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ON THE SINGULAR LOCUS OF FOLIATIONS OVER $\mathbb{P}^2$

Shi Xu

Abstract. For a foliation $\mathcal{F}$ of degree r over $\mathbb{P}^2$, we can regard it as a maximal invertible sheaf $N_{\mathcal{F}}^{\vee}$ of $\Omega_{\mathbb{P}^2}$, which is represented by a section $s \in H^0(\Omega_{\mathbb{P}^2}(r+2))$. The singular locus Sing$\mathcal{F}$ of $\mathcal{F}$ is the zero dimensional subscheme $Z(s)$ of $\mathbb{P}^2$ defined by s. Campillo and Olivares have given some characterizations of the singular locus by using some cohomology groups. In this paper, we will give some different characterizations. For example, the singular locus of a foliation over $\mathbb{P}^2$ can be characterized as the residual subscheme of r collinear points in a complete intersection of two curves of degree $r+1$.

1. Introduction

The main purpose of this paper is to try to answer the following question: *Given a zero dimensional subscheme Δ of $\mathbb{P}^2$, when is it the singular locus of a foliation over $\mathbb{P}^2$?* The following are the main theorems.

Theorem 1.1. *Suppose Δ is a zero dimensional subscheme of $\mathbb{P}^2$, and r is a non-negative integer. Then the following conditions are equivalent.*

(1) *Δ is the singular locus of a foliation $\mathcal{F}$ in $\mathbb{P}^2$ of degree r.*
(2) *Δ is the residual subscheme of r collinear points in a complete intersection of two curves F_1 and F_2 of degree $r+1$. We write it as*

$$\Delta = F_1F_2 - \{\, r \text{ collinear points}\}.$$

Equivalently,

$$\Delta = F_1F_2 - F_1F_2H,$$

where $\deg\Delta = r^2+r+1$, $\deg F_1 = \deg F_2 = r+1$, $\deg H = 1$.

Note that the implication (1) $\Longrightarrow$ (2) is in ([4], Section 1, p.98). The proof of Theorem 1.1 is given in Theorem 3.3. By Theorem 1.1, Δ is the singular locus of a foliation $\mathcal{F}$ of degree $r=0$, if and only if Δ is a point. Δ is the singular

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locus of a foliation $\mathcal{F}$ of degree $r = 1$, if and only if Δ is a non-collinear zero-dimensional subscheme of degree 3. (See [4], Theorem 4.1 or Corollary 3.5.) We will discuss only the case that $\deg \mathcal{F} = r \geq 2$.

Suppose Δ is a zero dimensional subscheme of $\mathbb{P}^2$ of degree $\deg \Delta = r^2 + r + 1 > 3$, and consider the following conditions:

(a1) $h^1(\mathcal{I}_\Delta(2r-2)) = 1$ and $h^1(\mathcal{I}_{\Delta'}(2r-2)) = 0$ for any $\Delta' \subset \Delta$ of degree $r^2 + r$.

(a2) $h^1(\mathcal{I}_{\Delta'}(2r-2)) = 0$ for any $\Delta' \subset \Delta$ of degree $r^2 + r$.

(a3) $h^0(\mathcal{I}_{\Delta'}(r)) = 0$ for any $\Delta' \subset \Delta$ of degree $r^2 + r$.

(b1) $h^0(\mathcal{I}_\Delta(r+1)) \geq 2$, and the linear system $\mathbb{P}\big(H^0(\mathcal{I}_\Delta(r+1))\big)$ has no base curve.

(b2) $h^0(\mathcal{I}_\Delta(r+1)) \geq 3$, and the linear system $\mathbb{P}\big(H^0(\mathcal{I}_\Delta(r+1))\big)$ has no base curve.

(b3) $h^0(\mathcal{I}_\Delta(r+1)) \geq 3$, and $h^0(\mathcal{I}_\Delta(r)) = 0$.

In fact, the condition (a1) means that, Δ is the zero subscheme given by a global section of a rank 2 locally free sheaf $\mathcal{E}$ with $c_1(\mathcal{E}) \equiv (2r+1)H$ and $c_2(\mathcal{E}) = \Delta$ (see Remark 3.7). Then we have the following theorem.

Theorem 1.2. *Suppose Δ is a zero dimensional subscheme of $\mathbb{P}^2$ with $\deg \Delta = r^2 + r + 1 > 3$. Then the following conditions are equivalent*:

(1) Δ *is the singular locus of a foliation $\mathcal{F}$ in $\mathbb{P}^2$ of degree r.*
(2) Δ *satisfies the conditions* (*a*1) *and* (*b*1).
(3) Δ *satisfies the conditions* (*a*2) *and* (*b*2).
(4) Δ *satisfies the conditions* (*a*3) *and* (*b*2).
(5) Δ *satisfies the conditions* (*a*1) *and* (*b*3).

Note that the equivalence of (1) and (4) is a theorem of Campillo-Olivares ([4], Theorem 4.5). The proofs of Theorem 1.2 are given in Theorems 3.6, 3.8, 3.9, 3.14, respectively.

2. Notation and preliminaries

2.1. Foliations over $\mathbb{P}^2$

Let $\mathbb{P}^2 = \operatorname{Proj} \mathbb{C}[X, Y, Z]$ be the projective plane over $\mathbb{C}$ and let $\mathcal{O}_{\mathbb{P}^2}, T_{\mathbb{P}^2}, \Omega_{\mathbb{P}^2}$ denote its structure, tangent and cotangent sheaves. A foliation $\mathcal{F}$ over $\mathbb{P}^2$ is given by a short exact sequence

$$0 \longrightarrow T_{\mathcal{F}} \longrightarrow T_{\mathbb{P}^2} \longrightarrow \mathcal{I}_\Delta \cdot N_{\mathcal{F}} \longrightarrow 0, \tag{1}$$

or

$$0 \longrightarrow N_{\mathcal{F}}^\vee \longrightarrow \Omega_{\mathbb{P}^2} \longrightarrow \mathcal{I}_\Delta \cdot K_{\mathcal{F}} \longrightarrow 0, \tag{2}$$

where $T_{\mathcal{F}}$ (resp. $N_{\mathcal{F}}^\vee$) is a maximal sub-invertible sheaf of $T_{\mathbb{P}^2}$ (resp. $\Omega_{\mathbb{P}^2}$). We call $K_{\mathcal{F}} = T_{\mathcal{F}}^\vee$ (resp. $\mathcal{N}_{\mathcal{F}}^\vee$, Δ) the *canonical sheaf* (resp. *conormal sheaf*, *singular locus*) of $\mathcal{F}$. It is clear that Δ is a local complete intersection. (See [3] for more details.)

Recall the famous Bott's formula (see [2] or [8], p.4).

Lemma 2.1 (Bott)**.**

$$h^p(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k)) = \begin{cases} \frac{1}{2}(k+2)(k+1), & \textit{if } p=0 \textit{ and } k \geq 0, \\ \frac{1}{2}(k+2)(k+1), & \textit{if } p=2 \textit{ and } k \leq -3, \\ 0, & \textit{others.} \end{cases}$$

$$h^p(\mathbb{P}^2, \Omega_{\mathbb{P}^2}(k)) = \begin{cases} k^2-1, & \textit{if } p=0 \textit{ and } k \geq 2, \\ 1, & \textit{if } p=1 \textit{ and } k=0, \\ k^2-1, & \textit{if } p=2 \textit{ and } k \leq -2, \\ 0, & \textit{others.} \end{cases}$$

Let $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(-r-2)$ be a maximal sub-invertible sheaf of $\Omega_{\mathbb{P}^2}$. By Bott's formula, we see that $r \geq 0$. Such $\mathcal{L}$ defines a foliation $\mathcal{F}$ with $\mathcal{N}_{\mathcal{F}}^{\vee} = \mathcal{L}$. We call r the *degree* of $\mathcal{F}$. Consider the natural projection

$$\pi : \mathbb{C}^3 - \{0\} \longrightarrow \mathbb{P}^2,$$

via $(X, Y, Z) \mapsto [X, Y, Z]$.

Proposition 2.2 ([4], p.99)**.** *$\mathcal{F}$ corresponds to a global section ω of $\Omega_{\mathbb{C}^3-\{0\}}$:*

$$\omega = A\mathrm{d}X + B\mathrm{d}Y + C\mathrm{d}Z,$$

where $A, B, C \in \mathbb{C}[X, Y, Z]$ are homogeneous polynomial of degree $r+1$ with no common factors, and the so-called Euler's condition

$$X \cdot A + Y \cdot B + Z \cdot C = 0 \tag{3}$$

holds.

By ([4], Remark 3.4), we see that

$$\gcd(B, C) = 1 \,\mathrm{or}\, X, \quad \gcd(A, C) = 1 \,\mathrm{or}\, Y, \quad \gcd(A, B) = 1 \,\mathrm{or}\, Z.$$

To characterize the singular locus $\Delta = \mathrm{Sing}\mathcal{F}$, we recall the definition of the *residual subscheme* (see [5]).

Definition. Let Γ be a zero dimensional scheme with coordinate ring $A(\Gamma)$. Let $\Gamma' \subset \Gamma$ be a closed subscheme and $\mathcal{I}_{\Gamma'} \subset A(\Gamma)$ its ideal. By the subscheme Γ'' of Γ residual to Γ' we shall mean the subscheme of Γ defined by the ideal $\mathcal{I}_{\Gamma''} = \mathrm{Ann}(\mathcal{I}_{\Gamma'}/\mathcal{I}_{\Gamma})$.

Lemma 2.3 ([5])**.** *Let $\Gamma, \Gamma', \Gamma''$ be as above. If Γ is a local complete intersection, then we have $\mathcal{I}_{\Gamma'} = \mathrm{Ann}(\mathcal{I}_{\Gamma''}/\mathcal{I}_{\Gamma})$ and $\deg\Gamma = \deg\Gamma' + \deg\Gamma''$. In this case, we always write $\Gamma'' = \Gamma - \Gamma'$.*

For hypersurfaces F_j $(j = 1, \ldots, n)$ of $\mathbb{P}^2$, we denote by $F_1F_2\cdots F_n$ the subscheme $F_1 \cap F_2 \cap \cdots \cap F_n$ of $\mathbb{P}^2$. We have the following proposition (see also [4], p.98).

Proposition 2.4. *Suppose Δ is the singular locus of a foliation $\mathcal{F}$ in $\mathbb{P}^2$ of degree r. Then we can write Δ as*

$$\Delta = F_1F_2 - F_1F_2H,$$

where F_1, F_2, H are curves of degree $r+1, r+1, 1$ in $\mathbb{P}^2$ and F_1, F_2 have no common components. In particular, $\deg\Delta = r^2 + r + 1$.

Proof. By choosing a suitable coordinate, we assume $\Delta \cap (X = 0) = \emptyset$. Then B, C have no common factors. Otherwise, it follows from the equation (3) and the fact $\mathcal{I}_\Delta = \langle A, B, C\rangle$ that we can write B, C as $B = -XB'$ and $C = -XC'$, where B', C' have no common factors. This induces $A = YB' + ZC'$. So $\Delta \cap (X = 0) = (A = 0) \cap (X = 0) \neq \emptyset$, which is a contradiction. So we can choose $F_1 = Z(B), F_2 = Z(C)$ and $H = Z(X)$. Then we see that

$$\Delta = F_1F_2 - F_1F_2H,$$

which is equivalent to the following claim.

Claim: For any point $p \in F_1F_2H$, $(\mathcal{I}_{F_1F_2})_p = (\mathcal{I}_{F_1F_2H})_p$, which implies $\mathcal{I}_{F_1F_2}|_H = \mathcal{I}_{F_1F_2H}$.

It suffices to consider the case $p \in (Z \neq 0) := U$, say $p = [0, a, 1]$. Then we have

$$\mathcal{I}_{F_1F_2}|_U = \langle B(x,y,1), C(x,y,1)\rangle, \quad \mathcal{I}_{F_1F_2H}|_U = \langle B(x,y,1), C(x,y,1), x\rangle.$$

It is clear that

$$xA(x,y,1) = -yB(x,y,1) - C(x,y,1) \in \mathcal{I}_{F_1F_2}|_U.$$

Since $p \in Z(B, C)$ but $p \notin \Delta$, we have $A(p) \neq 0$. So $x \in (\mathcal{I}_{F_1F_2})_p$. This implies $(\mathcal{I}_{F_1F_2})_p = (\mathcal{I}_{F_1F_2H})_p$ clearly.

By the equation $\Delta = F_1F_2 - F_1F_2H$, the degree of Δ can be computed as follows. We can write B, C as

$$\begin{aligned} B(X,Y,Z) &= XB_1(X,Y,Z) + B_2(Y,Z) \\ C(X,Y,Z) &= XC_1(X,Y,Z) + C_2(Y,Z), \end{aligned}$$

where $B_2 \neq 0$ or $C_2 \neq 0$. By the relation equation $XA + YB + ZC = 0$, we have

$$X(A + YB_1 + ZC_1) + YB_2 + ZC_2 = 0,$$

which implies $YB_2 + ZC_2 = 0$. So we have

$$B_2(Y,Z) = Z \cdot G(Y,Z), \quad C_2(Y,Z) = -Y \cdot G(Y,Z)$$

for some nonzero homogeneous polynomial $G(Y,Z)$ of degree r. Then we see

$$\mathcal{I}_{F_1F_2H} = \langle X, B_2(Y,Z), C_2(Y,Z)\rangle = \langle X, G(Y,Z)\rangle,$$

which implies $\deg(F_1F_2H) = \deg_H G = r$. Hence,

$$\deg\Delta = \deg(F_1F_2) - \deg(F_1F_2H) = (r+1)^2 - r = r^2 + r + 1. \qquad \square$$

2.2. The Cayley-Bacharach property

For the proof of the main theorem, we will use some results about the Cayley-Bacharach property. So we recall the following theorem ([9], Theorem 1).

Theorem 2.5 ([9]). *Let $\mathcal{E}$ be a locally free sheaf on a complex projective surface X of rank 2, and s be a section of $\mathcal{E}$ whose zero subscheme $\Delta = Z(s)$ is of dimension 0. Let $\Delta'' \subset \Delta'$ be two subschemes of Δ and let L be a divisor. Then there exists a complex of vector spaces*

$$0 \longrightarrow H^0(\mathcal{I}_{\Delta-\Delta''}(\det \mathcal{E} - L)) \xrightarrow{\alpha} H^0(\mathcal{I}_{\Delta-\Delta'}(\det \mathcal{E} - L)) \xrightarrow{\mu}$$
$$H^1(\mathcal{I}_{\Delta'}(K_X + L)) \xrightarrow{\beta} H^1(\mathcal{I}_{\Delta''}(K_X + L)) \longrightarrow 0,$$

exact except at $H^1(\mathcal{I}_{\Delta'}(K_X + L))$. In particular, if

$$H^1(X, \mathcal{E}^{\vee}(\det \mathcal{E} - L)) = H^1(X, \mathcal{E}(-L)) = 0,$$

then the complex is exact everywhere.

Remark 2.6. (1) Let $\mathcal{E} = \mathcal{O}_X(F_1) \oplus \mathcal{O}_X(F_2)$ where F_1, F_2 are effective divisors over X and $\dim(F_1 \cap F_2) = 0$, and let $s \in H^0(\mathcal{E})$ with $\Delta = Z(s) = F_1F_2$.

If $H^1(\mathcal{O}_X(F_i - L)) = 0$ for all $i = 1, 2$, for example, $X = \mathbb{P}^2$, then we have the following exact sequence

$$0 \longrightarrow H^0(\mathcal{I}_{\Delta-\Delta''}(F_1 + F_2 - L)) \xrightarrow{\alpha} H^0(\mathcal{I}_{\Delta-\Delta'}(F_1 + F_2 - L)) \xrightarrow{\mu}$$
$$H^1(\mathcal{I}_{\Delta'}(K_X + L)) \xrightarrow{\beta} H^1(\mathcal{I}_{\Delta''}(K_X + L)) \longrightarrow 0.$$

Moreover, if we have $H^1(K_X + L) = 0$, then

$$h^0(\mathcal{I}_{\Delta-\Delta'}(F_1 + F_2 - L)) - h^0(\mathcal{I}_{\Delta}(F_1 + F_2 - L)) = h^1(\mathcal{I}_{\Delta'}(K_X + L)).$$

(2) Let $X = \mathbb{P}^2$, $\mathcal{E} = \Omega_{\mathbb{P}^2}(r+2)$ and $L \equiv sH$ $(s \neq r+2)$. In this case, Δ is the singular locus of a foliation of degree r in $\mathbb{P}^2$. So

$$\det \mathcal{E} - L \equiv (2r + 1 - s)H, \quad K_X + L \equiv (s-3)H,$$

and

$$H^1(\mathcal{E}(-L)) = H^1(\Omega_{\mathbb{P}^2}(r + 2 - s)) = 0$$

for all $s \neq r+2$. Then for all $s \neq r+2$, we have the following exact sequence

$$0 \longrightarrow H^0(\mathcal{I}_{\Delta-\Delta''}(2r + 1 - s)) \xrightarrow{\alpha} H^0(\mathcal{I}_{\Delta-\Delta'}(2r + 1 - s)) \xrightarrow{\mu}$$
$$H^1(\mathcal{I}_{\Delta'}(s-3)) \xrightarrow{\beta} H^1(\mathcal{I}_{\Delta''}(s-3)) \longrightarrow 0.$$

Corollary 2.7 (Cayley-Bacharach). *Let F_1, F_2 be curves in $\mathbb{P}^2$ of degree d_1, d_2, and suppose that the intersection subscheme $\Delta = F_1 \cap F_2$ is zero-dimensional. Let $\Delta'' \subset \Delta'$ be subschemes of Δ and set $e = d_1 + d_2 - 3$. For all s, we have*

$$h^0(\mathbb{P}^2, \mathcal{I}_{\Delta-\Delta'}(s)) - h^0(\mathbb{P}^2, \mathcal{I}_{\Delta-\Delta''}(s)) = h^1(\mathbb{P}^2, \mathcal{I}_{\Delta'}(e-s)) - h^1(\mathbb{P}^2, \mathcal{I}_{\Delta''}(e-s)).$$

In particular,

$$h^0(\mathbb{P}^2, \mathcal{I}_{\Delta-\Delta'}(s)) - h^0(\mathbb{P}^2, \mathcal{I}_{\Delta}(s)) = h^1(\mathbb{P}^2, \mathcal{I}_{\Delta'}(e-s)).$$

Corollary 2.8. *Suppose Δ is the singular locus of a foliation of degree r in $\mathbb{P}^2$. Let $\Delta'' \subset \Delta'$ be subschemes of Δ. For all $s \neq r-1$, we have*

$$\begin{aligned} &h^0(\mathbb{P}^2, \mathcal{I}_{\Delta-\Delta'}(s)) - h^0(\mathbb{P}^2, \mathcal{I}_{\Delta-\Delta''}(s)) \\ &= h^1(\mathbb{P}^2, \mathcal{I}_{\Delta'}(2r-2-s)) - h^1(\mathbb{P}^2, \mathcal{I}_{\Delta''}(2r-2-s)). \end{aligned}$$

In particular, for any $r \neq 1$ and $\Delta' \subsetneq \Delta$, we have

$$h^1(\mathbb{P}^2, \mathcal{I}_{\Delta}(2r-2)) - h^1(\mathbb{P}^2, \mathcal{I}_{\Delta'}(2r-2)) = 1.$$

2.3. Some results of cohomology groups about $\Delta = \mathrm{Sing}\mathcal{F}$

We have the following proposition (see also [4], Theorem 3.2).

Proposition 2.9. *Let $\mathcal{F}$ be a foliation of degree $r \geq 0$ on $\mathbb{P}^2$, and let $\mathcal{I}_\Delta$ be the sheaf of ideals of its singular subscheme $\Delta = \mathrm{Sing}\mathcal{F}$.*

(i) *If $r = 0$, then for any integer $s \geq 1$, we have $\Delta = p$ for some point $p \in \mathbb{P}^2$ and*

$$h^0(\mathbb{P}^2, \mathcal{I}_\Delta(s)) = h^0(\mathbb{P}^2, \mathcal{O}(s)) - 1 = \frac{1}{2}(s+1)(s+2) - 1.$$

(ii) *If $r \geq 1$, then for any integer $s \geq 0$, we have*

$$h^0(\mathbb{P}^2, \mathcal{I}_\Delta(s)) = \begin{cases} 0, & \text{if } s \leq r, \\ (s-r)(s-r+2), & \text{if } r+1 \leq s \leq 2r, \\ \frac{1}{2}(s+1)(s+2) - (r^2+r+1), & \text{if } s > 2r. \end{cases}$$

In particular,

$$h^0(\mathbb{P}^2, \mathcal{I}_\Delta(r+1)) = \begin{cases} 2, & \text{if } r = 0, \\ 3, & \text{if } r \geq 1. \end{cases}$$

Proof. The part (i) is clear and next we will prove the part (ii). Consider the following exact sequence from the sequence (2):

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \Omega_{\mathbb{P}^2}(r+2) \longrightarrow \mathcal{I}_\Delta(2r+1) \longrightarrow 0,$$

and we have

$$0 \longrightarrow H^0(\mathcal{O}(s-2r-1)) \longrightarrow H^0(\Omega_{\mathbb{P}^2}(s-r+1)) \longrightarrow H^0(\mathcal{I}_\Delta(s)) \longrightarrow 0.$$

So

$$\begin{aligned} h^0(\mathbb{P}^2, \mathcal{I}_\Delta(s)) &= h^0(\Omega_{\mathbb{P}^2}(s+r-1)) - h^0(\mathcal{O}(s-2r-1)) \\ &= \begin{cases} 0, & \text{if } s \leq r, \\ (s-r)(s-r+2), & \text{if } r+1 \leq s \leq 2r, \\ \frac{1}{2}(s+1)(s+2) - (r^2+r+1), & \text{if } s > 2r. \end{cases} \end{aligned}$$

Here what we need is just the Bott's formula (Lemma 2.1). □

Proposition 2.10. *Let Δ be the singular locus of a foliation of degree r $(\neq 1)$ in $\mathbb{P}^2$. Then*

(i) $h^1(\mathcal{I}_\Delta(2r-2)) = 1$,
(ii) $h^1(\mathcal{I}_{\Delta'}(2r-2)) = 0$ *for any* $\Delta' \subsetneq \Delta$.

Proof. From the natural exact sequence

$$0 \longrightarrow \mathcal{I}_\Delta(2r-2) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(2r-2) \longrightarrow \mathcal{O}_\Delta \longrightarrow 0,$$

we have

$$h^1(\mathcal{I}_\Delta(2r-2)) = \deg\Delta - h^0(\mathcal{O}(2r-2)) + h^0(\mathcal{I}_\Delta(2r-2)) = 1.$$

Then by Corollary 2.8, for any $r \neq 1$ and any $\Delta' \subset \Delta$ with $\deg\Delta' = \deg\Delta - 1$, we have

$$h^1(\mathcal{I}_{\Delta'}(2r-2)) = h^1(\mathcal{I}_\Delta(2r-2)) - 1 = 0. \qquad \square$$

Remark 2.11. If $r = 1$, then $\deg\Delta = 3$. It is clear that

$$h^0(\mathcal{I}_{\Delta'}(2r-2)) = h^0(\mathcal{I}_{\Delta'}) = \deg\Delta' - 1,$$

for any $\Delta' \subset \Delta$ with $\deg\Delta' \geq 1$.

3. Proof of main theorems

3.1. Proof of the Theorem 1.1

Lemma 3.1. *Let* $F_1 = Z(B), F_2 = Z(C)$ *be curves of degree* $r+1$ (≥ 1) *over* $\mathbb{P}^2 = \operatorname{Proj}\mathbb{C}[X,Y,Z]$*, with no common component. Let* $H = Z(X)$ *be a hyperplane over* $\mathbb{P}^2$*. If* $\deg(F_1F_2H) = r$*, then we can write* B, C *as*

$$B(X,Y,Z) = X \cdot B_1(X,Y,Z) + a \cdot ZG(Y,Z),$$
$$C(X,Y,Z) = X \cdot C_1(X,Y,Z) + b \cdot YG(Y,Z),$$

after a coordinate transformation over Y, Z*, where* $a, b \neq 0$.

Proof. Firstly we can write B, C as

$$B(X,Y,Z) = X \cdot B_1(X,Y,Z) + B_2(Y,Z)$$
$$C(X,Y,Z) = X \cdot C_1(X,Y,Z) + C_2(Y,Z)$$

where $B_2 \neq 0$ and $C_2 \neq 0$ clearly. Let $p_1, \ldots, p_{r+1}$ (resp. $q_1, \ldots, q_{r+1}$) be the zeros of $B_2(Y,Z)$ (resp. $C_2(Y,Z)$) in $H = \operatorname{Proj}\mathbb{C}[Y,Z]$. Then $\deg(F_1F_2H) = r$ is equivalent to say that there exist i, j such that

$$p_i \notin \{q_1, \ldots, q_{r+1}\}, \quad q_j \notin \{p_1, \ldots, p_{r+1}\},$$
$$\{p_1, \ldots, \hat{p}_i, \ldots, p_{r+1}\} = \{q_1, \ldots, \hat{q}_j, \ldots, q_{r+1}\} := \Gamma.$$

After a coordinate transformation over Y, Z, we can assume $p_i = [1,0] = Z(Z)$, $q_i = [0,1] = Z(Y)$ and $\mathcal{I}_\Gamma = I(G)$ for a homogeneous polynomial $G \in \mathbb{C}[X,Y]$ with $\deg G = r$. So

$$B(X,Y,Z) = X \cdot B_1(X,Y,Z) + a \cdot ZG(Y,Z),$$
$$C(X,Y,Z) = X \cdot C_1(X,Y,Z) + b \cdot YG(Y,Z),$$

where $a, b \neq 0$. $\square$

Proposition 3.2. *Let F_1, F_2 be two curves of degree $r+1$ (≥ 1) over $\mathbb{P}^2$ with no common component. Let H be a hyperplane over $\mathbb{P}^2$. Consider the zero dimensional subscheme Δ of $\mathbb{P}^2$:*

$$\Delta = F_1F_2 - F_1F_2H.$$

Suppose $\deg\Delta = r^2+r+1$. *Then* $\Delta = \mathrm{Sing}\mathcal{F}$, *for some foliation $\mathcal{F}$ of degree r over $\mathbb{P}^2$. In particular, Δ is a local complete intersection.*

Proof. By Lemma 3.1, we can choose a coordinate such that $F_1 = Z(B), F_2 = Z(C), H = Z(X)$, where

$$B(X,Y,Z) = -X\cdot B_1(X,Y,Z) + ZG(Y,Z),$$
$$C(X,Y,Z) = -X\cdot C_1(X,Y,Z) - YG(Y,Z).$$

Here $G \in \mathbb{C}[Y,Z]$ is a nonzero homogeneous polynomial of degree r. Now let

$$A(X,Y,Z) = Y\cdot B_1(X,Y,Z) + Z\cdot C_1(X,Y,Z).$$

Then A, B, C have no common factors and

$$(*) \qquad X\cdot A(X,Y,Z) + Y\cdot B(X,Y,Z) + Z\cdot C(X,Y,Z) = 0.$$

Claim: $\Delta = F_1F_2F_3$, where $F_3 = Z(A)$.

By Definition 2.1, $\mathcal{I}_\Delta = [\mathcal{I}_{F_1F_2} : \mathcal{I}_{F_1F_2H}]$. Then the equation $(*)$ implies $A, B, C \in \mathcal{I}_\Delta$ clearly. So $\Delta \subset F_1F_2F_3$. By Corollary 2.7, we have

$$h^1(\mathcal{I}_{F_1F_2-F_1F_2F_3}(r-2)) = h^0(\mathcal{I}_{F_1F_2F_3}(r+1)) - h^0(\mathcal{I}_{F_1F_2}(r+1)) \geq 3-2 = 1.$$

Since $\mathcal{O}_{\mathbb{P}^2}(k)$ is k-very ample, we see that $\deg(F_1F_2 - F_1F_2F_3) \geq r$, which implies

$$(4) \qquad \deg(F_1F_2F_3) \leq r^2+r+1 = \deg\Delta.$$

So $\Delta = F_1F_2F_3$ and the Claim follows.

Now consider

$$\omega = AdX + BdX + CdZ \in \Omega_{\mathbb{C}^3-\{0\}},$$

which corresponds to a foliation $\mathcal{F}$ of degree r over $\mathbb{P}^2$, and it is clear that $\Delta = \mathrm{Sing}\mathcal{F}$. □

Now we have the following theorem, which is nothing but Theorem 1.1.

Theorem 3.3. *Suppose Δ is a zero dimensional subscheme of $\mathbb{P}^2$. Then the following conditions are equivalent.*

(1) *Δ is the singular locus of a foliation $\mathcal{F}$ in $\mathbb{P}^2$ of degree r.*
(2) *We can write Δ as*

$$\Delta = F_1F_2 - F_1F_2H,$$

with $\deg\Delta = r^2+r+1$, *where F_1, F_2 are two curves of degree $r+1(\geq 1)$ over $\mathbb{P}^2$ with no common component and H is a hyperplane over $\mathbb{P}^2$.*

Proof. (1) $\Longrightarrow$ (2) : Proposition 2.4. (2) $\Longrightarrow$ (1) : Proposition 3.2. □

Remark 3.4. In the context of Lemma 3.1, consider the syzygy exact sequence

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{O}(-F_1) \oplus \mathcal{O}(-F_2) \oplus \mathcal{O}(-H) \xrightarrow{(f_1,f_2,x)} \mathcal{I}_{F_1F_2H} \longrightarrow 0,$$

where $f_1 \in H^0(\mathcal{O}(F_1)), f_2 \in H^0(\mathcal{O}(F_2))$ and $x \in H^0(\mathcal{O}(H))$. Note that $\mathcal{E}_1$ is a subsheaf of a locally free sheaf with a torsion-free quotient, so it is reflexive (see [7]), and moreover it is locally free of rank 2. Considering the composition

$$\phi : \mathcal{E}_1 \to \mathcal{O}(-F_1) \oplus \mathcal{O}(-F_2) \oplus \mathcal{O}(-H) \to \mathcal{O}(-H),$$

we can see that the image of ϕ in $\mathcal{O}(-H)$ is $\mathcal{I}_\Delta(-H)$, by the definition of $\mathcal{I}_\Delta$, where $\Delta = F_1F_2 - F_1F_2H$. Thus $\ker\phi$ is an invertible sheaf. By comparing the first Chern classes, we get

$$0 \to \mathcal{O}(-F_1 - F_2) \to \mathcal{E}_1 \to \mathcal{I}_\Delta(-H) \to 0.$$

(See [9], Section 5 for more details.) Let $\mathcal{E} = \mathcal{E}_1(2r + 2)$, then

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{I}_\Delta(2r + 1) \to 0.$$

So

$$c_1(\mathcal{E}) \equiv (2r + 1)H, \quad c_2(\mathcal{E}) = \Delta = F_1F_2 - F_1F_2H.$$

Since for any $r \geq 0$ and $r \neq 1$,

$$\dim Ext^1(\mathcal{I}_\Delta(2r + 1), \mathcal{O}) = h^1(\mathcal{I}_\Delta(2r - 2)) = 1,$$

where the first equality is from Serre duality and ([6], p.234, Proposition 6.3) and the second equality is from Proposition 2.10, we have $\Omega_{\mathbb{P}^2}(r + 2) = \mathcal{E} = \mathcal{E}_1(2r + 2)$. So we have the following exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^2} \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(r - 1) \longrightarrow \mathcal{I}_{F_1F_2H}(r) \longrightarrow 0, \tag{5}$$

where $r \geq 0$, $r \neq 1$ and $\deg(F_1F_2H) = r$. In particular, if $r = 0$, then $\mathcal{I}_{F_1F_2H} = \mathcal{O}$ and we get

$$0 \longrightarrow \Omega_{\mathbb{P}^2} \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow 0,$$

which is just the Euler exact sequence.

Corollary 3.5 ([4], Theorem 4.1)**.** *Suppose Δ is a zero dimensional subscheme of $\mathbb{P}^2$.*

(1) *Δ is the singular locus of a foliation $\mathcal{F}$ in $\mathbb{P}^2$ of degree 0, if and only if $\deg\Delta = 1$.*
(2) *Δ is the singular locus of a foliation $\mathcal{F}$ in $\mathbb{P}^2$ of degree 1, if and only if $\deg\Delta = 3$ and $h^0(\mathcal{I}_\Delta(1)) = 0$.*

Proof. The "only if" parts are clear. Next we will prove the "if" parts.

(1) If $\deg\Delta = 0$, then $\Delta = p$ for some point $p \in \mathbb{P}^2$. So we can choose three lines, L_1, L_2 through p and L_3 linearly independent from them. Thus we can write Δ as $\Delta = L_1L_2 - L_1L_2L_3$, which implies $\Delta = \mathrm{Sing}\mathcal{F}$ for some foliation $\mathcal{F}$ of degree 0.

(2) If $\deg\Delta = 3$, then $h^1(\mathcal{I}_\Delta(2)) = 0$, since $\mathcal{O}_{\mathbb{P}^2}(2)$ is 2-very ample. So we see $h^0(\mathcal{I}_\Delta(2)) = 3$, say $H^0(\mathcal{I}_\Delta(2)) = \mathbb{C}\{A, B, C\}$. Since $h^0(\mathcal{I}_\Delta(1)) = 0$, A, B, C have no common factors. Hence we assume $\gcd(A, B) = 1$. Let $F_1 = Z(A), F_2 = Z(B)$, then $Z = F_1F_2$ is a complete intersection containing Δ with $\deg Z = 4$. Then $\deg(Z - \Delta) = 1$, say $p = Z - \Delta \subset Z$. It is clear that we can choose a line H passing through p such that $\deg(ZH) = 1$. So we can write Δ as $\Delta = F_1F_2 - F_1F_2H$, which implies $\Delta = \mathrm{Sing}\mathcal{F}$ for some foliation $\mathcal{F}$ of degree 1. □

Theorem 3.6. *Suppose Δ is a zero dimensional subscheme of $\mathbb{P}^2$ with $\deg\Delta = r^2 + r + 1 > 3$. Δ is the singular locus of a foliation $\mathcal{F}$ in $\mathbb{P}^2$ of degree r, if and only if the following conditions hold*:

(a1) *$h^1(\mathcal{I}_\Delta(2r-2)) = 1$ and $h^1(\mathcal{I}_{\Delta'}(2r-2)) = 0$ for any $\Delta' \subset \Delta$ of degree $r^2 + r$,*

(b1) *$h^0(\mathcal{I}_\Delta(r+1)) \geq 2$ and the linear system $\mathbb{P}\big(H^0(\mathcal{I}_\Delta(r+1))\big)$ has no base curve.*

Proof. The "only if" part is clear. Next we will prove the "if" part.

By the condition (b1), we can choose $A, B \in H^0(\mathcal{I}_\Delta(r+1))$ with $\gcd(A, B) = 1$. Let $F_1 = Z(A), F_2 = Z(B)$. Then $Z = F_1F_2$ is a complete intersection containing Δ with $\deg Z = (r+1)^2$. By Corollary 2.7, we have

$$h^0(\mathcal{I}_{Z-\Delta}(1)) - h^0(\mathcal{I}_{Z-\Delta'}(1)) = h^1(\mathcal{I}_\Delta(2r-2)) - h^1(\mathcal{I}_{\Delta'}(2r-2)) = 1,$$

for any $\Delta' \subsetneq \Delta$. Since $\deg(Z - \Delta) = r \geq 2$, $h^0(\mathcal{I}_{Z-\Delta}(1)) \leq 1$. So we have $h^0(\mathcal{I}_{Z-\Delta}(1)) = 1$ and $h^0(\mathcal{I}_{Z-\Delta'}(1)) = 0$, for any $\Delta' \subsetneq \Delta$. So we can choose a line H such that $ZH = Z - \Delta$. Hence we can write Δ as $\Delta = F_1F_2 - F_1F_2H$, which implies $\Delta = \mathrm{Sing}\mathcal{F}$ for some foliation $\mathcal{F}$ of degree r. □

Remark 3.7. In fact, the condition (a1) implies that, Δ is the zero subscheme given by a global section of a rank 2 locally free sheaf $\mathcal{E}$ with $c_1(\mathcal{E}) \equiv (2r+1)H$ and $c_2(\mathcal{E}) = \Delta$. Since $h^1(\mathcal{I}_\Delta(2r-2)) = 1$, we have an extension

$$0 \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow \mathcal{I}_\Delta(2r+1) \longrightarrow 0,$$

which corresponds to the identity map

$$\begin{aligned} \mathrm{id} \in\ & Hom\big(H^1(\mathcal{I}_\Delta(2r-2)), H^1(\mathcal{I}_\Delta(2r-2))\big) \\ \cong\ & H^1(\mathcal{I}_\Delta(2r-2))^\vee \\ \cong\ & Ext^1(\mathcal{I}_\Delta(2r+1), \mathcal{O}). \end{aligned}$$

Tyurin ([10], Lemma 1.2 and Corollary 1) said that the extension E is locally free if and only if Δ is $(2r-2)$-stable, i.e, $h^1(\mathcal{I}_\Delta(2r-2)) > h^1(\mathcal{I}_{\Delta'}(2r-2))$ for any $\Delta' \subsetneq \Delta$.

Now if we strengthen the condition (b1) a little, then we can weaken the condition (a1) a little. More precisely, we have the following theorem.

Theorem 3.8. *Suppose Δ is a zero dimensional subscheme of $\mathbb{P}^2$ with* $\deg \Delta = r^2 + r + 1 > 3$. *$\Delta$ is the singular locus of a foliation $\mathcal{F}$ in $\mathbb{P}^2$ of degree r, if and only if the following conditions hold*:

- $(a2)$ $h^1(\mathcal{I}_{\Delta'}(2r-2)) = 0$, *for any $\Delta' \subset \Delta$ of degree $r^2 + r$,*
- $(b2)$ $h^0(\mathcal{I}_\Delta(r+1)) \geq 3$, *and the linear system $\mathbb{P}\big(H^0(\mathcal{I}_\Delta(r+1))\big)$ has no base curve.*

Proof. The "only if" part is clear. Next we will prove the "if" part. By Theorem 3.6, it suffices to prove $h^1(\mathcal{I}_\Delta(2r-2)) = 1$.

Since $h^0(\mathcal{I}_\Delta(r+1)) \geq 3$ and $\mathbb{P}\big(H^0(\mathcal{I}_\Delta(r+1))\big)$ has no base curve, we can choose three linearly independent elements $A, B, C \in H^0(\mathcal{I}_\Delta(r+1))$ such that A, B have no common factors. Let $F_1 = Z(A), F_2 = Z(B), F_3 = Z(C)$. Then we see $\Delta \subset F_1F_2F_3$. In the other hand, we see $\deg(F_1F_2F_3) \leq r^2 + r + 1 = \deg \Delta$, from the inequality (4). So $\Delta = F_1F_2F_3$.

Consider

$$\widetilde{\Delta} := F_1F_2 - F_1F_2F_3 = Z - \Delta,$$

where $Z = F_1F_2$ is a complete intersection of degree $(r+1)^2$. Then we have

$$s(\widetilde{\Delta}) := \min\{m | h^0(\mathcal{I}_{\widetilde{\Delta}}(m)) \neq 0\} = 1.$$

(We can also see it in Remark 3.12.) Since $\deg \widetilde{\Delta} = r \geq 2$, we see $h^0(\mathcal{I}_{\widetilde{\Delta}}(1)) = 1$. Then applying the Cayley-Bacharach theorem (Corollary 2.7), we have

$$h^1(\mathcal{I}_\Delta(2r-2)) = h^0(\mathcal{I}_{\widetilde{\Delta}}(1)) - h^0(\mathcal{I}_Z(1)) = 1. \qquad \square$$

Moreover, we can replace the condition (a2) by the condition (a3) in the following theorem. In fact, they are talking about the same thing that the line passing through $\widetilde{\Delta} = Z - \Delta$ cannot pass through any another point in Δ.

Theorem 3.9 ([4]). *Suppose Δ is a zero dimensional subscheme of $\mathbb{P}^2$ with* $\deg \Delta = r^2 + r + 1 > 3$. *$\Delta$ is the singular locus of a foliation $\mathcal{F}$ in $\mathbb{P}^2$ of degree r, if and only if the following conditions hold*:

- (a3) $h^0(\mathcal{I}_{\Delta'}(r)) = 0$, *for any $\Delta' \subset \Delta$ of degree $r^2 + r$,*
- (b2) $h^0(\mathcal{I}_\Delta(r+1)) \geq 3$, *and the linear system $\mathbb{P}\big(H^0(\mathcal{I}_\Delta(r+1))\big)$ has no base curve.*

Proof. The "only if" part is clear. Next we will prove the "if" part. By Theorem 3.8, it suffices to prove $h^1(\mathcal{I}_{\Delta'}(2r-2)) = 0$ for any $\Delta' \subset \Delta$ of degree $r^2 + r$.

By the proof of the theorem above, (b2) implies that $\Delta = F_1F_2F_3$. where F_1, F_2, F_3 are curves of degree $r+1$ and F_1, F_2 have no common component. Let $Z = F_1F_2$ and $\widetilde{\Delta} = Z - \Delta$, then $h^0(\mathcal{I}_{\widetilde{\Delta}}(1)) = 1$.

Claim: $h^0(\mathcal{I}_{\widetilde{\Delta}'}(1)) = 0$ for any $\widetilde{\Delta} \subset \widetilde{\Delta}' \subset Z$ with $\deg \widetilde{\Delta}' = \deg \widetilde{\Delta} + 1 = r + 1$.

Indeed, if there exists a line $H \in H^0(\mathcal{I}_{\widetilde{\Delta}'}(1))$, then we have the following exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)(= \mathcal{I}_H) \longrightarrow \mathcal{I}_{\widetilde{\Delta}'} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-r-1)(= \mathcal{I}_{\widetilde{\Delta}'|H}) \longrightarrow 0,$$

which induces

$$0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^2}(r-2)) \longrightarrow H^0(\mathcal{I}_{\widetilde{\Delta}'}(r-1)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(-2)) = 0.$$

So

$$h^0(\mathcal{I}_{\widetilde{\Delta}'}(r-1)) = h^0(\mathcal{O}_{\mathbb{P}^2}(r-2)) = \frac{1}{2}r^2 - \frac{1}{2}r.$$

Then

$$\begin{aligned} h^1(\mathcal{I}_{\widetilde{\Delta}'}(r-1)) &= \deg \widetilde{\Delta}' - h^0(\mathcal{O}(r-1)) + h^0(\mathcal{I}_{\widetilde{\Delta}'}(r-1)) \\ &= (r+1) - \frac{1}{2}(r+1)r + (\frac{1}{2}r^2 - \frac{1}{2}r) = 1 > 0. \end{aligned}$$

Now let $\Delta' = Z - \widetilde{\Delta}'$, then it is clear that $\Delta' \subset \Delta$ and $\deg \Delta' = r^2 + r$. By the Cayley-Bacharach Theorem, we see

$$h^0(\mathcal{I}_{\Delta'}(r)) = h^1(\mathcal{I}_{\widetilde{\Delta}'}(r-1)) + h^0(\mathcal{I}_Z(r)) = 1 > 0,$$

which is a contradiction with the condition (b2). So the Claim is true.

Now for any $\Delta' \subset \Delta$ with $\deg \Delta' = r^2 + r$, by the Cayley-Bacharach theorem and the claim above, we have

$$h^1(\mathcal{I}_{\Delta'}(2r-2)) = h^0(\mathcal{I}_{Z-\Delta'}(1)) - h^0(\mathcal{I}_Z(1)) = 0. \qquad \square$$

This reproves a theorem of Campillo-Olivares ([4], Theorem 4.5).

3.2. Proof of the part (1) $\Longleftrightarrow$ (5) of Theorem 1.2

During the proof of Theorem 1.1, it is important to find a complete intersection of degree $(r+1)^2$ containing Δ. For this goal, we need one of the following conditions:

(b1) $h^0(\mathcal{I}_\Delta(r+1)) \geq 2$ *and the linear system* $\mathbb{P}\big(H^0(\mathcal{I}_\Delta(r+1))\big)$ *has no base curve*,

(b2) $h^0(\mathcal{I}_\Delta(r+1)) \geq 3$ *and the linear system* $\mathbb{P}\big(H^0(\mathcal{I}_\Delta(r+1))\big)$ *has no base curve.*

In this section, we will see that the condition (b1) or (b2) can be replaced by

(b3) $h^0(\mathcal{I}_\Delta(r+1)) \geq 3$, *and* $h^0(\mathcal{I}_\Delta(r)) = 0$.

Consider the following question:

Question 3.10. Given a zero dimensional subscheme Δ of $\mathbb{P}^2$ and a curve F, how many points in Δ can be passed through by F?

We denote by $\deg(\Delta F)$ the number of points in Δ passed through by F, where ΔF is a zero dimensional subscheme of $\mathbb{P}^2$ defined by the ideal sheaf $\mathcal{I}_{\Delta F}$.

Lemma 3.11. *Let $\mathcal{E}$ be a locally free sheaf on $\mathbb{P}^2$ with rank 2, let s be a global section of $\mathcal{E}$, and let $\Delta = Z(s) \subset \mathbb{P}^2$ be its zero subscheme. Let* $s(\Delta) =$

$\min\{m|h^0(\mathcal{I}_\Delta(m)) \neq 0\}$. *Then for any curve F with $\deg(F) \leq s(\Delta)$, we have $\deg(\Delta F) \leq \phi(\deg F)$, where*

$$\phi(\deg F) = \begin{cases} \deg \Delta - \big[\deg \mathcal{E} - s(\Delta)\big] \cdot \big[s(\Delta) - \deg F\big], & \text{if } s(\Delta) < \deg(\mathcal{E}), \\ \deg \Delta - \frac{1}{4}(\deg \mathcal{E} - \deg F)^2, & \text{if } s(\Delta) = \deg(\mathcal{E}). \end{cases}$$

In other words, for any $\Delta' \subset \Delta$ and $i > 0$, if $\deg \Delta' \geq \phi(i)+1$ then $h^0(\mathcal{I}_{\Delta'}(i)) = 0$.

Proof. Note that, under the assumption above, we have $\deg F \leq s(\Delta) \leq \deg \mathcal{E}$.

Consider the morphism

$$\pi = (s^\vee, f) : \mathcal{E}^\vee \oplus \mathcal{O}(-F) \longrightarrow \mathcal{O},$$

where $f \in H^0(\mathcal{O}(F))$, $s \in H^0(\mathcal{E})$ and $\pi\big((x,y)\big) = x \cdot s^\vee + y \cdot f$. It is clear that $\mathrm{Im}(\pi) = \mathcal{I}_{\Delta F}$. And let $\mathcal{E}_1 = \mathrm{Ker}(\pi)$, which is a reflexive sheaf over an algebraic surface. So we see $\mathcal{E}_1$ is locally free of rank 2 and we have the following exact sequence

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}^\vee \oplus \mathcal{O}(-F) \xrightarrow[(s^\vee, f)]{\pi} \mathcal{I}_{\Delta F} \longrightarrow 0.$$

Then we have

$$c(\mathcal{E}_1) = c(\mathcal{I}_{\Delta F})^{-1} \cdot c\big(\mathcal{E}^\vee \oplus \mathcal{O}(-F)\big)$$

from which, the Chern classes of $\mathcal{E}_1$ can be computed easily:

$$\begin{aligned} c_1(\mathcal{E}_1) &= -c_1(\mathcal{E}) - c_1(F), \\ c_2(\mathcal{E}_1) &= c_2(\mathcal{E}) + c_1(\mathcal{E}) \cdot c_1(F) - c_2(\mathcal{I}_{\Delta F}). \end{aligned}$$

Now let $\widetilde{\Delta} = \Delta - \Delta F$ and $\widetilde{\mathcal{E}} = \mathcal{E}_1(\det \mathcal{E})$, similar to Remark 3.4, we see

$$c_1(\widetilde{\mathcal{E}}) = c_1(\mathcal{E}) - c_1(F) \equiv (\deg \mathcal{E} - \deg F)H, \quad c_2(\widetilde{\mathcal{E}}) = \widetilde{\Delta}.$$

In other words, we have the following exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \widetilde{\mathcal{E}} \longrightarrow \mathcal{I}_{\widetilde{\Delta}}\big(c_1(\widetilde{\mathcal{E}})\big) \longrightarrow 0.$$

Recall the discriminant of $\widetilde{\mathcal{E}}$:

$$\mathbb{D}(\widetilde{\mathcal{E}}) = c_1(\widetilde{\mathcal{E}})^2 - 4c_2(\widetilde{\mathcal{E}}) = (\deg \mathcal{E} - \deg F)^2 - 4 \deg \Delta + 4 \deg(\Delta F).$$

Suppose $\mathbb{D}(\widetilde{\mathcal{E}}) \leq 0$. Then

$$\begin{aligned} \deg(\Delta F) &\leq \deg \Delta - \frac{1}{4}(\deg \mathcal{E} - \deg F)^2 \\ &\leq \deg \Delta - \big[\deg \mathcal{E} - s(\Delta)\big] \cdot \big[s(\Delta) - \deg F\big], \end{aligned}$$

where the second inequality is from $\deg F \leq s(\Delta) \leq \deg \mathcal{E}$.

Suppose $\mathbb{D}(\widetilde{\mathcal{E}}) > 0$. Then by the Bogmolov's instability theorem ([1], p. 500), there exists a saturated line bundle $\mathcal{M} \subset \widetilde{\mathcal{E}}$ such that (i) $2c_1(\mathcal{M}) - c_1(\widetilde{\mathcal{E}}) > 0$ and (ii) $(2c_1(\mathcal{M}) - c_1(\widetilde{\mathcal{E}}))^2 \geq \mathbb{D}(\widetilde{\mathcal{E}})$. So

$$\deg \mathcal{M} \geq \frac{1}{2}\big(\deg \widetilde{\mathcal{E}} + \sqrt{\mathbb{D}(\widetilde{\mathcal{E}})}\big).$$

In the other hand, since $\mathcal{M}$ is a maximal sub-line bundle of $\widetilde{\mathcal{E}}$, the induced composition $\mathcal{M} \to \widetilde{\mathcal{E}} \to \mathcal{I}_{\widetilde{\Delta}}\big(c_1(\widetilde{\mathcal{E}})\big)$ is non-zero, and therefore we have a non-zero morphism

$$\mathcal{O} \longrightarrow \mathcal{I}_{\widetilde{\Delta}}\big(c_1(\widetilde{\mathcal{E}}) - c_1(\mathcal{M})\big).$$

Hence

$$s(\widetilde{\Delta}) \leq \deg \widetilde{\mathcal{E}} - \deg \mathcal{M} \leq \frac{1}{2}\big(\deg \widetilde{\mathcal{E}} - \sqrt{\mathbb{D}(\widetilde{\mathcal{E}})}\big).$$

So

$$\begin{aligned} s(\Delta) &\leq s(\widetilde{\Delta}) + s(\Delta F) \\ &\leq \frac{1}{2}\big(\deg \widetilde{\mathcal{E}} - \sqrt{\mathbb{D}(\widetilde{\mathcal{E}})}\big) + \deg F \\ (*) \qquad &= \frac{1}{2}\big(\deg \mathcal{E} + \deg F - \sqrt{(\deg \mathcal{E} - \deg F)^2 - 4\deg \Delta + 4\deg(\Delta F)}\big). \end{aligned}$$

Then

$$\begin{aligned} \deg(\Delta F) &\leq \deg \Delta - \frac{1}{4}(\deg \mathcal{E} - \deg F)^2 + \frac{1}{4}(\deg \mathcal{E} + \deg F - 2s(\Delta))^2 \\ &= \deg \Delta - \big[\deg \mathcal{E} - s(\Delta)\big] \cdot \big[s(\Delta) - \deg F\big]. \end{aligned}$$

Note that, by the inequality $(*)$, $\deg F \leq s(\Delta)$ and $\mathbb{D}(\widetilde{\mathcal{E}}) > 0$ imply

$$s(\Delta) < \frac{1}{2}(\deg \mathcal{E} + \deg F) \leq \frac{1}{2}(\deg \mathcal{E} + s(\Delta)).$$

Thus $s(\Delta) < \deg \mathcal{E}$. In other words, for the case that $s(\Delta) = \deg \mathcal{E}$, we have $\mathbb{D}(\widetilde{\mathcal{E}}) \leq 0$. Next is clear. □

Remark 3.12. Consider the case that $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(F_1) \oplus \mathcal{O}_{\mathbb{P}^2}(F_2)$, where $\deg F_1 = \deg F_2 = r+1$ and F_1, F_2 have no common component. And let $\deg F = r+1$. Then

$$\deg \widetilde{\mathcal{E}} = r+1, \quad c_2(\widetilde{\mathcal{E}}) = \widetilde{\Delta} = F_1F_2 - F_1F_2F.$$

And $\mathbb{D}(\widetilde{\mathcal{E}}) = (\deg \widetilde{\mathcal{E}})^2 - 4\deg \widetilde{\Delta}$.

If $r \neq 1$ and $\deg \widetilde{\Delta} = r$, then $\mathbb{D}(\widetilde{\mathcal{E}}) = (r-1)^2 > 0$. So by the results in the proof above, we have

$$s(\widetilde{\Delta}) \leq \frac{1}{2}\big(\deg \widetilde{\mathcal{E}} - \sqrt{\mathbb{D}(\widetilde{\mathcal{E}})}\big) = 1.$$

or say, $h^0(\mathcal{I}_{\widetilde{\Delta}}(1)) > 0$.

Corollary 3.13. *Let Δ be the singular locus of a foliation $\mathcal{F}$ in $\mathbb{P}^2$ of degree $r(\geq 2)$. For any curve F with $\deg(F) = j \leq r$, we have*

$$\deg(\Delta F) \leq rj + 1$$

In other words, for any $\Delta' \subset \Delta$ and $j > 0$, if $\deg \Delta' \geq rj+2$ then $h^0(\mathcal{I}_{\Delta'}(j)) = 0$.

Proof. Just consider the locally free sheaf $\mathcal{E} = \Omega_{\mathbb{P}^2}(r+2)$, where

$$(**)\qquad \begin{cases} c_1(\mathcal{E}) \equiv (2r+1)H, \quad c_2(\mathcal{E}) = \Delta, \\ \deg \Delta = r^2 + r + 1, \\ s(\Delta) = r + 1. \end{cases}$$

Then it is clear. (In fact, the equations $(**)$ are enough for the results.) □

Now we return to the proof of the part (1) $\Longleftrightarrow$ (5) of Theorem 1.2.

Theorem 3.14. *Suppose Δ is a zero dimensional subscheme of $\mathbb{P}^2$ with* $\deg \Delta = r^2+r+1 > 3$. *$\Delta$ is the singular locus of a foliation $\mathcal{F}$ in $\mathbb{P}^2$ of degree $r(\geq 2)$, if and only if the following conditions hold*:

(a1) $h^1(\mathcal{I}_\Delta(2r-2)) = 1$ *and* $h^1(\mathcal{I}_{\Delta'}(2r-2)) = 0$, *for any* $\Delta' \subset \Delta$ *of degree* $r^2 + r$,

(b3) $h^0(\mathcal{I}_\Delta(r+1)) \geq 3$, *and* $h^0(\mathcal{I}_\Delta(r)) = 0$.

Proof. The "only if" part is clear. Next we will prove the "if" part. By Theorem 3.6, it suffices to prove that: the linear system $\mathbb{P}\big(H^0(\mathcal{I}_\Delta(r+1))\big)$ has no base curve.

Firstly, the condition (a1) can imply that Δ is the zero subscheme of a global section of a rank 2 locally free sheaf $\mathcal{E}$, where

$$(**)\qquad \begin{cases} c_1(\mathcal{E}) \equiv (2r+1)H, \quad c_2(\mathcal{E}) = \Delta, \\ \deg \Delta = r^2 + r + 1, \\ s(\Delta) = r + 1, \quad \text{(by the condition (b3))}. \end{cases}$$

Secondly, we assume that $\mathbb{P}\big(H^0(\mathcal{I}_\Delta(r+1))\big)$ has a base curve, say G, with $\deg G = j(>0)$. Since $h^0(\mathcal{I}_\Delta(r+1)) \geq 3$, we see $j \leq r$ and we can choose three linearly independent elements $A, B, C \in H^0(\mathcal{I}_\Delta(r+1))$. Let

$$A = G \cdot A', \quad B = G \cdot B', \quad C = G \cdot C',$$

where $\deg A' = \deg B' = \deg C' = r - j + 1$. Similar to the inequality (4), we have the following claim.

Claim: $\deg(F_1F_2F_3) \leq (r-j)^2 + (r-j) + 1$, where $F_1 = Z(A'), F_2 = Z(B')$ and $F_3 = Z(C')$.

By the definition, it is clear that $\Delta - \Delta G \subset F_1F_2F_3$, so we have

$$(I)\qquad \begin{aligned} \deg(\Delta G) &= \deg \Delta - \deg(\Delta - \Delta G) \\ &\geq (r^2 + r + 1) - \big[(r-j)^2 + (r-j) + 1\big] \\ &= 2rj - j^2 + j. \end{aligned}$$

In the other hand, by Corollary 3.13, the equations $(**)$ imply

$$(II)\qquad \deg(\Delta G) \leq rj + 1.$$

Since

$$(2rj - j^2 + j) - (rj + 1) = rj - j^2 + j - 1 > 0,$$

for any $0 < j < r+1$, the inequalities (I) and (II) give a contradiction. Hence there exists no base curves in the linear system $\mathbb{P}(H^0(\mathcal{I}_\Delta(r+1)))$. $\square$

4. Applications

4.1. Foliations of degree $r \neq 1$ on $\mathbb{P}^2$ are uniquely determined by its singular locus

Question 4.1. Given a zero dimensional subscheme Δ satisfying the condition (2) in Theorem 1.1, how many foliations $\mathcal{F}$ are there such that $\Delta = \mathrm{Sing}\mathcal{F}$?

For this question, Campillo and Olivares have given the answer (see [4]). Next we will give a different proof, by using the results of Theorem 1.1.

Lemma 4.2. *Let $r \neq 1$. If we fix Δ and H and assume $\Delta \cap H = \emptyset$, then the solution (F_1, F_2) of the equation $\Delta = F_1F_2 - F_1F_2H$ is unique. Here the "unique" means that if (F_1', F_2') satisfies $\Delta = F_1'F_2' - F_1'F_2'H$, then*

$$\mathbb{C}\{B', C'\} = \mathbb{C}\{B, C\} = H^0(\mathbb{P}^2, \mathcal{I}_{F_1F_2}(r+1))$$

as $\mathbb{C}$-vector spaces, where $F_1 = Z(B), F_2 = Z(C), F_1' = Z(B'), F_2' = Z(C')$.

Proof. If $r = 0$, then $h^0(\mathcal{I}_\Delta(r+1)) = 2$, which implies

$$H^0(\mathcal{I}_\Delta(r+1)) = H^0(\mathcal{I}_{F_1F_2}(r+1)) = \mathbb{C} \cdot \{B, C\}.$$

So $\mathbb{C}\{B', C'\} = \mathbb{C}\{B, C\}$ is clear.

Next we consider the case that $r \geq 2$. It suffices to assume that $F_1 = Z(B)$, $F_2 = Z(C)$ and $H = Z(X)$. And we can choose some homogeneous polynomial A of degree $r+1$ such that

$$H^0(\mathcal{I}_\Delta(r+1)) = \mathbb{C} \cdot \{A, B, C\}.$$

Let

$$\bar{A} = A(0, Y, Z), \quad \bar{B} = B(0, Y, Z), \quad \bar{C} = C(0, Y, Z).$$

Recall the proof of Proposition 3.2, we can write $\bar{B}, \bar{C}$ as

$$\bar{B} = Z \cdot G(Y, Z), \quad \bar{C} = -Y \cdot G(Y, Z),$$

after a suitable coordinate transformation.

Let $F_1' = (B' = 0)$ and $F_2' = (C' = 0)$, so $B', C' \in H^0(\mathcal{I}_\Delta(r+1))$. Thus we can write them as

$$B' = \alpha_1 A + \alpha_2 B + \alpha_3 C,$$
$$C' = \beta_1 A + \beta_2 B + \beta_3 C.$$

So

$$\bar{B}' := B'(0, Y, Z) = \alpha_1 \bar{A} + (\alpha_2 Z - \alpha_3 Y)G,$$
$$\bar{C}' := C'(0, Y, Z) = \beta_1 \bar{A} + (\beta_2 Z - \beta_3 Y)G.$$

(i) If $\alpha_1 = \beta_1 = 0$, then $\mathbb{C} \cdot \{B', C'\} = \mathbb{C} \cdot \{B, C\}$.

(ii) Suppose $\alpha_1 \neq 0$ or $\beta_1 \neq 0$. We say $\alpha_1 \neq 0$. (The case that $\beta_1 \neq 0$ is similar.) In this case,

$$\langle \bar{B}', \bar{C}' \rangle = \langle \bar{B}', \bar{C}' - \frac{\beta_1}{\alpha_1} \bar{B}' \rangle = \langle \bar{B}', (\gamma_2 Z - \gamma_3 Y) G \rangle,$$

where $\gamma_i = \beta_i - \frac{\beta_1}{\alpha_1} \alpha_i$, for $i = 2, 3$. So

$$\langle B', C', X \rangle = \langle \bar{B}', (\gamma_2 Z - \gamma_3 Y) G, X \rangle.$$

If (F_1', F_2') satisfies $\Delta = F_1' F_2' - F_1' F_2' H$, then

$$\begin{aligned} r = \deg(F_1' F_2' H) &= \sum_{p \in \mathbb{P}^2} \dim_{\mathbb{C}} \frac{\mathcal{O}_p}{\langle B', C', X \rangle} \\ &= \sum_{p \in H} \dim_{\mathbb{C}} \frac{\mathcal{O}_{p,H}}{\langle \bar{B}', (\gamma_2 Z - \gamma_3 Y) G \rangle}, \end{aligned}$$

which implies that there are r common zeros of $\bar{B}'$ and $(\gamma_2 Z - \gamma_3 Y) G = 0$. So there is at leat $r - 1$ common zeros of $\bar{B}'$ and G, which is denoted by Δ' as a zero dimensional subscheme. Note that

$$B' \in H^0(\mathcal{I}_{\Delta + \Delta'}(r+1)), \quad \Delta + \Delta' \subset F_1 F_2.$$

Since $\deg(F_1 F_2) = (r+1)^2$, $\deg(\Delta + \Delta') = r^2 + r + 1 + r - 1 = (r+1)^2 - 1$, we can choose a point $p \in H$ such that

$$p = F_1 F_2 - (\Delta + \Delta').$$

By the Cayley-Bacharach theorem (Corollary 2.7), we have

$$(*) \qquad h^0(\mathcal{I}_{\Delta + \Delta'}(r+1)) - h^0(\mathcal{I}_{F_1 F_2}(r+1)) = h^1(\mathcal{I}_p(r-2)) = 0, \quad \text{for } r \geq 2.$$

So $B' \in H^0(\mathcal{I}_{F_1 F_2}(r+1))$ which is generated by B, C. Thus $\alpha_1 = 0$, a contradiction. ☐

Remark 4.3. For $r = 1$, the lemma above is not true. In this case,

$$h^0(\mathcal{I}_{F_1 F_2 - p}(2)) - h^0(\mathcal{I}_{F_1 F_2}(2)) = h^1(\mathcal{I}_p(-1)) = 1.$$

So the equation $(*)$ in the proof of Lemma 4.2 above does not hold.

Proposition 4.4. *Suppose $\mathcal{F}$ is a foliation of degree $r (\neq 1)$ in $\mathbb{P}^2$. Then $\mathcal{F}$ is determined uniquely by its singular locus $\Delta = \mathrm{Sing}\mathcal{F}$.*

Proof. Firstly, by taking a coordinate transformation over X, we can assume $\Delta \cap (X = 0) = \emptyset$. Suppose $\mathcal{F}$ and $\mathcal{F}'$ are two different foliations such that $\Delta = \mathrm{Sing}\mathcal{F} = \mathrm{Sing}(\mathcal{F}')$. Then the foliation $\mathcal{F}$ (resp. $\mathcal{F}'$) corresponds to

$$(6) \qquad \omega = A \mathrm{d}X + B \mathrm{d}Y + C \mathrm{d}Z \quad (\text{resp. } \omega' = A' \mathrm{d}X + B' \mathrm{d}Y + C' \mathrm{d}Z),$$

where B, C (resp. B', C') have no common components and

$$XA + YB + ZC = 0 \quad (\text{resp. } XA' + YB' + ZC' = 0),$$

Then we can write Δ as

$$\Delta = F_1F_2 - F_1F_2H = F_1'F_2' - F_1'F_2'H,$$

where $F_1 = Z(B), F_2 = Z(C), F_1' = Z(B'), F_2' = Z(C')$ and $H = Z(X)$.

Secondly, by a coordinate transformation over Y, Z, we can write B, C as

$$B = XB_1 + ZG(Y,Z), \quad C = XC_1 - YG(Y,Z). \tag{7}$$

Then by Lemma 4.2, we can write

$$B' = \alpha_1 B + \alpha_2 C = X(\alpha_1 B_1 + \alpha_2 C_1) + (\alpha_1 Z - \alpha_2 Y)G,$$
$$C' = \gamma_1 B + \gamma_2 C = X(\gamma_1 B_1 + \gamma_2 C_1) + (\gamma_1 Z - \gamma_2 Y)G.$$

Since $X|(YB' + ZC')$, we have

$$0 = Y(\alpha_1 Z - \alpha_2 Y) + Z(\gamma_1 Z - \gamma_2 Y) = \gamma_1 Z^2 - \alpha_2 Y^2 + (\alpha_1 - \gamma_2)YZ,$$

which implies $\alpha_2 = \gamma_1 = 0$ and $\alpha_1 = \gamma_2 := \beta$. So

$$(A', B', C') = \beta \cdot (A, B, C)$$

which induces $\mathcal{F} = \mathcal{F}'$. □

4.2. Foliations of degree 1

Next we consider the case $r = 1$:

Lemma 4.5. *Let p be a point on $\mathbb{P}^2$ with $p \notin \Delta$. Then $\Delta + p$ is a complete intersection if and only if $h^0(\mathcal{I}_{\Delta'+p}(1)) = 0$ for any $\Delta' \subset \Delta$ with* $\deg \Delta' = 2$.

Proof. If $\Delta + p$ is a complete intersection, say F_1F_2, then by Corollary 2.7, we have

$$h^0(\mathcal{I}_{\Delta'+p}(1)) = h^0(\mathcal{I}_{F_1F_2}(1)) + h^1(\mathcal{I}_q) = 0,$$

where $\Delta' \subset \Delta$ with $\deg \Delta' = 2$ and $q = F_1F_2 - (\Delta' + p) = \Delta - \Delta'$.

Next we will show the "if" part. Consider the following exact sequence

$$0 \longrightarrow \mathcal{I}_{\Delta+p} \longrightarrow \mathcal{I}_\Delta \longrightarrow \mathcal{O}_p \longrightarrow 0,$$

and we get

$$h^0(\mathcal{I}_{\Delta+p}(2)) \geq h^0(\mathcal{I}_\Delta(2)) - 1 = 2.$$

So we can choose two elements $B, C \in H^0(\mathcal{I}_{\Delta+p}(2))$ such that $\{B, C\}$ spans a sub-vector space of dimension 2.

We claim that B, C have no common component. Otherwise, we write $B = B_1P$ and $C = C_1P$, for some P of $\deg P = 1$. Since $\deg(B_1C_1) = 1$ and $h^0(\mathcal{I}_\Delta(1)) = 0$ (by Corollary 3.5), we see that P must pass through a subscheme $\Delta' \subset \Delta$ with $\deg \Delta' = 2$, which is a contradiction with the assumption.

Let $F_1 = (B = 0), F_2 = (C = 0)$, then $F_1F_2 = \Delta + p$ is a complete intersection. □

Recall the classification of such Δ in [4], Section 4:

(a) Three different points q_0, q_1, q_2 in $\mathbb{P}^2$, not lying on the same line. Let L_1 (resp. L_2, L_3) denote the line $\overline{q_1q_2}$ (resp. $\overline{q_0q_2}$, $\overline{q_0q_1}$).

(b) Two points q_1, q_2 in $\mathbb{P}^2$, plus an infinitely near one q_0 over (say) q_2, and which does not correspond to the direction of the line joining q_1 with q_2. Let L'_1 (resp. L'_2) denote the line $\overline{q_1q_2}$ (resp. $\overline{q_0q_2}$).

(c) One single point q_2 in $\mathbb{P}^2$, plus an infinitely near one q_0 over q_2, plus another infinitely near one q_1 over q_0, which does not correspond neither to the direction of the exceptional divisor of the blow up of q_2, nor to the strict transform of the line joining q_2 with q_0 in $\mathbb{P}^2$. Let L''_1 (resp. L''_2) denote the line $\overline{q_0q_2}$ (resp. $\overline{q_1q_0}$).

Corollary 4.6. *For any point $p \in \mathbb{P}^2$ with $p \notin \Delta$, $\Delta + p$ is a complete intersection except in the following cases*:

(i) *Δ belongs to the case (a) and $p \in L_1$ or $p \in L_2$ or $p \in L_3$,*
(ii) *Δ belongs to the case (b) and $p \in L'_1$ or $p \in L'_2$,*
(iii) *Δ belongs to the case (c) and $p \in L''_1$ or $p \in L''_2$.*

For any foliation $\mathcal{F}$ on $\mathbb{P}^2$ such that $\mathrm{Sing}\mathcal{F} = \Delta$. By a coordinate transformation, we can assume that $(X = 0) \cap \Delta = \emptyset$. So $\mathcal{F}$ corresponds to

$$\omega = A\mathrm{d}X + B\mathrm{d}Y + C\mathrm{d}Z,$$

where B, C have no common component and $XA + YB + ZC = 0$. Let p be the unique zero of $B = C = X = 0$. So $F_1F_2H = p$, where $F_1 = (B = 0)$, $F_2 = (C = 0)$ and $H = (X = 0)$.

Note that such p determines a unique foliation. The proof is similar to that of Proposition 4.4: Indeed, we have equation (7), where now $G(Y, Z)$ is a polynomial of degree $r = 1$, and $Z(G)$ is the point p for this foliation $\mathcal{F}$. Say $G(Y, Z) = Y + aZ$, so that $p = [0, -a, 1]$, $a \neq 0$. It follows from Remark 4.3 that there exists $E \in H^0(\mathcal{I}_\Delta(2)) - H^0(\mathcal{I}_{F_1F_2}(2))$ such that $\{B, C, E\}$ spans $H^0(\mathcal{I}_\Delta(2))$. Hence, given a foliation $\mathcal{F}$ with $\mathcal{F}'$ with $\mathrm{Sing}\mathcal{F}' = \Delta$ as in equation (6), we can write its coefficients B' and C' as linear combinations of B, C and E and compute the point $p' = F_1F_2H$. It turns out that $p = p'$ if and only if $\omega' = \beta\omega$.

Hence the set of foliations $\mathcal{F}$ such that $\Delta = \mathrm{Sing}\mathcal{F}$, say $\mathfrak{M}(1, \Delta)$, is parameterized by the set

$$\mathfrak{S} = \{p \in H | \Delta + p \text{ is a complete intersection.}\}$$

By Corollary 4.4 above, we see that

(1) for the case (a), $\mathfrak{S} = H - \{q_1, q_2, q_3\} \cong \mathbb{C} - \{0, 1\}$, where $q_i = L_i \cap H$ for $i = 1, 2, 3$,

(2) for the case (b), $\mathfrak{S} = H - \{q'_1, q'_2\} \cong \mathbb{C} - \{0\}$, where $q'_i = L'_i \cap H$ for $i = 1, 2$,

(3) for the case (c), $\mathfrak{S} = H - \{q''_1, q''_2\} \cong \mathbb{C} - \{0\}$, where $q''_i = L''_i \cap H$ for $i = 1, 2$.

In [4], Section 4, the authors gave an algebraic characterization by discussing the 3 cases of Δ directly.

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Shi Xu
School of Mathematical Sciences
East China Normal University
Shanghai 200241, P. R. China
Email address: 572463515@qq.com