gamma$ [2] if and only if $f$ satisfies

$$\left| \arg\left( \frac{zf'(z)}{f(z)} - \gamma \right) \right| < \frac{\pi}{2}\delta, \quad z \in \mathbb{D}. \tag{1.2}$$

We denote by $\mathcal{S}^*_\beta(\alpha)$ and $\mathcal{SS}^*_\gamma(\delta)$ the classes of functions satisfying the condition (1.1) and (1.2), respectively. In particular, functions in $\mathcal{S}^*_0(\alpha)$ and $\mathcal{S}^*_\beta(0) \equiv \mathcal{SS}^*_\beta(1)$ are called $\alpha$-spirallike and starlike of order $\beta$, respectively. Also, by strongly starlike functions of order $\delta$ we call functions in $\mathcal{SS}^*_0(\delta)$. Especially, we have $\mathcal{S}^*_0(0) \equiv \mathcal{SS}^*_0(1) \equiv \mathcal{S}^*$, where $\mathcal{S}^*$ is the well-known class of starlike univalent functions. We note that all functions in $\mathcal{S}^*_\beta(\alpha)$ or $\mathcal{SS}^*_\gamma(\delta)$ are univalent. Moreover, it holds that

$$\mathcal{S}^*_\beta(\alpha) \subset \mathcal{S}^*_0(\alpha) \subset \mathcal{S},$$

$$\mathcal{SS}^*_\gamma(1) \equiv \mathcal{S}^*_\gamma(0) \subset \mathcal{S}^* \subset \mathcal{S}$$

and

$$\mathcal{SS}^*_\gamma(\delta) \subset \mathcal{SS}^*_0(\delta) \subset \mathcal{S}^* \subset \mathcal{S}.$$

We note that, by setting $J_f(z) := zf'(z)/f(z)$, we have the following equivalence

$$f \in \mathcal{S}^*_\beta(\alpha) \Longleftrightarrow J_f \in \mathcal{P}_\beta(\alpha)$$

and

$$f \in \mathcal{SS}^*_\gamma(\delta) \Longleftrightarrow J_f \in \mathcal{Q}_\gamma(\delta).$$

We also note that

$$\mathcal{P}_\beta(\alpha) \cap \mathcal{P}_\beta(-\alpha) \subset \mathcal{Q}_\beta\left(1 - \frac{2}{\pi}\alpha\right). \tag{1.3}$$

So it holds that

$$\mathcal{S}^*_\beta(\alpha) \cap \mathcal{S}^*_\beta(-\alpha) \subset \mathcal{SS}^*_\beta\left(1 - \frac{2}{\pi}\alpha\right).$$

In Section 2, we will find some sufficient conditions for $p \in \mathcal{H}_1$ to satisfy $p \in \mathcal{P}_\beta(\alpha)$ or $p \in \mathcal{Q}_\gamma(\delta)$. We consider a region of functional $(1-\kappa)p(z)+\kappa p^2(z)+\kappa\lambda zp'(z)$ for $p$ to be in the class $\mathcal{P}_\beta(\alpha)$. Also, for $p \in \mathcal{H}_1$ satisfying $\eta zp'(z) + P(z)p(z) = 1$, we will obtain some conditions for $P(z)$ to $p \in \mathcal{P}_\beta(\alpha)$. Then, as direct consequences of these results, new criteria for $\alpha$-spirallike functions of order $\beta$ or strongly starlike functions of order $\delta$ and type $\gamma$ will be listed in Section 3.

For analytic functions $f$ and $g$, we say that $f$ is subordinate to $g$, denoted by $f \prec g$, if there is an analytic function $\omega : \mathbb{D} \to \mathbb{D}$ with $|\omega(z)| \leq |z|$ such that $f(z) = g(\omega(z))$. Further, if $g$ is univalent, then the definition of subordination $f \prec g$ simplifies to the conditions $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$ (see [5, p. 36]).

Let $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ be the closure and boundary of $\mathbb{D}$, respectively. We denote by $\mathcal{R}$ the class of functions $q$ that are analytic and injective on $\overline{\mathbb{D}} \setminus E(q)$, where

$$E(q) = \left\{ \zeta : \zeta \in \partial\mathbb{D} \quad \text{and} \quad \lim_{z \to \zeta} q(z) = \infty \right\},$$

and are such that

$$q'(\zeta) \neq 0 \quad (\zeta \in \partial\mathbb{D} \setminus E(q)).$$

Furthermore, let the subclass of $\mathcal{R}$ for which $q(0) = a$ be denote by $\mathcal{R}(a)$. We recall that the following lemma which will be used for our results.

**Lemma 1.1** ([3, p. 24]). *Let $q \in \mathcal{R}(a)$ and let*

$$p(z) = a + a_n z^n + \cdots \quad (n \geq 1)$$

*be an analytic function in $\mathbb{D}$ with $p(0) = a$. If $p$ is not subordinate to $q$, then there exist points $z_0 \in \mathbb{D}$ and $\zeta_0 \in \partial\mathbb{D} \setminus E(q)$ for which*

(i) $p(z_0) = q(\zeta_0)$;
(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$ $(m \geq n \geq 1)$.

## 2. Main results

**Theorem 2.1.** *Let $\kappa$, $\alpha$, $\beta$ and $\lambda$ be real numbers such that $\kappa > 0$, $-\pi/2 < \alpha < \pi/2$, $0 \leq \beta < \cos\alpha$, $\lambda > 0$ and*

$$\lambda > \frac{-2(\cos\alpha - \beta)\cos 2\alpha}{\cos\alpha}. \tag{2.1}$$

*If an analytic function $p$ with $p(0) = 1$ satisfies*

$$\operatorname{Re}\{(1-\kappa)p(z) + \kappa p^2(z) + \kappa\lambda z p'(z)\} > \Lambda(\kappa, \alpha, \beta, \lambda), \quad z \in \mathbb{D}, \tag{2.2}$$

*where*

$$\begin{aligned} &\Lambda(\kappa, \alpha, \beta, \lambda) \\ &= \beta(1-\kappa)\cos\alpha + \kappa\beta^2\cos^2\alpha + \frac{\kappa\lambda\cos\alpha(2\beta\cos\alpha - 1 - \beta^2)}{2(\cos\alpha - \beta)} \\ &\quad + \frac{\sin^2\alpha[(\cos\alpha - \beta)(1-\kappa) + \kappa\cos\alpha(4\beta(\cos\alpha - \beta) + \lambda)]^2}{2\kappa(\cos\alpha - \beta)[2(\cos\alpha - \beta)\cos 2\alpha + \lambda\cos\alpha]}, \end{aligned} \tag{2.3}$$

*and $\Lambda(\kappa, \alpha, \beta, \lambda) < 1$, then $p \in \mathcal{P}_\beta(\alpha)$. Furthermore, if $0 \leq \alpha < \pi/2$, then $p \in \mathcal{Q}_\beta(1 - (2/\pi)\alpha)$.*

*Proof.* Let us define two functions $q$ and $h : \mathbb{D} \to \mathbb{C}$ by

$$q(z) = \mathrm{e}^{\mathrm{i}\alpha} p(z) \tag{2.4}$$

and

$$h(z) = \frac{\mathrm{e}^{\mathrm{i}\alpha} + (\mathrm{e}^{-\mathrm{i}\alpha} - 2\beta)z}{1-z}. \tag{2.5}$$

Then, the functions $q$ and $h$ are analytic in $\mathbb{D}$ with

$$q(0) = h(0) = \mathrm{e}^{\mathrm{i}\alpha} \in \mathbb{C} \quad \text{and} \quad h(\mathbb{D}) = \{w \in \mathbb{C} : \mathrm{Re}\,\{w\} > \beta\}.$$

Now, suppose that $q$ is not subordinate to $h$. Then by Lemma 1.1, there exist points $z_0 \in \mathbb{D}$ and $\zeta_0 \in \partial\mathbb{D} \setminus \{1\}$ such that

$$q(z_0) = h(\zeta_0) = \beta + \mathrm{i}\rho \ (\rho \in \mathbb{R}). \tag{2.6}$$

Furthermore, by a logarithmic differentiation of (2.5), we have

$$\frac{h'(z)}{h(z)} = \frac{\mathrm{e}^{-\mathrm{i}\alpha} - 2\beta}{\mathrm{e}^{\mathrm{i}\alpha} + (\mathrm{e}^{-\mathrm{i}\alpha} - 2\beta)z} + \frac{1}{1-z}$$

and

$$\begin{aligned} zh'(z) &= zh(z)\left[\frac{\mathrm{e}^{-\mathrm{i}\alpha} - 2\beta}{\mathrm{e}^{\mathrm{i}\alpha} + (\mathrm{e}^{-\mathrm{i}\alpha} - 2\beta)z} + \frac{1}{1-z}\right] \\ &= [\mathrm{e}^{-\mathrm{i}\alpha} - 2\beta + h(z)] \cdot \frac{z}{1-z}. \end{aligned} \tag{2.7}$$

From $h(\zeta_0) = \beta + \mathrm{i}\rho$, we have

$$\zeta_0 = \frac{\beta + \mathrm{i}\rho - \mathrm{e}^{\mathrm{i}\alpha}}{\mathrm{e}^{-\mathrm{i}\alpha} - \beta + \mathrm{i}\rho} \quad \text{and} \quad \frac{\zeta_0}{1-\zeta_0} = \frac{\beta + \mathrm{i}\rho - \mathrm{e}^{\mathrm{i}\alpha}}{2(\cos\alpha - \beta)}. \tag{2.8}$$

By taking (2.8) into account of (2.7), we get

$$\begin{aligned} \zeta_0 h'(\zeta_0) &= [\mathrm{e}^{-\mathrm{i}\alpha} - 2\beta + h(\zeta_0)] \cdot \frac{\zeta_0}{1-\zeta_0} \\ &= \frac{(\beta + \mathrm{i}\rho - \mathrm{e}^{\mathrm{i}\alpha})(\mathrm{e}^{-\mathrm{i}\alpha} - \beta + \mathrm{i}\rho)}{2(\cos\alpha - \beta)} \\ &= \frac{-\rho^2 + 2\rho\sin\alpha + 2\beta\cos\alpha - 1 - \beta^2}{2(\cos\alpha - \beta)} =: \sigma. \end{aligned} \tag{2.9}$$

Thus, by Lemma 1.1, we get

$$z_0 q'(z_0) = m\zeta_0 h'(\zeta_0) = m\sigma \quad (m \geq 1), \tag{2.10}$$

where $\sigma$ is given in (2.9).

Using (2.4), (2.6) and (2.10), we obtain

$$\begin{aligned} &(1-\kappa)p(z_0) + \kappa p^2(z_0) + \kappa\lambda z_0 p'(z_0) \\ &= (1-\kappa)(\beta\cos\alpha + \rho\sin\alpha) + \kappa((\beta^2 - \rho^2)\cos 2\alpha + 2\beta\rho\sin 2\alpha) \\ &\quad + \kappa\lambda m\sigma\cos\alpha + \mathrm{i}[(1-\kappa)(\rho\cos\alpha - \beta\sin\alpha) \\ &\quad + \kappa(2\beta\rho\cos 2\alpha - (\beta^2 - \rho^2)\sin 2\alpha) - \kappa\lambda m\sigma\sin\alpha]. \end{aligned} \tag{2.11}$$

By taking real parts in the above and using the inequality $\kappa\lambda m\sigma\cos\alpha \leq \kappa\lambda\sigma\cos\alpha$, we obtain

$$\mathrm{Re}\{(1-\kappa)p(z_0)+\kappa p^2(z_0)+\kappa\lambda z_0 p'(z_0)\} \leq \frac{1}{2(\cos\alpha-\beta)}g(\rho), \tag{2.12}$$

where $g(\rho)=k_2\rho^2+k_1\rho+k_0$ with

$$\begin{aligned} k_2 &= -\kappa[2(\cos\alpha-\beta)\cos 2\alpha+\lambda\cos\alpha], \\ k_1 &= 2(\cos\alpha-\beta)(1-\kappa)\sin\alpha+4(\cos\alpha-\beta)\kappa\beta\sin 2\alpha+2\kappa\lambda\sin\alpha\cos\alpha, \\ k_0 &= 2\beta(\cos\alpha-\beta)(1-\kappa)\cos\alpha+2\kappa\beta^2(\cos\alpha-\beta)\cos 2\alpha \\ &\quad +\kappa\lambda\cos\alpha(2\beta\cos\alpha-1-\beta^2). \end{aligned}$$

Since $\kappa>0$, from the condition (2.1), we have $k_2<0$. So, the function $g$ is a quadratic concave function in $\mathbb{R}$, and $g$ has the unique local maximum at $\rho^*=-k_1/(2k_2)$. Thus we have

$$g(\rho)\leq g(\rho^*)=-\frac{k_1^2}{4k_2}+k_0, \quad \rho\in\mathbb{R}. \tag{2.13}$$

Hence, by (2.12) and (2.13), we obtain

$$\mathrm{Re}\{(1-\kappa)p(z_0)+\kappa p^2(z_0)+\kappa\lambda z_0 p'(z_0)\} \leq \frac{1}{2(\cos\alpha-\beta)}g(\rho^*)=\Lambda(\kappa,\alpha,\beta,\lambda).$$

This is a contradiction to (2.2). Therefore we obtain $q\prec h$ in $\mathbb{D}$ and it follows that the inequality $\mathrm{Re}\left\{e^{i\alpha}p(z)\right\}>\beta$ holds for all $z\in\mathbb{D}$ and $p\in\mathcal{P}_\beta(\alpha)$.

Furthermore, for $0\leq\alpha<\pi/2$, it is clear that $\Lambda(\kappa,\alpha,\beta,\lambda)=\Lambda(\kappa,-\alpha,\beta,\lambda)$ holds. So, we have $p\in\mathcal{P}_\beta(-\alpha)$, and that $p\in\mathcal{Q}_\beta(1-(2/\pi)\alpha)$ follows from (1.3). □

By taking $\kappa=1$, $\lambda=1$ and $\beta=0$ in Theorem 2.1, we have the following result.

**Corollary 2.2.** *Let $\alpha\in(-\pi/2,\pi/2)$ with $13\sin^2\alpha<9$. If an analytic function $p$ with $p(0)=1$ satisfies*

$$\mathrm{Re}\left\{p^2(z)+zp'(z)\right\}>-\frac{1}{2}+\frac{\sin^2\alpha}{2(3-4\sin^2\alpha)}, \quad z\in\mathbb{D},$$

*then $\mathrm{Re}\{e^{i\alpha}p(z)\}>0$ for all $z\in\mathbb{D}$, and*

$$|\arg\{p(z)\}|<\frac{\pi}{2}-\alpha, \quad z\in\mathbb{D}.$$

**Theorem 2.3.** *Let $\kappa$, $\alpha$, $\beta$ and $\lambda$ be real numbers such that $\kappa>0$, $0<\alpha<\pi/2$, $0\leq\beta<\cos\alpha$ and $\lambda>0$. If an analytic function $p$ with $p(0)=1$ satisfies*

$$\mathrm{Im}\{(1-\kappa)p(z)+\kappa p^2(z)+\kappa\lambda zp'(z)\}<\Lambda(\kappa,\alpha,\beta,\lambda), \tag{2.14}$$

*where*

$$\begin{aligned} &\Lambda(\kappa,\alpha,\beta,\lambda) \\ = &-(1-\kappa)\beta\sin\alpha-\kappa\beta^2\sin 2\alpha+\frac{\kappa\lambda\sin\alpha}{2(\cos\alpha-\beta)}(-2\beta\cos\alpha+1+\beta^2) \\ &-\frac{[(\cos\alpha-\beta)[(1-\kappa)\cos\alpha+2\kappa\beta\cos 2\alpha]-\kappa\lambda\sin^2\alpha]^2}{2\kappa(\cos\alpha-\beta)[2(\cos\alpha-\beta)\sin 2\alpha+\lambda\sin\alpha]} \end{aligned} \tag{2.15}$$

*and* $\Lambda(\kappa,\alpha,\beta,\lambda)>0$*, then* $p\in\mathcal{P}_\beta(\alpha)$*.*

*Proof.* Let us define $q$ and $h$ as given (2.4) and (2.5), respectively. And suppose that $q$ is not subordinate to $h$. Then we have $z_0\in\mathbb{D}$ and $\zeta_0\in\partial\mathbb{D}\backslash\{1\}$ satisfying (2.6) and (2.10). Also, we get (2.11).

Using the inequality $\kappa\lambda m\sigma\le\kappa\lambda\sigma$, we obtain

$$\operatorname{Im}\{(1-\kappa)p(z_0)+\kappa p^2(z_0)+\kappa\lambda z_0p'(z_0)\}\ge k_2\rho^2+k_1\rho+k_0=:g(\rho),$$

where

$$\begin{aligned} k_2&=\kappa\sin\alpha\left(2\cos\alpha+\frac{\lambda}{2(\cos\alpha-\beta)}\right),\\ k_1&=(1-\kappa)\cos\alpha+2\kappa\beta\cos 2\alpha-\frac{\kappa\lambda\sin^2\alpha}{\cos\alpha-\beta},\\ k_0&=-(1-\kappa)\beta\sin\alpha-\kappa\beta^2\sin 2\alpha+\frac{\kappa\lambda\sin\alpha}{2(\cos\alpha-\beta)}[-2\beta\cos\alpha+1+\beta^2]. \end{aligned}$$

Since $k_2>0$, $g$ has the unique local minimum at $\rho=\rho^*:=-k_1/(2k_2)$. Thus we have

$$g(\rho)\ge g(\rho^*)=k_0-\frac{k_1^2}{4k_2}=\Lambda(\kappa,\alpha,\beta,\lambda)$$

for all $\rho\in\mathbb{R}$. Hence we obtain

$$\operatorname{Im}\{(1-\kappa)p(z_0)+\kappa p^2(z_0)+\kappa\lambda z_0p'(z_0)\}\ge\Lambda(\kappa,\alpha,\beta,\lambda),$$

which is a contradiction to (2.14). Therefore we obtain $q\prec h$ in $\mathbb{D}$ and it follows that the inequality $\operatorname{Re}\left\{e^{i\alpha}p(z)\right\}>\beta$ holds for all $z\in\mathbb{D}$ and $p\in\mathcal{P}_\beta(\alpha)$. □

By taking $\kappa=1$, $\lambda=1$ and $\beta=0$ in Theorem 2.3, we have the following result.

**Corollary 2.4.** *Let* $0<\alpha<\pi/2$*. If an analytic function* $p$ *with* $p(0)=1$ *satisfies*

$$\operatorname{Im}\{p^2(z)+zp'(z)\}<\frac{5\cos\alpha\sin\alpha}{2+8\cos^2\alpha},\quad z\in\mathbb{D},$$

*then* $p\in\mathcal{P}_0(\alpha)$*.*

**Theorem 2.5.** *Let* $\kappa$*,* $\alpha$*,* $\beta$ *and* $\lambda$ *be real numbers such that* $\kappa>0$*,* $-\pi/2<\alpha<0$*,* $0\le\beta<\cos\alpha$ *and* $\lambda>0$*. If an analytic function* $p$ *with* $p(0)=1$ *satisfies*

$$\operatorname{Im}\{(1-\kappa)p(z)+\kappa p^2(z)+\kappa\lambda zp'(z)\}>\Lambda(\kappa,\alpha,\beta,\lambda),$$

*where* $\Lambda(\kappa,\alpha,\beta,\lambda)$ *is given by* (2.15) *and* $\Lambda(\kappa,\alpha,\beta,\lambda)<0$*, then* $p\in\mathcal{P}_\beta(\alpha)$.

By taking $\kappa=1$, $\lambda=1$ and $\beta=0$ in Theorem 2.5, we have the following result.

**Corollary 2.6.** *Let* $-\pi/2<\alpha<0$*. If an analytic function* $p$ *with* $p(0)=1$ *satisfies*

$$\operatorname{Im}\{p^2(z)+zp'(z)\}>\frac{5\cos\alpha\sin\alpha}{2+8\cos^2\alpha},\quad z\in\mathbb{D},$$

*then* $p\in\mathcal{P}_0(\alpha)$.

Also, combining Corollaries 2.4 and 2.6 leads to the following result.

**Corollary 2.7.** *Let* $0<\alpha<\pi/2$*. If an analytic function* $p$ *with* $p(0)=1$ *satisfies*

$$|\operatorname{Im}\{p^2(z)+zp'(z)\}|<\frac{5\cos\alpha\sin\alpha}{2+8\cos^2\alpha},\quad z\in\mathbb{D},$$

*then* $p\in\mathcal{Q}_0(1-(2/\pi)\alpha)$.

Now we consider a differential equation of $p$ defined by

$$\eta zp'(z)+P(z)p(z)=1 \tag{2.16}$$

for some $P:\mathbb{D}\to\mathbb{C}$. In what follows, we find some sufficient conditions for $p\in\mathcal{H}_1$ satisfying (2.16) to belong to $\mathcal{P}_\beta(\alpha)$ or $\mathcal{Q}_\gamma(\delta)$.

**Theorem 2.8.** *Let* $\alpha$*,* $\beta$ *and* $\eta$ *be real numbers such that* $-\pi/2<\alpha<\pi/2$*,* $0\le\beta<\cos\alpha$ *and* $\eta>0$*. Let*

$$\Delta=\Delta(\alpha,\beta,\eta):=\min_{\rho\in\mathbb{R}}\frac{\eta^2\rho^4+a_3\rho^3+a_2\rho^2+a_1\rho+a_0}{\beta^2+\rho^2}, \tag{2.17}$$

*where*

$$\begin{aligned}
a_0 &= 4(\cos\alpha-\beta)^2+\eta(2\beta\cos\alpha-1-\beta^2)[-4\cos\alpha(\cos\alpha-\beta)\\
&\qquad +\eta(2\beta\cos\alpha-1-\beta^2)],\\
a_1 &= 4\eta\sin\alpha[-2\cos\alpha(\cos\alpha-\beta)+\eta(2\beta\cos\alpha-1-\beta^2)],\\
a_2 &= 4\eta\cos\alpha(\cos\alpha-\beta)+2\eta^2(2\sin^2\alpha-2\beta\cos\alpha+1+\beta^2),\\
a_3 &= -4\eta^2\sin\alpha.
\end{aligned} \tag{2.18}$$

*Assume that* $\sqrt{\Delta}>2(\cos\alpha-\beta)$*, and let* $P:\mathbb{D}\to\mathbb{C}$ *with*

$$|P(z)|<\frac{\sqrt{\Delta}}{2(\cos\alpha-\beta)}=:\tilde{\Delta},\quad z\in\mathbb{D}.$$

*If* $p$ *is analytic in* $\mathbb{D}$*,* $p(0)=1$ *and* $p$ *satisfies* (2.16)*, then* $\operatorname{Re}\{e^{i\alpha}p(z)\}>\beta$ *for all* $z\in\mathbb{D}$*. Furthermore, if* $0\le\alpha<\pi/2$*, then* $p\in\mathcal{Q}_\beta(1-(2/\pi)\alpha)$.

*Proof.* Let us define $q$ and $h$ as given (2.4) and (2.5), respectively. And suppose that $q$ is not subordinate to $h$. Then we have $z_0 \in \mathbb{D}$ and $\zeta_0 \in \partial\mathbb{D}\backslash\{1\}$ satisfying (2.6) and (2.10).

By (2.16), we get

$$P(z_0) = \frac{\mathrm{e}^{\mathrm{i}\alpha} - \eta z_0 q'(z_0)}{q(z_0)} = \frac{\mathrm{e}^{\mathrm{i}\alpha} - \eta m\sigma}{\beta + \mathrm{i}\rho}. \tag{2.19}$$

Moreover, since $m \geq 1$ and $\sigma < 0$, by (2.9), we have

$$\begin{aligned} &|\mathrm{e}^{\mathrm{i}\alpha} - \eta m\sigma|^2 \\ &= (\cos\alpha - \eta m\sigma)^2 + \sin^2\alpha \\ &\geq (\cos\alpha - \eta\sigma)^2 + \sin^2\alpha \\ &= 1 - 2\eta\sigma\cos\alpha + \eta^2\sigma^2 \\ &= \frac{\eta^2\rho^4 + a_3\rho^3 + a_2\rho^2 + a_1\rho + a_0}{4(\cos\alpha - \beta)^2}, \end{aligned} \tag{2.20}$$

where $a_i$, $i \in \{0, 1, 2, 3\}$ are given by (2.18). Hence, combining (2.19) and (2.20) yields

$$|P(z_0)|^2 = \frac{|\mathrm{e}^{\mathrm{i}\alpha} - \eta m\sigma|^2}{\beta^2 + \rho^2} \geq \frac{\rho^4 + a_3\rho^3 + a_2\rho^2 + a_1\rho + a_0}{4(\cos\alpha - \beta)^2(\beta^2 + \rho^2)}.$$

Thus we get

$$|P(z_0)|^2 \geq \frac{\Delta}{4(\cos\alpha - \beta)^2},$$

which contradicts the assumption of Theorem 2.8. Therefore we obtain $q \prec h$ in $\mathbb{D}$ and $\operatorname{Re}\{\mathrm{e}^{\mathrm{i}\alpha}p(z)\} > \beta$ for $z \in \mathbb{D}$.

Furthermore, let $0 \leq \alpha < \pi/2$. Then it is easy to see that

$$\begin{aligned} \Delta(-\alpha, \beta, \eta) &= \min_{\rho\in\mathbb{R}} \frac{\eta^2\rho^4 - a_3\rho^3 + a_2\rho^2 - a_1\rho + a_0}{\beta^2 + \rho^2} \\ &= \min_{\tilde\rho\in\mathbb{R}} \frac{\eta^2\tilde\rho^4 + a_3\tilde\rho^3 + a_2\tilde\rho^2 + a_1\tilde\rho + a_0}{\beta^2 + \tilde\rho^2} \\ &= \Delta(\alpha, \beta, \eta), \end{aligned}$$

where $\tilde\rho = -\rho \in \mathbb{R}$. So, it follows that $p \in \mathcal{P}_\beta(-\alpha)$ and $p \in \mathcal{Q}_\beta(1 - (2/\pi)\alpha)$ by (1.3). □

By putting $\alpha = \beta = 0$ in Theorem 2.8, we have the following result.

**Corollary 2.9.** *Let $\eta \in \mathbb{R}$ with $\eta > -1 + \sqrt{2}$. Let $P : \mathbb{D} \to \mathbb{C}$ with $|P(z)| < \sqrt{\eta^2+2\eta}$. If $p$ is analytic in $\mathbb{D}$, $p(0) = 1$ and $p$ satisfies* (2.16)*, then* $\operatorname{Re}\{p(z)\} > 0$ *for all $z \in \mathbb{D}$.*

We give tables which give the approximate values of $\tilde\Delta$ in Theorem 2.8 for the following cases:

(a) $\alpha = 0$, $\eta = 1$ and $\beta = j/10$ $(j = 1, 2, \ldots, 9)$,

(b) $\alpha = 0$, $\beta = 1/2$ and $\eta = 1, 2, \dots, 10$,
(c) $\beta = 1/2$, $\eta = 1$ and $\alpha = j/10$ ($j = 0, 1, 2, \dots, 10$).

TABLE 1. The approximate values of $\tilde{\Delta}$ in the case (a)

| $\beta$ | $\tilde{\Delta}$ | $\beta$ | $\tilde{\Delta}$ |
|---|---|---|---|
| 0 | 1.73205 | 0.5 | 2 |
| 0.1 | 1.79161 | 0.6 | 1.93649 |
| 0.2 | 1.85405 | 0.7 | 1.64286 |
| 0.3 | 1.91663 | 0.8 | 1.375 |
| 0.4 | 1.97203 | 0.9 | 1.16667 |

TABLE 2. The approximate values of $\tilde{\Delta}$ in the case (b)

| $\eta$ | $\tilde{\Delta}$ | $\eta$ | $\tilde{\Delta}$ |
|---|---|---|---|
| 1 | 2 | 6 | 5 |
| 2 | 2.82843 | 7 | 5.5 |
| 3 | 3.4641 | 8 | 6 |
| 4 | 4 | 9 | 6.5 |
| 5 | 4.5 | 10 | 7 |

TABLE 3. The approximate values of $\tilde{\Delta}$ in the case (c)

| $\alpha$ | $\tilde{\Delta}$ | $\alpha$ | $\tilde{\Delta}$ |
|---|---|---|---|
| 0 | 2 | 0.6 | 1.24888 |
| 0.1 | 1.83753 | 0.7 | 1.16618 |
| 0.2 | 1.69343 | 0.8 | 1.09589 |
| 0.3 | 1.56427 | 0.9 | 1.04047 |
| 0.4 | 1.44783 | 1.0 | 1.00543 |
| 0.5 | 1.34286 | | |

The following result is a sufficient condition for $p \in \mathcal{P}_0(\alpha)$.

**Theorem 2.10.** *Let $\alpha$ and $\eta$ be real numbers such that $-\pi/2 < \alpha < \pi/2$, $\eta > 0$ and $\cos^2\alpha(\eta\cos^2\alpha - 1) + \eta^2(1 - \sin\alpha)^2 > 0$. Let $P : \mathbb{D} \to \mathbb{C}$ with*

$$|P(z)| \leq \frac{\sqrt{\eta\cos^4\alpha + \eta^2(1 - \sin\alpha)^2}}{\cos\alpha}, \quad z \in \mathbb{D}. \tag{2.21}$$

*If $p$ is analytic in $\mathbb{D}$, $p(0) = 1$ and $p$ satisfies the differential equation* (2.16), *then $p \in \mathcal{P}_0(\alpha)$. Furthermore, if $0 \leq \alpha < \pi/2$, then $p \in \mathcal{Q}_0(1 - (2/\pi)\alpha)$.*

*Proof.* Let $q(z) = \mathrm{e}^{\mathrm{i}\alpha} p(z)$ and

$$h_1(z) = \frac{\mathrm{e}^{\mathrm{i}\alpha} + \mathrm{e}^{-\mathrm{i}\alpha} z}{1 - z}.$$

Suppose that $q$ is not subordinate to $h_1$. By Lemma 1.1, there exist points $z_0 \in \mathbb{D}$ and $\zeta_0 \in \partial\mathbb{D} \setminus \{1\}$ such that

$$\begin{aligned} q(z_0) = h_1(\zeta_0) = \mathrm{i}\rho \ (\rho \in \mathbb{R} \setminus \{0\}) \quad \text{and} \\ z_0 q'(z_0) = m\zeta_0 h_1'(\zeta_0) = m\sigma_1 \ (m \geq 1), \end{aligned} \tag{2.22}$$

where

$$\sigma_1 = \frac{-\rho^2 + 2\rho \sin\alpha - 1}{2\cos\alpha}.$$

Therefore, from (2.16) and (2.22), we have

$$P(z_0) = \frac{\mathrm{e}^{\mathrm{i}\alpha} - \eta m \sigma_1}{\mathrm{i}\rho}.$$

Moreover, since $m \geq 1$ and $\sigma_1 < 0$, we have

$$|\mathrm{e}^{\mathrm{i}\alpha} - \eta m \sigma_1|^2 \geq 1 - 2\eta\sigma_1 \cos\alpha + \eta^2 \sigma_1^2 = g(\rho),$$

where

$$g(x) = 1 + \eta(x^2 - 2x\sin\alpha + 1) + \frac{\eta^2}{4\cos^2\alpha}(x^2 - 2x\sin\alpha + 1)^2.$$

For $x > 0$, we have

$$\frac{1}{x^2} > 0, \quad \frac{x^2 - 2x\sin\alpha + 1}{x^2} \geq 1 - \sin^2\alpha = \cos^2\alpha$$

and

$$\frac{x^2 - 2x\sin\alpha + 1}{x} \geq 2(1 - \sin\alpha).$$

Using the above inequalities, we obtain

$$\frac{g(x)}{x^2} > \frac{\eta\cos^4\alpha + \eta^2(1 - \sin\alpha)^2}{\cos^2\alpha}, \quad x > 0. \tag{2.23}$$

By a similar method with the above, we also obtain

$$\frac{g(x)}{x^2} > \frac{\eta\cos^4\alpha + \eta^2(1 + \sin\alpha)^2}{\cos^2\alpha} \geq \frac{\eta\cos^4\alpha + \eta^2(1 - \sin\alpha)^2}{\cos^2\alpha}, \quad x < 0. \tag{2.24}$$

Therefore, by (2.23) and (2.24), we get

$$|P(z_0)|^2 \geq \frac{g(\rho)}{\rho^2} > \frac{\eta\cos^4\alpha + \eta^2(1 - \sin\alpha)^2}{\cos^2\alpha},$$

which contradicts (2.21). Therefore we obtain $q \prec h_1$ in $\mathbb{D}$ and $\mathrm{Re}\left\{\mathrm{e}^{\mathrm{i}\alpha} p(z)\right\} > 0$ for $z \in \mathbb{D}$ as we asserted.

Also, for $0 \leq \alpha < \pi/2$, we have $\mathrm{Re}\left\{\mathrm{e}^{-\mathrm{i}\alpha} p(z)\right\} > 0$. Therefore, we get $p \in \mathcal{Q}_0(1 - (2/\pi)\alpha)$. □

In particular, the case $\alpha = 0$ in Theorem 2.10 induces the following result.

**Corollary 2.11.** *Let $\eta \in \mathbb{R}$ with $\eta > (-1+\sqrt{5})/2$, and let $P : \mathbb{D} \to \mathbb{C}$ with $|P(z)| \leq \sqrt{\eta(1+\eta)}$ for $z \in \mathbb{D}$. If $p$ is analytic in $\mathbb{D}$, $p(0) = 1$ and $p$ satisfies the differential equation* (2.16), *then* $\mathrm{Re}\{p(z)\} > 0$ *for all* $z \in \mathbb{D}$.

By taking $\eta = 1$ and $P(z) = 1 + (\sqrt{2}-1)z^n$ in Corollary 2.11, we have the following result.

**Corollary 2.12.** *Let $n \in \mathbb{N}$. If $p$ is analytic in $\mathbb{D}$, $p(0) = 1$ and $p$ satisfies the differential equation*

$$p(z) + [1 + (\sqrt{2}-1)z^n]zp'(z) = 1, \quad z \in \mathbb{D},$$

*then* $\mathrm{Re}\{p(z)\} > 0$ *for* $z \in \mathbb{D}$.

**Theorem 2.13.** *For given $\alpha$, $\beta$ and $\eta$ be real numbers such that $-\pi/2 < \alpha < \pi/2$, $0 < \beta < \cos\alpha$ and $\eta > 0$, let*

$$\begin{aligned} b_0 &= 2\beta\cos\alpha(\cos\alpha - \beta) + \beta\eta(-2\beta\cos\alpha + 1 + \beta^2),\\ b_1 &= \sin\alpha(\cos\alpha - \beta - \beta\eta),\\ b_2 &= \beta\eta. \end{aligned}$$

*Assume that $\Xi_1 > 2(\cos\alpha - \beta)$, where*

$$\Xi_1 := \Xi_1(\alpha,\beta,\eta) = \begin{cases} \min\{b_0/\beta^2, b_2\}, & \text{when } \alpha = 0 \text{ or } \cos\alpha = \beta(1+\eta),\\ \min\{g(\rho_1), g(\rho_2), b_2\}, & \text{otherwise} \end{cases} \tag{2.25}$$

*with*

$$g(x) = \frac{b_2x^2 + 2b_1x + b_0}{x^2 + \beta^2} \tag{2.26}$$

*and*

$$\rho_i = \frac{b_2\beta^2 - b_0 + (-1)^i\sqrt{(b_2\beta^2 - b_0)^2 + 4b_1^2\beta^2}}{2b_1}, \quad i \in \{1,2\}.$$

*Let $P : \mathbb{D} \to \mathbb{C}$ with*

$$\mathrm{Re}\{P(z)\} < \frac{\Xi_1}{2(\cos\alpha - \beta)}, \quad z \in \mathbb{D}. \tag{2.27}$$

*If $p$ is analytic in $\mathbb{D}$, $p(0) = 1$ and $p$ satisfies the differential equation* (2.16), *then $p \in \mathcal{P}_\beta(\alpha)$. Furthermore, if $0 \leq \alpha < \pi/2$, then $p \in \mathcal{Q}_\beta(1 - (2/\pi)\alpha)$.*

*Proof.* Let us define $q$ and $h$ as given (2.4) and (2.5), respectively. And suppose that $q$ is not subordinate to $h$. Then we have $z_0 \in \mathbb{D}$ and $\zeta_0 \in \partial\mathbb{D}\backslash\{1\}$ satisfying (2.6) and (2.10).

By (2.16), we get

$$\mathrm{Re}\{P(z_0)\} = \mathrm{Re}\left\{\frac{e^{i\alpha} - \eta m\sigma}{\beta + i\rho}\right\} = \frac{\beta(\cos\alpha - m\eta\sigma) + \rho\sin\alpha}{\beta^2 + \rho^2}, \tag{2.28}$$

where $\sigma$ is given by (2.9). Since $m \geq 1$, $\sigma < 0$ and $\eta > 0$, (2.28) implies

$$\mathrm{Re}\{P(z_0)\} \geq \frac{\beta(\cos\alpha - \eta\sigma) + \rho\sin\alpha}{\beta^2 + \rho^2} = \frac{g(\rho)}{2(\cos\alpha - \beta)},$$

where $g$ is given by (2.26). Therefore, it is sufficient to show that $g(\rho) \geq \Xi_1$ for all $\rho \in \mathbb{R}$, which leads a contraction to (2.27).

Assume that $\alpha = 0$ or $\cos\alpha = \beta(1+\eta)$. Then we have $b_1 = 0$. For the case $b_0/\beta^2 = b_2$, we have

$$g(\rho) = b_2 = \min\{b_0/\beta^2, b_2\} = \Xi_1, \quad \rho \in \mathbb{R}.$$

For the case $b_0/\beta^2 \neq b_2$, we note that $g'(\rho) = 0$ occurs only at $\rho = 0$. Also, we have

$$\lim_{\rho\to\infty} g(\rho) = \lim_{\rho\to-\infty} g(\rho) = b_2. \tag{2.29}$$

Therefore, we get

$$g(\rho) \geq \min\{b_0/\beta^2, b_2\} = \Xi_1, \quad \rho \in \mathbb{R},$$

which contradicts (2.27).

Now we assume that $\alpha \neq 0$ and $\cos\alpha \neq \beta(1+\eta)$. Then we have $b_1 \neq 0$ and $g'(\rho) = 0$ occurs when $\rho = \rho_1$ or $\rho_2$. Since the equalities in (2.29) hold again, we get

$$g(\rho) \geq \min\{g(\rho_1), g(\rho_2), b_2\} = \Xi_1, \quad \rho \in \mathbb{R},$$

which contradicts (2.27). Thus we have $\mathrm{Re}\{e^{i\alpha}p(z)\} > \beta$ for all $z \in \mathbb{D}$.

For $0 \leq \alpha < 1$, it holds that

$$\begin{aligned}\Xi_1(-\alpha, \beta, \eta) &= \min_{x\in\mathbb{R}} \frac{b_2x^2 - 2b_1x + b_0}{x^2 + \beta^2}\\ &= \min_{\tilde{x}\in\mathbb{R}} \frac{b_2\tilde{x}^2 + 2b_1\tilde{x} + b_0}{\tilde{x}^2 + \beta^2}\\ &= \Xi_1(\alpha, \beta, \eta),\end{aligned}$$

where $\tilde{x} = -x$. Thus we have $\mathrm{Re}\{e^{-i\alpha}p(z)\} > \beta$ for all $z \in \mathbb{D}$, which follows that $p \in \mathcal{Q}_\beta(1 - (2/\pi)\alpha)$. It completes the proof of Theorem 2.13. □

Next, we give a similar result with Theorem 2.13 for the case $\eta < 0$. We omit the proof of following result because it is so analogous to the proof of Theorem 2.13.

**Theorem 2.14.** *Let $\eta \in \mathbb{R}$ with $\eta < 0$. And let $\alpha$, $\beta$, $b_0$, $b_1$, $b_2$, $\rho_1$ and $\rho_2$ be the quantities defined as in Theorem 2.13. Assume that $\Xi_2 < 2(\cos\alpha - \beta)$, where*

$$\Xi_2 := \Xi_2(\alpha, \beta, \eta) = \begin{cases}\max\{b_0/\beta^2, b_2\}, & \text{when } \alpha = 0 \text{ or } \cos\alpha = \beta(1+\eta),\\ \max\{g(\rho_1), g(\rho_2), b_2\}, & \text{otherwise},\end{cases} \tag{2.30}$$

*where g is defined by* (2.26). *Let* $P : \mathbb{D} \to \mathbb{C}$ *with*

$$\operatorname{Re}\{P(z)\} > \frac{\Xi_2}{2(\cos\alpha - \beta)}, \quad z \in \mathbb{D}.$$

*If p is analytic in* $\mathbb{D}$, $p(0) = 1$ *and p satisfies the differential equation* (2.16), *then* $p \in \mathcal{P}_\beta(\alpha)$. *Furthermore, if* $0 \le \alpha < \pi/2$, *then* $p \in \mathcal{Q}_\beta(1 - (2/\pi)\alpha)$.

If we put $\alpha = 0$ in Theorems 2.13 and 2.14, then we have the following corollaries.

**Corollary 2.15.** *Let* $P : \mathbb{D} \to \mathbb{C}$ *with* $\operatorname{Re}\{P(z)\} < \Theta_1$, *where*

$$\Theta_1 = \begin{cases} \frac{\beta\eta}{2(1-\beta)}, & \text{when} \begin{cases} 0 < \beta \le 1/2 \text{ and } \eta > 2(1-\beta)/\beta, \\ 1/2 < \beta < 1 \text{ and } 2(1-\beta)/\beta < \eta < 2(1-\beta)/(2\beta - 1), \end{cases} \\ \frac{2+\eta(1-\beta)}{2\beta}, & \text{when } 1/2 < \beta < 1 \text{ and } \eta \ge 2(1-\beta)/(2\beta-1). \end{cases}$$

*If p is analytic in* $\mathbb{D}$, $p(0) = 1$ *and p satisfies the differential equation* (2.16), *then* $\operatorname{Re}\{p(z)\} > \beta$ *for all* $z \in \mathbb{D}$.

**Corollary 2.16.** *Let* $P : \mathbb{D} \to \mathbb{C}$ *with* $\operatorname{Re}\{P(z)\} > \Theta_2$, *where*

$$\Theta_2 = \begin{cases} \frac{\beta\eta}{2(1-\beta)}, & \text{when } 1/2 < \beta < 1 \text{ and } \eta \le 2(1-\beta)/(1-2\beta), \\ \frac{2+\eta(1-\beta)}{2\beta}, & \text{when} \begin{cases} 0 < \beta \le 1/2 \text{ and } \eta < -2, \\ 1/2 < \beta < 2/3 \text{ and } 2(1-\beta)/(1-2\beta) \le \eta < -2. \end{cases} \end{cases}$$

*If p is analytic in* $\mathbb{D}$, $p(0) = 1$ *and p satisfies the differential equation* (2.16), *then* $\operatorname{Re}\{p(z)\} > \beta$ *for all* $z \in \mathbb{D}$.

## 3. Sufficient conditions for spirallike and strongly starlike functions

**Corollary 3.1.** *Let* $\kappa$, $\alpha$, $\beta$ *and* $\lambda$ *be real numbers such that* $\kappa > 0$, $-\pi/2 < \alpha < \pi/2$, $0 \le \beta < \cos\alpha$, $\lambda > 0$ *and*

$$\lambda > \frac{-2(\cos\alpha - \beta)\cos 2\alpha}{\cos\alpha}.$$

*If* $f \in \mathcal{A}$ *satisfies*

$$\operatorname{Re}\left\{\left(\frac{zf'(z)}{f(z)}\right)\left[1 - \kappa + \kappa(1-\lambda)\frac{zf'(z)}{f(z)} + \kappa\lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]\right\} > \Lambda(\kappa, \alpha, \beta, \lambda), \quad z \in \mathbb{D},$$

*and* $\Lambda(\kappa, \alpha, \beta, \lambda) < 1$, *where* $\Lambda$ *is given by* (2.3), *then* $f \in \mathcal{S}^*_\beta(\alpha)$, *or* $f \in \mathcal{SS}^*_\beta(1 - (2/\pi)\alpha)$.

In particular, by putting $\lambda = 1$ and $\alpha = 0$ or $\kappa = 1$ and $\alpha = 0$ in Corollary 3.1, we have the following corollaries.

**Corollary 3.2.** *If a function $f \in \mathcal{A}$ satisfies the condition*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)} + \kappa\frac{z^2f''(z)}{f(z)}\right\} > \beta + \kappa\left(-\frac{1}{2} - \frac{\beta}{2} + \beta^2\right), \quad z \in \mathbb{D},$$

*then $f \in \mathcal{S}^*_\beta(0)$.*

**Corollary 3.3.** *If a function $f \in \mathcal{A}$ satisfies the condition*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]\right\} > \frac{1}{2}(2\beta^2 + \beta\lambda - 1), \quad z \in \mathbb{D},$$

*then $f \in \mathcal{S}^*_\beta(0)$.*

For $f \in \mathcal{A}$, setting $p(z) = zf'(z)/f(z)$ in (2.16) gives a differential equation

$$(\eta + P(z))zf(z)f'(z) + \eta z^2 f(z)f''(z) - \eta z^2(f'(z))^2 = (f(z))^2. \tag{3.1}$$

So, by Theorems 2.8, 2.10, 2.13 and 2.14, we have the following results.

**Corollary 3.4.** *Let $\alpha$, $\beta$ and $\eta$ be real numbers such that $-\pi/2 < \alpha < \pi/2$, $0 \le \beta < \cos\alpha$ and $\eta > 0$. Let $P : \mathbb{D} \to \mathbb{C}$ with*

$$|P(z)| < \frac{\sqrt{\Delta}}{2(\cos\alpha - \beta)}, \quad z \in \mathbb{D},$$

*where $\Delta$ is given by (2.17). If $f \in \mathcal{A}$ satisfies (3.1), then $f \in \mathcal{S}^*_\beta(\alpha)$, or $f \in \mathcal{SS}^*_\beta(1 - (2/\pi)\alpha)$.*

**Corollary 3.5.** *Let $\alpha$ and $\eta$ be real numbers such that $-\pi/2 < \alpha < \pi/2$ and $\eta > 0$. Let $P : \mathbb{D} \to \mathbb{C}$ with*

$$|P(z)| \le \frac{\sqrt{\eta\cos^4\alpha + \eta^2(1 - \sin\alpha)^2}}{\cos\alpha}, \quad z \in \mathbb{D}.$$

*If $f \in \mathcal{A}$ satisfies (3.1), then $f \in \mathcal{S}^*_0(\alpha)$, or $f \in \mathcal{SS}^*_0(1 - (2/\pi)\alpha)$.*

**Corollary 3.6.** *For given $\alpha$, $\beta$ and $\eta$ be real numbers such that $-\pi/2 < \alpha < \pi/2$, $0 < \beta < \cos\alpha$ and $\eta > 0$. Assume that $\Xi_1 > 2(\cos\alpha - \beta)$, where $\Xi_1$ is given by (2.25). Let $P : \mathbb{D} \to \mathbb{C}$ with*

$$\mathrm{Re}\{P(z)\} < \frac{\Xi_1}{2(\cos\alpha - \beta)}, \quad z \in \mathbb{D}.$$

*If $f \in \mathcal{A}$ satisfies (3.1), then $f \in \mathcal{S}^*_\beta(\alpha)$, or $f \in \mathcal{SS}^*_\beta(1 - (2/\pi)\alpha)$.*

**Corollary 3.7.** *For given $\alpha$, $\beta$ and $\eta$ be real numbers such that $-\pi/2 < \alpha < \pi/2$, $0 < \beta < \cos\alpha$ and $\eta > 0$. Assume that $\Xi_2 < 2(\cos\alpha - \beta)$, where $\Xi_2$ is given by (2.30). Let $P : \mathbb{D} \to \mathbb{C}$ with*

$$\mathrm{Re}\{P(z)\} > \frac{\Xi_2}{2(\cos\alpha - \beta)}, \quad z \in \mathbb{D}.$$

*If $f \in \mathcal{A}$ satisfies (3.1), then $f \in \mathcal{S}^*_\beta(\alpha)$, or $f \in \mathcal{SS}^*_\beta(1 - (2/\pi)\alpha)$.*

We end this paper with suggesting a geometric property of an integral operator defined on $\mathcal{A}$.

**Corollary 3.8.** *Let $f \in \mathcal{S}$ and $\beta$ and $\gamma$ be real numbers such as $\beta \neq 0$ and $\beta + \gamma > 0$. If*

$$\left|\beta \frac{zf'(z)}{f(z)} + \gamma\right| < \frac{\sqrt{(\beta+\gamma)\cos^4\alpha + (1-\sin\alpha)^2}}{\cos\alpha}, \quad z \in \mathbb{D},$$

*then*

$$\left|\arg\left(\beta \frac{zF'(z)}{F(z)} + \gamma\right)\right| < \frac{\pi}{2} - \alpha, \quad z \in \mathbb{D},$$

*where $F$ is the integral operator defined by*

$$F(z) = \left[\frac{\beta+\gamma}{z^\gamma}\int_0^z f^\beta(t) t^{\gamma-1}\right]^{1/\beta}. \tag{3.2}$$

*Proof.* Let

$$P(z) = \frac{1}{\beta+\gamma}\left(\beta \frac{zf'(z)}{f(z)} + \gamma\right)$$

and

$$p(z) = \frac{\beta+\gamma}{z^\gamma f^\beta(z)}\int_0^z f^\beta(t) t^{\gamma-1} dt. \tag{3.3}$$

Then $P$ and $p$ are analytic in $\mathbb{D}$ with $P(0) = p(0) = 1$. By a simple calculation, we have

$$\frac{1}{\beta+\gamma} zp'(z) + P(z)p(z) = 1.$$

By using Theorem 2.10 with $\eta = 1/(\beta+\gamma)$, we obtain that

$$|\arg p(z)| < \frac{\pi}{2} - \alpha, \quad z \in \mathbb{D}.$$

From (3.2) and (3.3), we easily see that $F(z) = f(z)[p(z)]^{1/\beta}$. Since

$$\beta \frac{zF'(z)}{F(z)} + \gamma = \frac{\beta+\gamma}{p(z)},$$

the conclusion of Corollary 3.8 immediately follows. □

## References

[1] A. W. Goodman, *Univalent Functions*, Mariner, Tampa, 1983.

[2] I. Hotta and M. Nunokawa, *On strongly starlike and convex functions of order $\alpha$ and type $\beta$*, Mathematica **53(76)** (2011), no. 1, 51–56.

[3] S. S. Miller and P. T. Mocanu, *Differential Subordinations*, Monographs and Textbooks in Pure and Applied Mathematics, 225, Marcel Dekker, Inc., New York, 2000.

[4] P. Montel, *Leçons sur Les Fonctions Univalentes on Multivalentes*, Gauthier-Villars, Paris, 1933.

[5] C. Pommerenke, *Univalent Functions*, Studia Mathematica/Mathematische Lehrbücher, Band XXV, Vandenhoeck & Ruprecht, Göttingen, 1975.

[6] M. S. Robertson, *Variational methods for functions with positive real part*, Trans. Amer. Math. Soc. **102** (1962), 82–93. `https://doi.org/10.2307/1993881`

[7] M. S. Robertson, *Extremal problems for analytic functions with positive real part and applications*, Trans. Amer. Math. Soc. **106** (1963), 236–253. `https://doi.org/10.2307/1993766`

[8] S. Ruscheweyh and V. Singh, *On certain extremal problems for functions with positive real part*, Proc. Amer. Math. Soc. **61** (1976), no. 2, 329–334. https://doi.org/10.2307/2041336

[9] K. Sakaguchi, *A variational method for functions with positive real part*, J. Math. Soc. Japan **16** (1964), 287–297. https://doi.org/10.2969/jmsj/01630287

[10] L. Špaček, *Contribution à la theorie des fonctions univalentes*, Casopis Pest. Mat. **62** (1932), 12–19.

[11] D. K. Thomas, N. Tuneski, and A. Vasudevarao, *Univalent Functions*, De Gruyter Studies in Mathematics, 69, De Gruyter, Berlin, 2018. https://doi.org/10.1515/9783110560961

[12] L.-M. Wang, *The tilted Carathéodory class and its applications*, J. Korean Math. Soc. **49** (2012), no. 4, 671–686. https://doi.org/10.4134/JKMS.2012.49.4.671

Nak Eun Cho
Department of Applied Mathematics
Pukyong National University
Busan 48513, Korea
*Email address*: necho@pknu.ac.kr

Inhwa Kim
Anheuser-Bush School of Business
Harris-Stowe State University
St. Louis, MO 63103, USA
*Email address*: ihyahootn@gmail.com, kimi@hssu.edu

Young Jae Sim
Department of Artificial Intelligence and Mathematics
Kyungsung University
Busan 48434, Korea
*Email address*: yjsim@ks.ac.kr