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WEAK BOUNDEDNESS FOR THE COMMUTATOR OF n-DIMENSIONAL ROUGH HARDY OPERATOR ON HOMOGENEOUS HERZ SPACES AND CENTRAL MORREY SPACES

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Abstract. In this paper, we study the boundedness of the commutator H^b_Ω formed by the rough Hardy operator H_Ω and a locally integrable function b from homogeneous Herz spaces to homogeneous weak Herz spaces. In addition, the weak boundedness of H^b_Ω on central Morrey spaces is also established.

1. Introduction

The classical Hardy operator, initially introduced by Hardy [19], was extended to the n-dimensional setting by Christ and Grafakos [5]:

$$Hf(x) := \frac{1}{|x|^n}\int_{|t|<|x|} f(t)dt,\ x\in\mathbb{R}^n\backslash\{0\},$$

where f is a locally integrable function on $\mathbb{R}^n$. The dual operator of H, denoted by H^*, is defined by

$$H^*f(x) = \int_{|t|\geq|x|}\frac{f(t)}{|t|^n}dt,\ x\in\mathbb{R}^n\backslash\{0\}.$$

Obviously, H and H^* satisfy

$$\int_{\mathbb{R}^n} g(x)Hf(x)dx = \int_{\mathbb{R}^n} f(x)H^*g(x)dx$$

for some suitable functions g.

It was proven in [5] that H is bounded on $L^p(\mathbb{R}^n)$, and so is H^* by duality. Hardy-type operators, as basic average operators, have wide applications in

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harmonic analysis and some related fields, see [2,6,15,16,20,29,31,33,37]. On the other hand, the study of the commutators has attracted much attention recently. In [11], the commutators of H and H^* are defined by

$$H_b f := b(Hf) - H(fb),$$

and

$$H_b^* f := b(H^* f) - H^*(fb),$$

respectively, where b is a locally integrable function on $\mathbb{R}^n$. The boundedness of H_b and H_b^* has been intensively studied, see e.g. [24,25]. Commonly, the symbol functions b in the commutators H_b and H_b^* are central bounded mean oscillation functions, since both H and H^* are centrosymmetric. Fu et al. [11] proved that H_b and H_b^* are bounded on $L^p(\mathbb{R}^n)$ if and only if $b \in \mathrm{CBMO}_{\max(p,p')}(\mathbb{R}^n)$, where $1 < p < \infty$. $\mathrm{CBMO}_p(\mathbb{R}^n)$ denotes the central bounded mean oscillation space introduced by Lu and Yang [26], which is given by the condition

$$\|f\|_{\mathrm{CBMO}_p} := \sup_{r>0} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^p dx \right)^{\frac{1}{p}} < \infty,$$

where $1 \le p < \infty$, and $B(0,r)$ denotes the ball centered at the origin with radius r. The space $\mathrm{CBMO}_p(\mathbb{R}^n)$ can be regarded as a local version of $\mathrm{BMO}(\mathbb{R}^n)$ at the origin. However, their properties may be quite different, since the absence of the famous John–Nirenberg inequality for the space $\mathrm{CBMO}_p(\mathbb{R}^n)$. Here, the space $\mathrm{BMO}(\mathbb{R}^n)$, initially introduced by Fefferman [7], is the bounded mean oscillation space defined similar to $\mathrm{CBMO}_p(\mathbb{R}^n)$, except that we take the supremum over all the balls in $\mathbb{R}^n$ instead of the balls centered at the origin.

Recently, the boundedness of H_b and H_b^* has been extended to several function spaces, such as central Morrey spaces [8,21,38] and homogeneous Herz spaces [10,11,36]. Moreover, the symbol functions b in H_b and H_b^* have been considered in different settings, such as λ-central bounded mean oscillation spaces [39], central Campanato spaces [32] and mixed central bounded mean oscillation spaces [36].

As is well known, the study of operators with rough kernels is an important branch in harmonic analysis. Inspired by the Calderón–Zygmund singular integral operator with rough kernels, Fu et al. [13] gave the definition of the n-dimensional rough Hardy operator H_Ω:

$$H_\Omega f(x) := \frac{1}{|x|^n} \int_{|t|<|x|} \Omega(x-t) f(t) dt, \ x \in \mathbb{R}^n \backslash \{0\},$$

where $\Omega \in L^s(S^{n-1})$ $(1 \le s < \infty)$ is homogeneous of degree zero. The commutator H_Ω^b formed by the n-dimensional rough Hardy operator H_Ω and a locally integrable function b was also defined in [13] as follows:

$$H_\Omega^b f(x) := \frac{1}{|x|^n} \int_{|t|<|x|} (b(x) - b(t)) \Omega(x-t) f(t) dt, \ x \in \mathbb{R}^n \backslash \{0\}.$$

The definitions of H_Ω^* and $H_\Omega^{b,*}$ can be formulated similarly, see [12]. When $\Omega \equiv 1$, we have $H_\Omega = H$ and $H_\Omega^b = H_b$. Fu et al. [13, Theorem 3.1] proved that H_Ω^b is bounded on $L^p(\mathbb{R}^n)$ for $b \in \mathrm{CBMO}_{\max(p,u)}(\mathbb{R}^n)$, where $1 < p < \infty$, $1/u = 1/p' - 1/s$ and $s > p'$. Besides, Fu et al. [12, Theorem 3.1] also established the boundedness of H_Ω^b and $H_\Omega^{b,*}$ on $L^p(\mathbb{R}^n)$ for $b \in \mathrm{CBMO}_{\max(p,u)}(\mathbb{R}^n)$, where $1 < p < \infty$ and $1/u = 1/p' - 1/s$ for some $s > 1$. Furthermore, the authors [13, Theorem 3.2] extended the boundedness of H_Ω^b to homogeneous Herz spaces (see Section 2 for the definition), which can be formulated as follows.

Proposition 1.1. *Suppose* $1 < q < \infty$, $0 < p_1 \le p_2 < \infty$ *and* $1/u = 1/q' - 1/s$. *Let* $s > q'$ *and* $b \in \mathrm{CBMO}_{\max(q,u)}(\mathbb{R}^n)$. *If* $\alpha < n/u$, *then* H_Ω^b *is bounded from* $\dot{K}_q^{\alpha,p_1}(\mathbb{R}^n)$ *to* $\dot{K}_q^{\alpha,p_2}(\mathbb{R}^n)$.

Note that the boundedness of H_Ω^b was also extended to central Morrey spaces in [13] and Morrey–Herz spaces in [14].

Recently, the commutators formed by $\mathrm{BMO}(\mathbb{R}^n)$ functions and some important operators in harmonic analysis have been proven to be weak bounded on several function spaces, see, for instance, [18, 34, 35]. To study the weak boundedness of H_b and H_b^*, Wang and Zhou [34] introduced the weak central bounded mean oscillation space $\mathrm{WCBMO}_p(\mathbb{R}^n)$. For $1 < p < \infty$, a locally integrable function f on $\mathbb{R}^n$ is said to belong to $\mathrm{WCBMO}_p(\mathbb{R}^n)$ if

$$\begin{aligned}&\|f\|_{\mathrm{WCBMO}_p}\\ &:= \sup_{r>0} \frac{1}{|B(0,r)|^{\frac{1}{p}}} \sup_{\eta>0} \eta |\{x \in B(0,r) : |f(x) - f_{B(0,r)}| > \eta\}|^{\frac{1}{p}} < \infty,\end{aligned}$$

where $B(0,r)$ is the ball centered at the origin with radius r. Briefly, $\mathrm{W}_p(\mathbb{R}^n) := \mathrm{WCBMO}_p(\mathbb{R}^n)$. Obviously, $\mathrm{CBMO}_p(\mathbb{R}^n) \subseteq \mathrm{W}_p(\mathbb{R}^n)$ for $1 < p < \infty$. Moreover, $\mathrm{W}_{p_2}(\mathbb{R}^n) \subseteq \mathrm{W}_{p_1}(\mathbb{R}^n)$ and the inclusion is proper if $1 < p_1 < p_2 < \infty$ by virtue of [34, Proposition 4.1]. Therefore, it is meaningful to consider the space $\mathrm{W}_p(\mathbb{R}^n)$. In [34, Theorem 5.1], the authors proved that H_b and H_b^* are bounded from $L^p(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$ if and only if $b \in \mathrm{CBMO}_{p'}(\mathbb{R}^n) \cap \mathrm{W}_p(\mathbb{R}^n)$, where $1 < p < \infty$ and $1/p + 1/p' = 1$. In [23, Theorem 3.1], we further extend this result by providing a similar characterization of the boundedness for H_b and H_b^* from central Morrey spaces to weak central Morrey spaces.

Inspired by [13, 23, 34], it is natural for us to consider the weak boundedness for the commutator of the rough Hardy operator H_Ω. More precisely, similar to Proposition 1.1, we give the sufficient conditions on the symbol b to guarantee the boundedness of H_Ω^b from homogeneous Herz spaces to homogeneous weak Herz spaces. In addition, we also obtain the boundedness of H_Ω^b from central Morrey spaces to weak central Morrey spaces. Throughout this paper, the letter C denotes constants which are independent of the main variables and may change from one occurrence to another.

2. Preliminaries and some lemmas

We first give some notations. Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $C_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Suppose $\chi_k = \chi_{C_k}$, where χ_E is the characteristic of set E. Following Lu and Yang [27], the homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is given by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) := \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}} := \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^q} \right\}^{\frac{1}{p}}$$

for $\alpha \in \mathbb{R}$ and $0 < p, q \le \infty$. The usual modifications are made when $p = \infty$ or $q = \infty$. Similar to the definition of weak Lebesgue spaces, Hu et al. [22] introduced the homogeneous weak Herz space $W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ endowed with the expression

$$\|f\|_{W\dot{K}_q^{\alpha,p}} := \sup_{\eta>0} \eta \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} |\{x \in C_k : |f(x)| > \eta\}|^{\frac{p}{q}} \right\}^{\frac{1}{p}} < \infty$$

for $\alpha \in \mathbb{R}$, $0 < q < \infty$ and $0 < p \le \infty$. The usual modifications are made when $p = \infty$.

Except for Herz spaces, Morrey spaces are also important extensions of Lebesgue spaces, which were introduced by Morrey [28] in 1938. Recently, the mapping properties of many important operators on Morrey-type spaces have been established, see, for instance, [3, 4, 9, 17, 30]. We now recall the definition of the central Morrey space $\dot{M}^{p,\lambda}(\mathbb{R}^n)$, which was introduced by Álvarez et al. [1]:

$$\dot{M}^{p,\lambda}(\mathbb{R}^n) := \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\dot{M}^{p,\lambda}} < \infty\},$$

where

$$\|f\|_{\dot{M}^{p,\lambda}} := \sup_{r>0} \frac{1}{|B(0,r)|^{\lambda}} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |f|^p dx \right)^{\frac{1}{p}}$$

for $1 < p < \infty$ and $-1/p \le \lambda < 0$. The weak central Morrey space $W\dot{M}^{p,\lambda}(\mathbb{R}^n)$ can be defined by

$$\|f\|_{W\dot{M}^{p,\lambda}} := \sup_{r>0} \frac{1}{|B(0,r)|^{\lambda+\frac{1}{p}}} \sup_{\eta>0} \eta |\{x \in B(0,r) : |f(x)| > \eta\}|^{\frac{1}{p}} < \infty$$

for $1 < p < \infty$ and $-1/p \le \lambda < 0$.

The following lemma gives the proper inclusion for $\mathrm{CBMO}_p(\mathbb{R}^n)$.

Lemma 2.1 ([11])**.** *If* $1 \le p < q < \infty$*, then* $\mathrm{CBMO}_q(\mathbb{R}^n) \subseteq \mathrm{CBMO}_p(\mathbb{R}^n)$ *and the inclusion is proper.*

We also have the following basic estimates for the space $\mathrm{CBMO}_1(\mathbb{R}^n)$ given by Fu et al. [11].

Lemma 2.2. *Suppose* $b \in \mathrm{CBMO}_1(\mathbb{R}^n)$ *and* $i, k \in \mathbb{Z}$. *Then*

$$|b(x) - b_{B_k}| \le |b(x) - b_{B_i}| + C|i-k|\|b\|_{\mathrm{CBMO}_1}.$$

3. Main theorems

Now we are in a position to present the weak boundedness of the commutator H^b_Ω. We can formulate the first main result as follows.

Theorem 3.1. *Suppose* $1 < q < \infty$, $0 < p_1 \le p_2 < \infty$, $s > q'$ *and* $1/u = 1/q' - 1/s$. *Let* $b \in \mathrm{CBMO}_u(\mathbb{R}^n) \cap \mathrm{W}_q(\mathbb{R}^n)$. *If* $\alpha < n/u$, *then* H^b_Ω *is bounded from* $\dot{K}^{\alpha,p_1}_q(\mathbb{R}^n)$ *to* $W\dot{K}^{\alpha,p_2}_q(\mathbb{R}^n)$.

Proof. For simplicity, we write

$$\sum_{i=-\infty}^{\infty} f(x)\chi_i(x) = \sum_{i=-\infty}^{\infty} f_i(x).$$

For $f \in \dot{K}^{\alpha,p_1}_q(\mathbb{R}^n)$, we deduce that

$$\begin{aligned}
&\eta^q|\{x \in C_k : |H^b_\Omega f(x)| > \eta\}| \\
&= \eta^q\left|\left\{x \in C_k : \left|\frac{1}{|x|^n}\int_{|t|<|x|}(b(x)-b(t))\Omega(x-t)f(t)dt\right| > \eta\right\}\right| \\
&\le C\eta^q\left|\left\{x \in C_k : 2^{-kn}\int_{B_k}|(b(x)-b(t))\Omega(x-t)f(t)|dt > \eta\right\}\right| \\
&\le C\eta^q\left|\left\{x \in C_k : 2^{-kn}\sum_{i=-\infty}^{k}\int_{C_i}|(b(x)-b(t))\Omega(x-t)f(t)|dt > \eta\right\}\right| \\
&\le C\eta^q\left|\left\{x \in C_k : 2^{-kn}\sum_{i=-\infty}^{k}\int_{C_i}|(b(x)-b_{B_k})\Omega(x-t)f(t)|dt > \frac{\eta}{2}\right\}\right| \\
&\quad + C\eta^q\left|\left\{x \in C_k : 2^{-kn}\sum_{i=-\infty}^{k}\int_{C_i}|(b(t)-b_{B_k})\Omega(x-t)f(t)|dt > \frac{\eta}{2}\right\}\right| \\
&=: I_1 + I_2.
\end{aligned}$$

For $x \in C_k$, $t \in C_i$ and $i \le k$, we have $0 \le |x-t| \le |x| + |t| \le 2^k + 2^i \le 2 \cdot 2^k = 2^{k+1}$, and then

$$\int_{C_i}|\Omega(x-t)|^s dt \le \int_0^{2^{k+1}}\int_{\mathbb{S}^{n-1}}|\Omega(x')|^s d\sigma(x')r^{n-1}dr \le C2^{kn}.$$

Note that $1/q + 1/s + 1/u = 1$, where $1/u = 1/q' - 1/s$. By Hölder's inequality, we get that

$$I_1 \le C\eta^q\left|\left\{x \in C_k : 2^{-kn}\sum_{i=-\infty}^{k}|b(x)-b_{B_k}|\int_{C_i}|\Omega(x-t)f(t)|dt > \frac{\eta}{2}\right\}\right|$$

$$\le C\eta^q \bigg|\bigg\{x \in C_k : 2^{-kn}|b(x)-b_{B_k}|$$
$$\times \sum_{i=-\infty}^{k} \left(\int_{C_i} |f(t)|^q dt\right)^{\frac{1}{q}} \left(\int_{C_i} |\Omega(x-t)|^s dt\right)^{\frac{1}{s}} |B_i|^{\frac{1}{u}} > \frac{\eta}{2}\bigg\}\bigg|$$
$$\le C\eta^q \bigg|\bigg\{x \in C_k : 2^{-kn}|b(x)-b_{B_k}| \sum_{i=-\infty}^{k} \|f_i\|_q 2^{\frac{kn}{s}} 2^{\frac{in}{u}} > \frac{\eta}{2}\bigg\}\bigg|$$
$$\le C\bigg(\sum_{i=-\infty}^{k} \|f_i\|_q 2^{-kn+\frac{kn}{s}+\frac{in}{u}} |B_k|^{\frac{1}{q}}\bigg)^q \frac{\eta^q}{|B_k|} |\{x \in B_k : |b(x)-b_{B_k}| > \eta\}|$$
$$\le C\|b\|^q_{\mathrm{W}_q} \bigg(\sum_{i=-\infty}^{k} 2^{\frac{(i-k)n}{u}} \|f_i\|_q\bigg)^q.$$

Applying Lemma 2.2, we have that

$$I_2 = C\int_{\{x\in C_k: 2^{-kn}\sum_{i=-\infty}^{k}\int_{C_i}|(b(t)-b_{B_k})\Omega(x-t)f(t)|dt > \frac{\eta}{2}\}} \eta^q dx$$
$$\le C\int_{C_k}\left(2^{-kn}\sum_{i=-\infty}^{k}\int_{C_i}|(b(t)-b_{B_k})\Omega(x-t)f(t)|dt\right)^q dx$$
$$\le C2^{-knq}\int_{C_k}\left(\sum_{i=-\infty}^{k}\int_{C_i}|(b(t)-b_{B_i})\Omega(x-t)f(t)|dt\right)^q dx$$
$$+ C2^{-knq}\|b\|^q_{\mathrm{CBMO}_1}\int_{C_k}\left(\sum_{i=-\infty}^{k}(k-i)\int_{C_i}|\Omega(x-t)f(t)|dt\right)^q dx$$
$$=: I_{21} + I_{22}.$$

By using Hölder's inequality with $1/q+1/s+1/u=1$, one has that

$$I_{21} \le C2^{-knq}\int_{C_k}\bigg\{\sum_{i=-\infty}^{k}\left(\int_{C_i}|f(t)|^q dt\right)^{\frac{1}{q}}$$
$$\times \left(\int_{C_i}|\Omega(x-t)|^s dt\right)^{\frac{1}{s}} \left(\int_{C_i}|b(t)-b_{B_i}|^u dt\right)^{\frac{1}{u}}\bigg\}^q dx$$
$$\le C2^{-knq}\int_{C_k}\bigg\{\sum_{i=-\infty}^{k}\|f_i\|_q 2^{\frac{kn}{s}}\left(\int_{C_i}|b(t)-b_{B_i}|^u dt\right)^{\frac{1}{u}}\bigg\}^q dx$$
$$\le C\bigg\{\sum_{i=-\infty}^{k} 2^{-kn}2^{\frac{kn}{q}}2^{\frac{kn}{s}}2^{\frac{in}{u}}\left(\frac{1}{|B_i|}\int_{B_i}|b(t)-b_{B_i}|^u dt\right)^{\frac{1}{u}}\|f_i\|_q\bigg\}^q$$
$$\le C\|b\|^q_{\mathrm{CBMO}_u}\bigg(\sum_{i=-\infty}^{k} 2^{\frac{(i-k)n}{u}}\|f_i\|_q\bigg)^q.$$

Similar to the estimate of I_{21}, using Lemma 2.1 and Hölder's inequality again, we conclude that

$$\begin{aligned} I_{22} &\leq C2^{-knq}\|b\|^q_{\mathrm{CBMO}_1} \\ &\quad\times \int_{C_k}\bigg\{\sum_{i=-\infty}^{k}(k-i)\bigg(\int_{C_i}|f(t)|^q dt\bigg)^{\frac{1}{q}}\bigg(\int_{C_i}|\Omega(x-t)|^s dt\bigg)^{\frac{1}{s}}|B_i|^{\frac{1}{u}}\bigg\}^q dx \\ &\leq C\|b\|^q_{\mathrm{CBMO}_1}\bigg(\sum_{i=-\infty}^{k}(k-i)2^{-kn}2^{\frac{kn}{q}}2^{\frac{kn}{s}}2^{\frac{in}{u}}\|f_i\|_q\bigg)^q \\ &\leq C\|b\|^q_{\mathrm{CBMO}_u}\bigg(\sum_{i=-\infty}^{k}(k-i)2^{\frac{(i-k)n}{u}}\|f_i\|_q\bigg)^q. \end{aligned}$$

In view of I_1, I_{21} and I_{22}, it is true for $0<p_1\leq p_2<\infty$ that

$$\begin{aligned} &\eta\bigg\{\sum_{k=-\infty}^{\infty}2^{k\alpha p_2}|\{x\in C_k:|H^b_\Omega f(x)|>\eta\}|^{\frac{p_2}{q}}\bigg\}^{\frac{1}{p_2}} \\ &=\bigg\{\sum_{k=-\infty}^{\infty}2^{k\alpha p_2}(\eta^q|\{x\in C_k:|H^b_\Omega f(x)|>\eta\}|)^{\frac{p_2}{q}}\bigg\}^{\frac{1}{p_2}} \\ &\leq\bigg\{\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}(\eta^q|\{x\in C_k:|H^b_\Omega f(x)|>\eta\}|)^{\frac{p_1}{q}}\bigg\}^{\frac{1}{p_1}} \\ &\leq C\bigg(\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\|b\|^{p_1}_{\mathrm{W}_q}\bigg(\sum_{i=-\infty}^{k}2^{\frac{(i-k)n}{u}}\|f_i\|_q\bigg)^{p_1}\bigg)^{\frac{1}{p_1}} \\ &\quad+C\bigg(\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\|b\|^{p_1}_{\mathrm{CBMO}_u}\bigg(\sum_{i=-\infty}^{k}2^{\frac{(i-k)n}{u}}\|f_i\|_q\bigg)^{p_1}\bigg)^{\frac{1}{p_1}} \\ &\quad+C\bigg(\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\|b\|^{p_1}_{\mathrm{CBMO}_u}\bigg(\sum_{i=-\infty}^{k}(k-i)2^{\frac{(i-k)n}{u}}\|f_i\|_q\bigg)^{p_1}\bigg)^{\frac{1}{p_1}} \\ &=: S. \end{aligned}$$

Therefore, we get

$$S\leq C\bigg(\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\bigg(\sum_{i=-\infty}^{k}(k-i)2^{\frac{(i-k)n}{u}}\|f_i\|_q\bigg)^{p_1}\bigg)^{\frac{1}{p_1}}.$$

When $0<p_1\leq 1$ and $\alpha<n/u$, we deduce that

$$S^{p_1}\leq C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\bigg(\sum_{i=-\infty}^{k}(k-i)2^{\frac{(i-k)n}{u}}\|f_i\|_q\bigg)^{p_1}$$

$$
\begin{aligned}
&= C \sum_{k=-\infty}^{\infty} \bigg(\sum_{i=-\infty}^{k} 2^{i\alpha} \|f_i\|_q (k-i) 2^{(i-k)(\frac{n}{u}-\alpha)} \bigg)^{p_1} \\
&\le C \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{k} 2^{i\alpha p_1} \|f_i\|_q^{p_1} (k-i)^{p_1} 2^{(i-k)(\frac{n}{u}-\alpha)p_1} \\
&= C \sum_{i=-\infty}^{\infty} 2^{i\alpha p_1} \|f_i\|_q^{p_1} \sum_{k=i}^{\infty} (k-i)^{p_1} 2^{(i-k)(\frac{n}{u}-\alpha)p_1} \\
&= C \|f\|_{\dot{K}_q^{\alpha,p_1}}^{p_1}.
\end{aligned}
$$

For $p_1 > 1$ and $\alpha < n/u$, it follows from Hölder's inequality that

$$
\begin{aligned}
S^{p_1} &\le C \sum_{k=-\infty}^{\infty} \bigg(\sum_{i=-\infty}^{k} 2^{i\alpha} \|f_i\|_q (k-i) 2^{(i-k)(\frac{n}{u}-\alpha)} \bigg)^{p_1} \\
&\le C \sum_{k=-\infty}^{\infty} \bigg(\sum_{i=-\infty}^{k} 2^{i\alpha p_1} \|f_i\|_q^{p_1} 2^{(i-k)(\frac{n}{u}-\alpha)\frac{p_1}{2}} \\
&\qquad \times \bigg(\sum_{i=-\infty}^{k} (k-i)^{p_1'} 2^{(i-k)(\frac{n}{u}-\alpha)\frac{p_1'}{2}} \bigg)^{\frac{p_1}{p_1'}} \bigg) \\
&= C \sum_{i=-\infty}^{\infty} 2^{i\alpha p_1} \|f_i\|_q^{p_1} \sum_{k=i}^{\infty} 2^{(i-k)(\frac{n}{u}-\alpha)\frac{p_1}{2}} \\
&= C \|f\|_{\dot{K}_q^{\alpha,p_1}}^{p_1}.
\end{aligned}
$$

As a consequence, we arrive at

$$S \le C \|f\|_{\dot{K}_q^{\alpha,p_1}}.$$

Thus, there holds

$$
\begin{aligned}
&\|H_\Omega^b f\|_{W\dot{K}_q^{\alpha,p_2}} \\
&= \sup_{\eta>0} \eta \bigg\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p_2} |\{x \in C_k : |H_\Omega^b f(x)| > \eta\}|^{\frac{p_2}{q}} \bigg\}^{\frac{1}{p_2}} \le C \|f\|_{\dot{K}_q^{\alpha,p_1}}. \qquad \square
\end{aligned}
$$

As for the boundedness of H_Ω^b from central Morrey spaces to weak central Morrey spaces, we have the following theorem.

Theorem 3.2. *Suppose $1 < p < \infty$, $-1/p \le \lambda < 0$, $s > p'$ and $1/u = 1/p' - 1/s$. Let $b \in \mathrm{CBMO}_u(\mathbb{R}^n) \cap \mathrm{W}_p(\mathbb{R}^n)$. Then H_Ω^b is bounded from $\dot{M}^{p,\lambda}(\mathbb{R}^n)$ to $W\dot{M}^{p,\lambda}(\mathbb{R}^n)$.*

Proof. For a fixed ball $B = B(0,r) \subset \mathbb{R}^n$, there is no loss of generality in assuming $B(0,r) = B_{k_0}$ with $k_0 \in \mathbb{Z}$. Keep in mind that

$$\sum_{i=-\infty}^{\infty} f(x)\chi_i(x) = \sum_{i=-\infty}^{\infty} f_i(x).$$

For $f \in \dot{M}^{p,\lambda}(\mathbb{R}^n)$, we have that

$$\begin{aligned}
&\frac{\eta^p}{|B_{k_0}|^{1+\lambda p}} |\{x \in B_{k_0} : |H_\Omega^b f(x)| > \eta\}| \\
&= \frac{\eta^p}{|B_{k_0}|^{1+\lambda p}} \left|\left\{x \in B_{k_0} : \left|\frac{1}{|x|^n} \int_{|t|<|x|} (b(x)-b(t))\Omega(x-t)f(t)dt\right| > \eta\right\}\right| \\
&\le \frac{C\eta^p}{|B_{k_0}|^{1+\lambda p}} \sum_{k=-\infty}^{k_0} \left|\left\{x \in C_k : 2^{-kn} \int_{B_k} |(b(x)-b(t))\Omega(x-t)f(t)|dt > \eta\right\}\right| \\
&\le \frac{C\eta^p}{|B_{k_0}|^{1+\lambda p}} \sum_{k=-\infty}^{k_0} \left|\left\{x \in C_k : 2^{-kn} \sum_{i=-\infty}^{k} \int_{C_i} |(b(x)-b(t))\Omega(x-t)f(t)|dt > \eta\right\}\right| \\
&\le \frac{C\eta^p}{|B_{k_0}|^{1+\lambda p}} \sum_{k=-\infty}^{k_0} \left|\left\{x \in C_k : 2^{-kn} \sum_{i=-\infty}^{k} \int_{C_i} |(b(x)-b_{B_k})\Omega(x-t)f(t)|dt > \frac{\eta}{2}\right\}\right| \\
&\quad + \frac{C\eta^p}{|B_{k_0}|^{1+\lambda p}} \sum_{k=-\infty}^{k_0} \left|\left\{x \in C_k : 2^{-kn} \sum_{i=-\infty}^{k} \int_{C_i} |(b(t)-b_{B_k})\Omega(x-t)f(t)|dt > \frac{\eta}{2}\right\}\right| \\
&=: I_1 + I_2.
\end{aligned}$$

Similar to the proof of Theorem 3.1, for $x \in C_k$, $t \in C_i$ and $i \le k$, there holds that $0 \le |x-t| \le 2^{k+1}$, and hence

$$\int_{C_i} |\Omega(x-t)|^s dt \le \int_0^{2^{k+1}} \int_{\mathbb{S}^{n-1}} |\Omega(x')|^s d\sigma(x') r^{n-1} dr \le C 2^{kn}.$$

Note that $1/p+1/s+1/u = 1$, where $1/u = 1/p' - 1/s$. The Hölder's inequality allows us to get that

$$\begin{aligned}
I_1 &\le \frac{C\eta^p}{|B_{k_0}|^{1+\lambda p}} \sum_{k=-\infty}^{k_0} \left|\left\{x \in C_k : 2^{-kn} \sum_{i=-\infty}^{k} |b(x)-b_{B_k}| \int_{C_i} |\Omega(x-t)f(t)|dt > \frac{\eta}{2}\right\}\right| \\
&\le \frac{C\eta^p}{|B_{k_0}|^{1+\lambda p}} \sum_{k=-\infty}^{k_0} \left|\left\{x \in C_k : 2^{-kn} |b(x)-b_{B_k}|\right.\right. \\
&\quad \left.\left. \times \sum_{i=-\infty}^{k} \left(\int_{C_i} |f(t)|^p dt\right)^{\frac{1}{p}} \left(\int_{C_i} |\Omega(x-t)|^s dt\right)^{\frac{1}{s}} |B_i|^{\frac{1}{u}} > \frac{\eta}{2}\right\}\right| \\
&\le \frac{C\eta^p}{|B_{k_0}|^{1+\lambda p}} \sum_{k=-\infty}^{k_0} \left|\left\{x \in C_k : 2^{-kn} |b(x)-b_{B_k}| \sum_{i=-\infty}^{k} \|f_i\|_p 2^{\frac{kn}{s}} 2^{\frac{in}{u}} > \frac{\eta}{2}\right\}\right| \\
&\le \frac{C}{|B_{k_0}|^{1+\lambda p}} \sum_{k=-\infty}^{k_0} \left(\sum_{i=-\infty}^{k} \|f_i\|_p 2^{-kn+\frac{kn}{s}+\frac{in}{u}} |B_k|^{\frac{1}{p}}\right)^p \frac{\eta^p}{|B_k|} |\{x \in B_k : |b(x)-b_{B_k}| > \eta\}| \\
&\le \frac{C}{|B_{k_0}|^{1+\lambda p}} \|b\|_{\mathrm{W}_p}^p \sum_{k=-\infty}^{k_0} \left(\sum_{i=-\infty}^{k} 2^{\frac{(i-k)n}{u}} \|f_i\|_p\right)^p.
\end{aligned}$$

Lemma 2.2 gives that

$$
\begin{aligned}
I_2 &= \frac{C}{|B_{k_0}|^{1+\lambda p}} \sum_{k=-\infty}^{k_0} \int_{\{x\in C_k: 2^{-kn}\sum_{i=-\infty}^{k}\int_{C_i}|(b(t)-b_{B_k})\Omega(x-t)f(t)|dt>\frac{\eta}{2}\}} \eta^p dx \\
&\le \frac{C}{|B_{k_0}|^{1+\lambda p}} \sum_{k=-\infty}^{k_0} \int_{C_k} \left(2^{-kn} \sum_{i=-\infty}^{k} \int_{C_i} |(b(t)-b_{B_k})\Omega(x-t)f(t)|dt\right)^p dx \\
&\le \frac{C}{|B_{k_0}|^{1+\lambda p}} \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{C_k} \left(\sum_{i=-\infty}^{k} \int_{C_i} |(b(t)-b_{B_i})\Omega(x-t)f(t)|dt\right)^p dx \\
&\quad + \frac{C}{|B_{k_0}|^{1+\lambda p}} \|b\|^p_{\mathrm{CBMO}_1} \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{C_k} \left(\sum_{i=-\infty}^{k} (k-i) \int_{C_i} |\Omega(x-t)f(t)|dt\right)^p dx \\
&=: I_{21} + I_{22}.
\end{aligned}
$$

For I_{21}, by using Hölder's inequality with $1/p+1/s+1/u=1$, we can obtain that

$$
\begin{aligned}
I_{21} &\le \frac{C}{|B_{k_0}|^{1+\lambda p}} \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{C_k} \left\{ \sum_{i=-\infty}^{k} \left(\int_{C_i} |f(t)|^p dt\right)^{\frac{1}{p}} \right. \\
&\qquad \times \left. \left(\int_{C_i} |\Omega(x-t)|^s dt\right)^{\frac{1}{s}} \left(\int_{C_i} |b(t)-b_{B_i}|^u dt\right)^{\frac{1}{u}} \right\}^p dx \\
&\le \frac{C}{|B_{k_0}|^{1+\lambda p}} \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{C_k} \left\{ \sum_{i=-\infty}^{k} \|f_i\|_p 2^{\frac{kn}{s}} 2^{\frac{in}{u}} \right. \\
&\qquad \times \left. \left(\frac{1}{|B_i|} \int_{B_i} |b(t)-b_{B_i}|^u dt\right)^{\frac{1}{u}} \right\}^p dx \\
&\le \frac{C}{|B_{k_0}|^{1+\lambda p}} \|b\|^p_{\mathrm{CBMO}_u} \sum_{k=-\infty}^{k_0} \left(\sum_{i=-\infty}^{k} 2^{-kn} 2^{\frac{kn}{p}} 2^{\frac{kn}{s}} 2^{\frac{in}{u}} \|f_i\|_p\right)^p \\
&= \frac{C}{|B_{k_0}|^{1+\lambda p}} \|b\|^p_{\mathrm{CBMO}_u} \sum_{k=-\infty}^{k_0} \left(\sum_{i=-\infty}^{k} 2^{\frac{(i-k)n}{u}} \|f_i\|_p\right)^p.
\end{aligned}
$$

For I_{22}, we use Lemma 2.1 and Hölder's inequality to get that

$$
\begin{aligned}
I_{22} &\le \frac{C}{|B_{k_0}|^{1+\lambda p}} \|b\|^p_{\mathrm{CBMO}_1} \sum_{k=-\infty}^{k_0} 2^{-knp} \\
&\quad \times \int_{C_k} \left(\sum_{i=-\infty}^{k} (k-i) \left(\int_{C_i} |f(t)|^p dt\right)^{\frac{1}{p}} \left(\int_{C_i} |\Omega(x-t)|^s dt\right)^{\frac{1}{s}} |B_i|^{\frac{1}{u}}\right)^p dx \\
&\le \frac{C}{|B_{k_0}|^{1+\lambda p}} \|b\|^p_{\mathrm{CBMO}_1} \sum_{k=-\infty}^{k_0} \left(\sum_{i=-\infty}^{k} (k-i) 2^{-kn} 2^{\frac{kn}{p}} 2^{\frac{kn}{s}} 2^{\frac{in}{u}} \|f_i\|_p\right)^p
\end{aligned}
$$

$$\leq \frac{C}{|B_{k_0}|^{1+\lambda p}}\|b\|^p_{\mathrm{CBMO}_u}\sum_{k=-\infty}^{k_0}\left(\sum_{i=-\infty}^{k}(k-i)2^{\frac{(i-k)n}{u}}\|f_i\|_p\right)^p.$$

Following the estimates of I_1, I_{21} and I_{22}, we only need to prove

$$\frac{1}{|B_{k_0}|^{1+\lambda p}}\sum_{k=-\infty}^{k_0}\left(\sum_{i=-\infty}^{k}(k-i)2^{\frac{(i-k)n}{u}}\|f_i\|_p\right)^p \leq C\|f\|^p_{\dot{M}^{p,\lambda}}.$$

Since $1<p<\infty$, a simple calculation yields that

$$\begin{aligned}
&\frac{1}{|B_{k_0}|^{1+\lambda p}}\sum_{k=-\infty}^{k_0}\left(\sum_{i=-\infty}^{k}(k-i)2^{\frac{(i-k)n}{u}}\|f_i\|_p\right)^p\\
&\leq \frac{1}{|B_{k_0}|^{1+\lambda p}}\sum_{k=-\infty}^{k_0}\left(\sum_{i=-\infty}^{k}\|f_i\|_p^p 2^{\frac{(i-k)np}{2u}}\times\left(\sum_{i=-\infty}^{k}(k-i)^{p'}2^{\frac{(i-k)np'}{2u}}\right)^{\frac{p}{p'}}\right)\\
&\leq \frac{C}{|B_{k_0}|^{1+\lambda p}}\sum_{i=-\infty}^{k_0}\|f_i\|_p^p\sum_{k=i}^{k_0}2^{\frac{(i-k)np}{2u}}\\
&\leq \frac{C}{|B_{k_0}|^{1+\lambda p}}\int_{\bigcup_{i=-\infty}^{k_0}(B_i\setminus B_{i-1})}|f(t)|^p dt\\
&= \frac{C}{|B_{k_0}|^{1+\lambda p}}\int_{B_{k_0}}|f(t)|^p dt\\
&\leq C\|f\|^p_{\dot{M}^{p,\lambda}}.
\end{aligned}$$

Therefore, we deduce

$$\begin{aligned}
&\|H^b_\Omega f\|^p_{W\dot{M}^{p,\lambda}}\\
&= \sup_{r>0}\frac{1}{|B(0,r)|^{1+\lambda p}}\sup_{\eta>0}\eta^p|\{x\in B(0,r):|H^b_\Omega f(x)|>\eta\}| \leq C\|f\|^p_{\dot{M}^{p,\lambda}}. \quad \square
\end{aligned}$$

By taken $\lambda=-1/p$ in Theorem 3.2, we can obtain the following corollary, which is also new and has its own interests in the study of the weak boundedness of operators.

Corollary 3.3. *Suppose* $1<p<\infty$ *and* $1/u=1/p'-1/s$. *Let* $s>p'$ *and* $b\in \mathrm{CBMO}_u(\mathbb{R}^n)\cap \mathrm{W}_p(\mathbb{R}^n)$. *Then* H^b_Ω *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^{p,\infty}(\mathbb{R}^n)$.

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