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PARAMETRIC EULER SUMS OF HARMONIC NUMBERS

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Abstract. In this paper, we define a parametric variant of generalized Euler sums and construct contour integration to give some explicit evaluations of these parametric Euler sums. In particular, we establish several explicit formulas of (Hurwitz) zeta functions, linear and quadratic parametric Euler sums. Furthermore, we also give an explicit evaluation of alternating double zeta values $\zeta(\overline{2j}; 2m+1)$ in terms of a combination of alternating Riemann zeta values by using the parametric Euler sums.

1. Introduction

We begin with some basic notations. Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}^-$ be the set of complex numbers, real numbers, integers, positive integers and negative integers, respectively, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{N}_0^- := \mathbb{N}^- \cup \{0\}$. For $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$, the *generalized harmonic number* $H_n^{(p)}$ is defined by

$$H_n^{(p)} := \sum_{k=1}^{n} \frac{1}{k^p} \quad \text{and} \quad H_0^{(p)} := 0. \tag{1}$$

If $p = 1$ then $H_n \equiv H_n^{(1)}$ is the *classical harmonic number*.

Between late 1742 and early 1743, Euler first touched on the subject of the *linear Euler sums* in a series of correspondence with Goldbach. In modern notation, these are defined as follows:

$$S_{p;q} := \sum_{n \geq k \geq 1} \frac{1}{k^p n^q} = \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q}, \tag{2}$$

where $p, q \in \mathbb{N}$ and $q \geq 2$ are to ensure convergence of the series.

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Euler returned to the same subject after about 30 years and discovered the now famous Euler's decomposition formula in [7]. More than two hundred years later, these objects were generalized to the so-called *generalized Euler sums* by Flajolet and Salvy [8]:

$$S_{\mathbf{p};q} := \sum_{n=1}^{\infty} \frac{H_n^{(p_1)} H_n^{(p_2)} \cdots H_n^{(p_r)}}{n^q}, \tag{3}$$

where $\mathbf{p} = (p_1, p_2, \ldots, p_r) \in \mathbb{N}^r$ with $p_1 \leq p_2 \leq \cdots \leq p_r$ and $q \in \mathbb{N} \setminus \{1\}$. The quantity $w := p_1 + \cdots + p_r + q$ is called the weight and the quantity r is called the degree (or order). Moreover, if $r > 1$ in (3), they were called *nonlinear Euler sums*. As usual, repeated summands in partitions are indicated by powers, so that for instance

$$S_{1^3 2^2 5;q} = S_{111225;q} = \sum_{n=1}^{\infty} \frac{H_n^3 [H_n^{(2)}]^2 H_n^{(5)}}{n^q}.$$

The Euler sums are in contrast to *multiple zeta values* (abbr. MZVs) defined in [9, 24] as follows:

$$\zeta(\boldsymbol{k}) \equiv \zeta(k_1, \ldots, k_r) := \sum_{n_1 > \cdots > n_r > 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}, \tag{4}$$

where $\boldsymbol{k} = (k_1, k_r, \ldots, k_r) \in \mathbb{N}^r$ and $k_1 > 1$ are to ensure convergence of the series. Here $|\boldsymbol{k}| := k_1 + \cdots + k_r$ and $\mathrm{dep}(\boldsymbol{k}) := r$ were called the depth and the weight of MZVs, respectively. Clearly, if $r = 1$ and $k_1 = k$, then it becomes the Riemann zeta value $\zeta(k)$ $(k \in \mathbb{N} \setminus \{1\})$. This theory indeed dates back to Hoffman [9] and Zagier [24] (independently at almost the same time), while recent research on this topic has been quite active. For instance, these quantities have appeared in several areas of mathematics and physics, and have a remarkable depth of algebraic structure in the past three decades (for example, see the book by Zhao [25]). Recently, several variants of classical multiple zeta values of level 2 called multiple t-values (abbr. MtVs), multiple T-values (abbr. MTVs) and multiple mixed values (abbr. MMVs) were introduced and studied in Hoffman [10], Kaneko-Tsumura [11] and Xu-Zhao [22]. It is clear that every MMV (MtV or MTV) can be written as a $\mathbb{Q}$-linear combination of colored MZVs of level two. The colored MZV (abbr. CMZV) of level N is defined for any $(k_1, \ldots, k_r) \in \mathbb{N}^r$ and Nth roots of unity $\eta_1, \ldots, \eta_r$ by (see Yuan-Zhao [23])

$$\mathrm{Li}_{k_1,\ldots,k_r}(\eta_1, \ldots, \eta_r) := \sum_{n_1 > \cdots > n_r > 0} \frac{\eta_1^{n_1} \cdots \eta_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}},$$

which converges if $(k_1, \eta_1) \neq (1, 1)$ (see [25, Ch. 15]), in which case we call $(\boldsymbol{k}, \boldsymbol{\eta})$ *admissible*. The study found that there should be rich connections between Euler sums and multiple zeta values.

For an early introduction and study on the evaluations of Euler sums, the readers may consult in Bailey-Borwein-Girgensohn's [2] and Flajolet-Salvy's paper [8], in which they have developed experimental method and a contour integral representation approach to the evaluation of Euler sums, respectively. For some recent progress of Euler sums, the readers are referred to [1, 4–6, 12–18, 21] and references therein. Recently, some parametric Euler sums were introduced and studied, see [3, 19, 20] and references therein. For example, in [3, Thm. 1], Borweins and Bradley proved the results

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{1}{n(n-a)} \sum_{k=1}^{n-1} \frac{1}{k-a}\\ &= \sum_{n=1}^{\infty} \frac{1}{n^2(n-a)} \quad (a \in \mathbb{C} \setminus \mathbb{N}).\end{aligned}$$

Setting $a = 0$, the well-known identity $\zeta(2,1) = \zeta(3)$ is obtained. In [19], Xu proved the formula ($a \in \mathbb{R} \setminus \mathbb{Z}$):

$$\begin{aligned}&\frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{(n^2-a^2)^2} - \left(\sum_{n=1}^{\infty} \frac{1}{n^2-a^2} \right)^2\\ &= 2a^2 \left\{ \sum_{n=1}^{\infty} \frac{1}{n^2-a^2} \sum_{n=1}^{\infty} \frac{1}{(n^2-a^2)^2} - \sum_{n=1}^{\infty} \frac{1}{(n^2-a^2)^3} \right\}.\end{aligned}$$

Setting $a = 0$ gives $\zeta(2)^2 = \frac{5}{2}\zeta(4)$. Therefore, it is possible to obtain many classical results of Euler sums and MZVs by studying the parametric Euler sums. In this paper, we will use the approach of contour integral representation to study the following (alternating) parametric Euler sums involving generalized harmonic numbers

$$\begin{aligned}&S^{\sigma}_{\mathbf{p};\mathbf{q}}(a_1, a_2, \ldots, a_m)\\ &:= \sum_{n=1}^{\infty} \frac{H_n^{(p_1)} H_n^{(p_2)} \cdots H_n^{(p_r)} \sigma^n}{(n+a_1)^{q_1} (n+a_2)^{q_2} \cdots (n+a_m)^{q_m}} \quad (a_1, \ldots, a_m \notin \mathbb{N}^-),\end{aligned} \tag{5}$$

where $\sigma \in \{\pm 1\}$, $\mathbf{p} = (p_1, \ldots, p_r) \in \mathbb{N}^r$ and $\mathbf{q} = (q_1, \ldots, q_m) \in \mathbb{N}_0^m$ with $q_1 + \cdots + q_m \geq 2$. Obviously, if $\sigma = 1, m = 1$ and $a_1 = 0, q_1 = q$ in (5), then it becomes the classical Euler sums $S_{\mathbf{p};q}$ in (3).

2. Main results

We define a complex kernel function $\xi(z)$ with two requirements: (i) $\xi(z)$ is meromorphic in the whole complex plane, (ii) $\xi(z)$ satisfies $\xi(z) = o(z)$ over an infinite collection of circles $|z| = \rho_k$ with $\rho_k \to \infty$. Applying these two conditions of kernel function $\xi(z)$, Flajolet and Salvy recalled the residue lemma under the premise of these two conditions concerning the kernel function $\xi(z)$.

Lemma 2.1. (cf. [8]) Let $\xi(z)$ be a kernel function and let $r(z)$ be a rational function which is $O(z^{-2})$ at infinity. Then

$$\sum_{\alpha\in E}\mathrm{Res}(r(z)\xi(z),\alpha)+\sum_{\beta\in S}\mathrm{Res}(r(z)\xi(z),\beta)=0,$$

where S is the set of poles of $r(z)$ and E is the set of poles of $\xi(z)$ that are not poles of $r(z)$. Here $\mathrm{Res}(r(z),\alpha)$ denotes the residue of $r(z)$ at $z=\alpha$.

Furthermore, Flajolet and Salvy [8, Eq. (2.4)] found the fact that any polynomial form in $\pi\cot\pi z$, $\frac{\pi}{\sin\pi z}$, $\psi^{(j)}(\pm z)$ is itself a kernel function with poles at a subset of the integers. Here, $\psi^{(j)}(z)$ stands for the polygamma function of order j defined as the $(j+1)$-st derivative of the logarithm of the gamma function:

$$\psi^{(j)}(z):=\frac{d^j}{dz^j}\psi(z)=\frac{d^{j+1}}{dz^{j+1}}\log\Gamma(z).$$

Thus

$$\psi^{(0)}(z)=\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$$

holds, where $\psi(x)$ is the digamma function and $\Gamma(z)$ is the usual gamma function. Observe that $\psi^{(j)}(z)$ satisfies the following relations:

$$\psi(z)=-\gamma+\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+z}\right),\ z\in\mathbb{C}\setminus\mathbb{N}_0^-,$$

$$\psi^{(j)}(z)=(-1)^{j+1}j!\sum_{k=0}^{\infty}1/(z+k)^{j+1},\ j\in\mathbb{N},\ z\in\mathbb{C}\setminus\mathbb{N}_0^-,$$

$$\psi(z+n)=\frac{1}{z}+\frac{1}{z+1}+\cdots+\frac{1}{z+n-1}+\psi(z),\ n\in\mathbb{N}.$$

In this context, γ denotes the Euler-Mascheroni constant, which is defined by

$$\gamma:=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)=-\psi(1)\approx 0.577215664901532860606512\cdots.$$

Moreover, from classical expansions and the properties of ψ function, they (see [8]) listed the following expressions of $\pi\cot\pi z$ and $\psi^{(j)}(-z)$ at an integer n.

Lemma 2.2. (cf. [8]) For integer p,

$$\pi\cot(\pi z)\overset{z\to n}{=}\frac{1}{z-n}-2\sum_{k=1}^{\infty}\zeta(2k)(z-n)^{2k-1}\quad(n\in\mathbb{Z}),\tag{6}$$

$$\frac{\pi}{\sin(\pi z)}\overset{z\to n}{=}(-1)^n\left(\frac{1}{z-n}+2\sum_{k=1}^{\infty}\bar{\zeta}(2k)(z-n)^{2k-1}\right)\quad(n\in\mathbb{Z}),\tag{7}$$

$$\frac{\psi^{(p-1)}(-z)}{(p-1)!}\overset{z\to n}{=}\frac{1}{(z-n)^p}\sum_{k=1}^{\infty}\binom{k+p-2}{p-1}\tag{8}$$

$$\times \left[(-1)^p\zeta(k+p-1)-(-1)^k H_n^{(k+p-1)}\right](z-n)^{k+p-1}$$
$$+\frac{1}{(z-n)^p} \quad (n\in\mathbb{N}_0),$$

$$\frac{\psi^{(p-1)}(-z)}{(p-1)!} \overset{z\to -n}{=} (-1)^p \sum_{k=1}^{\infty}\binom{p+k-2}{p-1}\left[\zeta(p+k-1)-H_{n-1}^{(p+k-1)}\right] \times (z+n)^{k-1} (n\in\mathbb{N}), \tag{9}$$

where $\zeta(1)$ should be interpreted as 0 and if $p=1$, replace $\psi(-z)$ by $\psi(-z)+\gamma$. $\bar\zeta(s)$ denotes the *alternating Riemann zeta function* which is defined by

$$\bar\zeta(s) := \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} \quad (\Re(s)>0). \tag{10}$$

In [8], Flajolet and Salvy used residue computations on large circular contour and specific functions to obtain more independent relations for classical Euler sums. These functions are of the form $\xi(z)r(z)$, where $r(z):=1/z^q$ and $\xi(z)$ is a product of cotangent (or cosecant) and polygamma function. In [19], Xu used the method of Flajolet and Salvy to obtain some explicit evaluations of parametric Euler sums. Next, we will use a similar method with the help of Lemmas 2.1 and 2.2 to give some explicit evaluations of the parametric Euler sums (5).

2.1. Linear parametric Euler sums

In this subsection, we apply the contour integral representation approach to consider the linear parametric Euler sums

$$\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{(n+a)(n+b)} \text{ and } \sum_{n=1}^{\infty}\frac{H_n^{(p)}}{(n+a)(n+b)}(-1)^n.$$

Theorem 2.3. For positive integer p and complexes a,b with $a\neq b$ and $a,b\in\mathbb{C}\setminus\mathbb{Z}$, we have

$$\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{(n+a)(n+b)}-(-1)^p\sum_{n=1}^{\infty}\frac{H_{n-1}^{(p)}}{(n-a)(n-b)}$$
$$=2\frac{(-1)^p}{b-a}\sum_{k=0}^{[p/2]}\zeta(2k)\Big(\zeta(p-2k+1,a)-\zeta(p-2k+1,b)\Big)$$
$$+\frac{(-1)^p}{b-a}\Big(\pi\cot(\pi a)\big(\zeta(p,a)-\zeta(p)\big)-\pi\cot(\pi b)\big(\zeta(p,b)-\zeta(p)\big)\Big), \tag{11}$$

where $\zeta(0):=-1/2$ and $\zeta(1,a):=-(\psi(a)+\gamma)$, and $\zeta(s,a)$ is *Hurwitz zeta function* defined by $(a\neq 0,-1,-2,-3,\ldots)$

$$\zeta(s,a) := \sum_{n=0}^{\infty}\frac{1}{(n+a)^s} \quad (\Re(s)>1).$$

Proof. Applying the method of contour integration in [19] (similar to the proof in [19, Thm. 2.4]), we need to consider the contour integral

$$\oint_{(\infty)} F(z)dz := \oint_{(\infty)} \frac{\pi\cot(\pi z)\psi^{(p-1)}(-z)}{(p-1)!(z+a)(z+b)}dz,$$

where $\oint_{(\infty)}$ denotes integral along large circles, that is, the limit of integrals $\oint_{|z|=R}$ as $R\to\infty$. Clearly, $\pi\cot(\pi z)\psi^{(p-1)}(-z)/(p-1)!$ is a kernel function. Hence, $\oint_{(\infty)} F(z)dz = 0$. Note that the function in the contour integration only have poles at $z=n$ and $-a,-b$. Applying Lemma 2.2, we can find that, at a nonnegative integer n, the pole has order $p+1$. Moreover, by a direct calculation, for $n\in\mathbb{N}_0$, we have

$$\begin{aligned} &F(z)\\ &\overset{z\to n}{=} \frac{1}{(z-n)^{p+1}}\\ &\quad\times\frac{1-2\sum\limits_{1\le k\le[p/2]}\zeta(2k)(z-n)^{2k}+\left((-1)^p\zeta(p)+H_n^{(p)}\right)(z-n)^p+o((z-n)^p)}{(z+a)(z+b)} \end{aligned}$$

and the residue is

$$\begin{aligned} &\mathrm{Res}(F(z),n)\\ &=\lim_{z\to n}\frac{1}{p!}\frac{d^p}{dz^p}\left\{(z-n)^{p+1}F(z)\right\}\\ &=2\frac{(-1)^p}{a-b}\sum_{k=0}^{[p/2]}\zeta(2k)\left(\frac{1}{(n+a)^{p-2k+1}}-\frac{1}{(n+b)^{p-2k+1}}\right)+\frac{(-1)^p\zeta(p)+H_n^{(p)}}{(n+a)(n+b)}. \end{aligned}$$

At a negative integer $-n$ and complexes $-a,-b$, the poles are simple and the residues are

$$\begin{aligned} \mathrm{Res}(F(z),-n)&=(-1)^p\frac{\zeta(p)-H_{n-1}^{(p)}}{(n-a)(n-b)},\\ \mathrm{Res}(F(z),-a)&=(-1)^p\frac{\pi\cot(\pi a)}{a-b}\zeta(p,a),\\ \mathrm{Res}(F(z),-b)&=(-1)^p\frac{\pi\cot(\pi b)}{b-a}\zeta(p,b). \end{aligned}$$

Hence, using Lemma 2.1, we have

$$\sum_{n=0}^{\infty}\mathrm{Res}(F(z),n)+\sum_{n=1}^{\infty}\mathrm{Res}(F(z),-n)+\mathrm{Res}(F(z),-a)+\mathrm{Res}(F(z),-b)=0.$$

Summing these four contributions yields the statement of the theorem. □

Corollary 2.4. (cf. [3,19]) For $a \in \mathbb{C} \setminus \mathbb{Z}$ and $m \in \mathbb{N}_0$,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n^{(2m+1)}}{n^2-a^2} &= \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^{2m+1}(n^2-a^2)} \\ &+ \frac{1}{2a}\sum_{k=0}^{m} \zeta(2k)\left(\zeta(2m-2k+2,a)-\zeta(2m-2k+2,-a)\right) \\ &+ \frac{1}{4a}\pi\cot(\pi a)\Big(\zeta(2m+1,a)+\zeta(2m+1,-a)-2\zeta(2m+1)\Big). \end{aligned} \tag{12}$$

Proof. Setting $b=-a$ and $p=2m+1$ in (11) yields the desired result. □

Corollary 2.5. For integer $m \geq 0$ and $a \in \mathbb{C} \setminus \mathbb{Z}$ with $a \neq 1/2$,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n^{(2m+1)}}{(n+a)(n+1-a)} &= \frac{1}{2a-1}\sum_{k=0}^{m} \zeta(2k)\Big(\zeta(2m-2k+2,a)-\zeta(2m-2k+2,1-a)\Big) \\ &+ \frac{\pi\cot(\pi a)}{2(2a-1)}\Big(\zeta(2m+1,a)+\zeta(2m+1,1-a)-2\zeta(2m+1)\Big). \end{aligned} \tag{13}$$

Proof. Setting $b=1-a$ and $p=2m+1$ in (11) yields the desired result. □

It is obvious that upon differentiating both members of (11) $k-1$ times with respect to a and $l-1$ times with respect to b, we obtain an explicit evaluation of the combined series

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{(n+a)^k(n+b)^l} - (-1)^{p+k+l}\sum_{n=1}^{\infty} \frac{H_{n-1}^{(p)}}{(n-a)^k(n-b)^l} \quad (k,l \geq 1).$$

For example, upon differentiating both members of (11) 1 times with respect to a and 1 times with respect to b, and noting the facts that $\psi^{(j)}(a) = (-1)^{j+1}j!\zeta(j+1,a)$ and $\zeta(1,a) := -(\psi(a)+\gamma)$, by a direct calculation, we deduce

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{H_n}{(n+a)^2(n+b)^2} + \sum_{n=1}^{\infty} \frac{H_{n-1}}{(n-a)^2(n-b)^2} \\ &= -\frac{2(-\psi^{(1)}(a)-\pi\cot(\pi a)(\psi(a)+\gamma)+\psi^{(1)}(b)+\pi\cot(\pi b)(\psi(b)+\gamma))}{(a-b)^3} \\ &\quad -\frac{\psi^{(2)}(a)+\pi\cot(\pi a)\psi^{(1)}(a)-\pi^2\csc^2(\pi a)(\psi(a)+\gamma)}{(a-b)^2} \\ &\quad -\frac{\psi^{(2)}(b)+\pi\cot(\pi b)\psi^{(1)}(b)-\pi^2\csc^2(\pi b)(\psi(b)+\gamma)}{(a-b)^2}. \end{aligned} \tag{14}$$

Then setting $b = 1 - a$ yields the following evaluation

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{H_n}{(n+a)^2(n+1-a)^2} \\
&= \frac{-\psi^{(1)}(1-a)+\psi^{(1)}(a)+\pi\cot(\pi a)(\psi(1-a)+\gamma)+\pi\cot(\pi a)(\psi(a)+\gamma)}{(2a-1)^3} \\
&\quad - \frac{\psi^{(2)}(1-a)-\pi\cot(\pi a)\psi^{(1)}(1-a)-\pi^2\csc^2(\pi a)(\psi(1-a)+\gamma)}{2(2a-1)^2} \\
&\quad - \frac{\psi^{(2)}(a)+\pi\cot(\pi a)\psi^{(1)}(a)-\pi^2\csc^2(\pi a)(\psi(a)+\gamma)}{2(2a-1)^2}. \qquad (15)
\end{aligned}$$

More general, we can get the following general theorem.

Theorem 2.6. For $p \in \mathbb{N}$,

$$\begin{aligned}
&\sum_{n=0}^{\infty}\Big((-1)^p\zeta(p)+H_n^{(p)}\Big)r(n)+(-1)^p\sum_{n=1}^{\infty}\Big(\zeta(p)-H_{n-1}^{(p)}\Big)r(-n) \\
&-2\sum_{k=0}^{[p/2]}\frac{1}{(p-2k)!}\zeta(2k)\sum_{n=0}^{\infty}r^{(p-2k)}(n)+\sum_{\beta\in S}\mathrm{Res}(f(z),\beta)=0, \qquad (16)
\end{aligned}$$

where $\zeta(0)$ and $\zeta(1)$ should be interpreted as $-1/2$ and 0 wherever they occur. $r^{(p)}(n)$ is defined as the p-st derivative of $r(z)$ with $z = n$, $r(z)$ is a rational function which is $O(z^{-2})$ at infinity, S is the set of poles of $r(z)$ and any integers n are not poles of $r(z)$. The function $f(z)$ is defined by

$$f(z) := \frac{\pi\cot(\pi z)\psi^{(p-1)}(-z)}{(p-1)!}r(z).$$

Proof. Considering the contour integral

$$\oint_{(\infty)} f(z)dz = 0.$$

By a similar argument as in the proof of Theorem 2.3, we obtain

$$\mathrm{Res}(f(z),n) = -2\sum_{k=0}^{[p/2]}\zeta(2k)\frac{r^{(p-2k)}(n)}{(p-2k)!}+\Big((-1)^p\zeta(p)+H_n^{(p)}\Big)r(n) \quad (n\in\mathbb{N}_0),$$

$$\mathrm{Res}(f(z),-n) = (-1)^p\Big(\zeta(p)-H_{n-1}^{(p)}\Big)r(-n) \quad (n\in\mathbb{N}).$$

Hence, applying Lemma 2.1, we have

$$\sum_{n=0}^{\infty}\mathrm{Res}(f(z),n)+\sum_{n=1}^{\infty}\mathrm{Res}(f(z),-n)+\sum_{\beta\in S}\mathrm{Res}(f(z),\beta)=0.$$

Thus, combining related identities yields the desired result. □

It is clear that the Theorem 2.3 follows immediately from Theorem 2.6 if we set $r(z) = 1/((z+a)(z+b))$.

Theorem 2.7. For positive integer p and complexes a, b with $a \neq b$ and $a, b \in \mathbb{C} \setminus \mathbb{Z}$, we have

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{(n+a)(n+b)}(-1)^n - (-1)^p \sum_{n=1}^{\infty} \frac{H_{n-1}^{(p)}}{(n-a)(n-b)}(-1)^n$$

$$= 2\frac{(-1)^p}{b-a} \sum_{k=0}^{[p/2]} \bar{\zeta}(2k)\Big(\bar{\zeta}(p-2k+1, b) - \bar{\zeta}(p-2k+1, a)\Big)$$

$$+ \frac{(-1)^p}{b-a}\pi \left(\frac{\zeta(p,a) - \zeta(p)}{\sin(\pi a)} - \frac{\zeta(p,b) - \zeta(p)}{\sin(\pi b)} \right), \tag{17}$$

where $\bar{\zeta}(0) := 1/2$ and $\bar{\zeta}(s, a)$ is *alternating Hurwitz zeta function* defined by ($a \in \mathbb{C} \setminus \mathbb{N}_0^-$)

$$\bar{\zeta}(s, a) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+a)^s} \quad (\Re(s) \geq 1).$$

Proof. The proof of Theorem 2.7 is similar to the proof of Theorem 2.3, thus we need to consider the contour integral

$$\oint_{(\infty)} G(z)dz := \oint_{(\infty)} \frac{\pi \psi^{(p-1)}(-z)}{(p-1)! \sin(\pi z)(z+a)(z+b)} dz.$$

Clearly, $\pi\psi^{(p-1)}(-z)/((p-1)!\sin(\pi z))$ is a kernel function. Hence,

$$\oint_{(\infty)} G(z)dz = 0.$$

Note that the function in the contour integral only has poles at $z = n$ and $-a, -b$. Applying Lemma 2.2, we can find that, at a nonnegative integer n, the pole has order $p+1$. Moreover, by a direct calculation, for $n \in \mathbb{N}_0$, we have

$$G(z) \overset{z \to n}{=} \frac{(-1)^n}{(z-n)^{p+1}} \times \frac{1 + 2\sum\limits_{1 \leq k \leq [p/2]} \bar{\zeta}(2k)(z-n)^{2k} + \Big((-1)^p \zeta(p) + H_n^{(p)}\Big)(z-n)^p + o((z-n)^p)}{(z+a)(z+b)}$$

and the residue is

$$\begin{aligned} \operatorname{Res}(G(z), n) &= \lim_{z \to n} \frac{1}{p!} \frac{d^p}{dz^p} \left\{ (z-n)^{p+1} G(z) \right\} \\ &= 2\frac{(-1)^p}{b-a} \sum_{k=0}^{[p/2]} \bar{\zeta}(2k) \left(\frac{(-1)^n}{(n+a)^{p-2k+1}} - \frac{(-1)^n}{(n+b)^{p-2k+1}} \right) \\ &\quad + \frac{(-1)^p \zeta(p) + H_n^{(p)}}{(n+a)(n+b)}(-1)^n. \end{aligned}$$

At a negative integer $-n$ and complexes $-a, -b$, the poles are simple and residues are

$$\operatorname{Res}(G(z), -n) = (-1)^p \frac{\zeta(p) - H_{n-1}^{(p)}}{(n-a)(n-b)}(-1)^n,$$
$$\operatorname{Res}(G(z), -a) = (-1)^p \frac{\pi\zeta(p,a)}{\sin(\pi a)(a-b)},$$
$$\operatorname{Res}(G(z), -b) = (-1)^p \frac{\pi\zeta(p,b)}{\sin(\pi b)(b-a)}.$$

Hence, using Lemma 2.1, we have

$$\sum_{n=0}^{\infty} \operatorname{Res}(G(z), n) + \sum_{n=1}^{\infty} \operatorname{Res}(G(z), -n) + \operatorname{Res}(G(z), -a) + \operatorname{Res}(G(z), -b) = 0.$$

Noting the fact that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+a)(n+b)} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-a)(n-b)} = \frac{\pi}{b-a}\left(\frac{1}{\sin(\pi a)} - \frac{1}{\sin(\pi b)}\right).$$

Summing these four contributions yields the statement of the theorem. □

Corollary 2.8. For $a \in \mathbb{C} \setminus \mathbb{Z}$ and $m \in \mathbb{N}_0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{H_n^{(2m+1)}}{n^2 - a^2}(-1)^n \\
&= \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2m+1}(n^2-a^2)} \\
&\quad + \frac{1}{2a}\sum_{k=0}^{m} \bar{\zeta}(2k)\left(\bar{\zeta}(2m-2k+2, -a) - \bar{\zeta}(2m-2k+2, a)\right) \\
&\quad + \frac{\pi}{4a\sin(\pi a)}\Big(\zeta(2m+1, a) + \zeta(2m+1, -a) - 2\zeta(2m+1)\Big). \qquad (18)
\end{aligned}$$

Proof. Setting $b = -a$ and $p = 2m+1$ in (17) yields the desired result. □

Corollary 2.9. For integer $m \geq 1$ and $a \in \mathbb{C} \setminus \mathbb{Z}$ with $a \neq 1/2$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{H_n^{(2m)}}{(n+a)(n+1-a)}(-1)^n \\
&= \frac{1}{1-2a}\sum_{k=0}^{m} \bar{\zeta}(2k)\Big(\bar{\zeta}(2m-2k+2, 1-a) - \bar{\zeta}(2m-2k+2, a)\Big) \\
&\quad + \frac{\pi}{2(1-2a)} \frac{\zeta(2m, a) - \zeta(2m, 1-a)}{\sin(\pi a)}. \qquad (19)
\end{aligned}$$

Proof. Setting $b = 1-a$ and $p = 2m$ in (17) yields the desired result. □

Similar to Theorem 2.6 , we can get the following general theorem.

Theorem 2.10. For $p \in \mathbb{N}$,

$$\begin{aligned}
&\sum_{n=0}^{\infty}\Big((-1)^p\zeta(p)+H_n^{(p)}\Big)(-1)^n r(n) \\
&+(-1)^p\sum_{n=1}^{\infty}\Big(\zeta(p)-H_{n-1}^{(p)}\Big)(-1)^n r(-n) \\
&+2\sum_{k=0}^{[p/2]}\frac{1}{(p-2k)!}\bar{\zeta}(2k)\sum_{n=0}^{\infty}(-1)^n r^{(p-2k)}(n) \\
&+\sum_{\beta\in T}\mathrm{Res}(g(z),\beta)=0, \qquad (20)
\end{aligned}$$

where $\zeta(0)$ and $\zeta(1)$ should be interpreted as $-1/2$ and 0 wherever they occur. $r^{(p)}(n)$ is defined as the p-th derivative of $r(z)$ with $z=n$, $r(z)$ is a rational function which is $O(z^{-2})$ at infinity, T is the set of poles of $r(z)$ and any integers n are not poles of $r(z)$. The function $g(z)$ is defined by

$$g(z):=\frac{\pi\psi^{(p-1)}(-z)}{(p-1)!\sin(\pi z)}r(z).$$

Proof. Consider the contour integral

$$\oint\limits_{(\infty)} g(z)dz=0.$$

By a similar argument as in the proof of Theorem 2.7, we obtain

$$\begin{aligned}
\mathrm{Res}(g(z),n)&=2\sum_{k=0}^{[p/2]}\bar{\zeta}(2k)\frac{r^{(p-2k)}(n)}{(p-2k)!}(-1)^n+\Big((-1)^p\zeta(p)+H_n^{(p)}\Big)(-1)^n r(n) \\
&\quad (n\in\mathbb{N}_0), \\
\mathrm{Res}(g(z),-n)&=(-1)^p\Big(\zeta(p)-H_{n-1}^{(p)}\Big)(-1)^n r(-n) \quad (n\in\mathbb{N}).
\end{aligned}$$

Hence, applying Lemma 2.1, we have

$$\sum_{n=0}^{\infty}\mathrm{Res}(g(z),n)+\sum_{n=1}^{\infty}\mathrm{Res}(g(z),-n)+\sum_{\beta\in T}\mathrm{Res}(g(z),\beta)=0.$$

Thus, combining related identities yields the desired result. □

Hence, Theorem 2.7 follows immediately from Theorem 2.10 if we set $r(z)=1/((z+a)(z+b))$.

In [3], D. Borwein, J.M. Borwein and D.M. Bradley used the evaluation (12) to obtain an explicit formula of double zeta values $\zeta(2j;2m+1)$ $(j\in\mathbb{N}, m\in\mathbb{N}_0)$ by using power series expansions and comparing the coefficients on both sides. Similarly, applying (18), we can also get the following corollary.

Corollary 2.11. For $j \in \mathbb{N}$ and $m \in \mathbb{N}_0$,

$$
\begin{aligned}
\zeta(\overline{2j}; 2m+1) = {} & \frac{1}{2}\bar{\zeta}(2m+2j+1) \\
& - \sum_{k=0}^{m} \binom{2j+2m-2k}{2j-1} \bar{\zeta}(2k)\bar{\zeta}(2m+2j+1-2k) \\
& + \sum_{l=0}^{j-1} \binom{2j+2m-2l}{2m} \bar{\zeta}(2l)\zeta(2m+2j+1-2l), \qquad (21)
\end{aligned}
$$

where $\zeta(\overline{2j}; 2m+1)$ is an alternating double zeta values defined by

$$
\zeta(\overline{2j}; 2m+1) := \sum_{n>k\geq 1} \frac{(-1)^n}{n^{2j} k^{2m+1}} = \sum_{n=1}^{\infty} \frac{H_{n-1}^{(2m+1)}}{n^{2j}} (-1)^n. \qquad (22)
$$

Proof. By direct calculations, for $|a| < 1$, we have

$$
\begin{aligned}
& \sum_{n=1}^{\infty} \frac{H_{n-1}^{(2m+1)}}{n^2 - a^2} (-1)^n = \sum_{j=1}^{\infty} \zeta(\overline{2j}; 2m+1) a^{2j-2}, \\
& \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2m+1}(n^2 - a^2)} = -\sum_{j=1}^{\infty} \bar{\zeta}(2m+2j+1) a^{2j-2}, \\
& \frac{\pi}{\sin(\pi a)} = \frac{1}{a} + 2a \sum_{j=1}^{\infty} \bar{\zeta}(2j) a^{2j-2}, \\
& \zeta(2m+1, a) + \zeta(2m+1, 1-a) - 2\zeta(2m+1) \\
= {} & 2 \sum_{j=1}^{\infty} \binom{2j+2m}{2j} \zeta(2m+2j+1) a^{2j}, \\
& \bar{\zeta}(2m-2k+2, -a) - \bar{\zeta}(2m-2k+2, a) \\
= {} & -2 \sum_{j=1}^{\infty} \binom{2j+2m-2k}{2j-1} \bar{\zeta}(2m+2j+1-2k) a^{2j-1}.
\end{aligned}
$$

Then, using (18) gives

$$
\begin{aligned}
& \sum_{j=1}^{\infty} \zeta(\overline{2j}; 2m+1) a^{2j-2} \\
= {} & \frac{1}{2} \sum_{j=1}^{\infty} \bar{\zeta}(2m+2j+1) a^{2j-2} \\
& - \sum_{j=1}^{\infty} \left\{ \sum_{k=0}^{m} \binom{2j+2m-2k}{2j-1} \bar{\zeta}(2k)\bar{\zeta}(2m+2j+1-2k) \right\} a^{2j-2}
\end{aligned}
$$

$$+\frac{1}{2}\sum_{j=1}^{\infty}\binom{2j+2m}{2j}\zeta(2m+2j+1)a^{2j-2}$$
$$+\sum_{j=1}^{\infty}\sum_{\substack{j_1+j_2=j,\\ j_1,j_2\geq 1}}\binom{2j_2+2m}{2j_2}\bar{\zeta}(2j_1)\zeta(2m+2j_2+1)a^{2j-2}.$$

Thus, comparing the coefficients of a^{2j-2} in above identity yields the desired evaluation. □

2.2. Quadratic parametric Euler sums

Now, we give some evaluations of quadratic parametric Euler sums by using the method of contour integral.

Theorem 2.12. For $p,m\in\mathbb{N}$ and $a,b\in\mathbb{C}\setminus\mathbb{Z}$ with $a\neq b$,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{H_n^{(p)}H_n^{(m)}}{(n+a)(n+b)}+(-1)^{p+m}\sum_{n=1}^{\infty}\frac{H_{n-1}^{(p)}H_{n-1}^{(m)}}{(n-a)(n-b)}\\
&+(-1)^m\zeta(m)\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{(n+a)(n+b)}+(-1)^p\zeta(p)\sum_{n=1}^{\infty}\frac{H_n^{(m)}}{(n+a)(n+b)}\\
&-(-1)^{p+m}\zeta(m)\sum_{n=1}^{\infty}\frac{H_{n-1}^{(p)}}{(n-a)(n-b)}-(-1)^{p+m}\zeta(p)\sum_{n=1}^{\infty}\frac{H_{n-1}^{(m)}}{(n-a)(n-b)}\\
&+(-1)^{p+m}\frac{\pi\cot(\pi b)}{b-a}\left\{\zeta(p,b)\zeta(m,b)-\zeta(p)\zeta(m)\right\}\\
&-(-1)^{p+m}\frac{\pi\cot(\pi a)}{b-a}\left\{\zeta(p,a)\zeta(m,a)-\zeta(p)\zeta(b)\right\}\\
&-2\frac{(-1)^{p+m}}{b-a}\sum_{k=0}^{[(p+m)/2]}\zeta(2k)\left\{\zeta(p+m-2k+1,a)-\zeta(p+m-2k+1,b)\right\}\\
&+2\frac{(-1)^{p+m}}{b-a}\sum_{\substack{2k_1+k_2\leq m+1,\\ k_1\geq 0,k_2\geq 1}}(-1)^{k_2}\binom{k_2+p-2}{p-1}\zeta(2k_1)\\
&\times\left\{\begin{array}{l}\zeta(k_2+p-1)\left[\zeta(m-2k_1-k_2+2,a)-\zeta(m-2k_1-k_2+2,b)\right]\\ -(-1)^{p+k_2}\sum\limits_{n=0}^{\infty}\left[\frac{H_n^{(k_2+p-1)}}{(n+a)^{m-2k_1-k_2+2}}-\frac{H_n^{(k_2+p-1)}}{(n+b)^{m-2k_1-k_2+2}}\right]\end{array}\right\}\\
&+2\frac{(-1)^{p+m}}{b-a}\sum_{\substack{2k_1+k_2\leq p+1,\\ k_1\geq 0,k_2\geq 1}}(-1)^{k_2}\binom{k_2+m-2}{m-1}\zeta(2k_1)\\
&\times\left\{\begin{array}{l}\zeta(k_2+m-1)\left[\zeta(p-2k_1-k_2+2,a)-\zeta(p-2k_1-k_2+2,b)\right]\\ -(-1)^{m+k_2}\sum\limits_{n=0}^{\infty}\left[\frac{H_n^{(k_2+m-1)}}{(n+a)^{p-2k_1-k_2+2}}-\frac{H_n^{(k_2+m-1)}}{(n+b)^{p-2k_1-k_2+2}}\right]\end{array}\right\}\\
&=0,
\end{aligned}\tag{23}$$

where $\zeta(1)$ and $\zeta(0)$ should be interpreted as 0 and $-1/2$, respectively, wherever they occur. $\zeta(1,a):=-(\psi(a)+\gamma)$.

Proof. Just as demonstrated in the proof of Theorem 2.3, we consider the contour integral

$$\oint\limits_{(\infty)} H(z)dz := \oint\limits_{(\infty)} \frac{\pi\cot(\pi z)\psi^{(p-1)}(-z)\psi^{(m-1)}(-z)}{(z+a)(z+b)(p-1)!(m-1)!}dz = 0.$$

Observe that $H(z)$ has poles only at $-a, -b$ and the integers. Applying Lemma 2.2, we can find that, at a nonnegative integer n, the pole has order $p+m+1$. Moreover, by a straightforward calculation, for $n \in \mathbb{N}_0$, we have

$$\begin{aligned}
&H(z)\\
&\overset{z\to n}{=} \frac{1}{(z-n)^{p+m+1}}\frac{1}{(z+a)(z+b)}\\
&\times\left\{\begin{array}{l}
1-2\sum\limits_{k=1}^{[(p+m)/2]}\zeta(2k)(z-n)^{2k}\\
+\sum\limits_{k=1}^{m+1}\binom{k+p-2}{p-1}\left[(-1)^p\zeta(k+p-1)-(-1)^kH_n^{(k+p-1)}\right](z-n)^{k+p-1}\\
-2\sum\limits_{\substack{2k_1+k_2\le m+1,\\ k_1,k_2\ge 1}}\binom{k_2+p-2}{p-1}\zeta(2k_1)\left[(-1)^p\zeta(k_2+p-1)\right.\\
\left.-(-1)^{k_2}H_n^{(k_2+p-1)}\right]\times(z-n)^{2k_1+k_2+p-1}\\
+\sum\limits_{k=1}^{p+1}\binom{k+m-2}{m-1}\left[(-1)^m\zeta(k+m-1)-(-1)^kH_n^{(k+m-1)}\right]\\
\times(z-n)^{k+m-1}\\
-2\sum\limits_{\substack{2k_1+k_2\le p+1,\\ k_1,k_2\ge 1}}\binom{k_2+m-2}{m-1}\zeta(2k_1)\left[(-1)^m\zeta(k_2+m-1)\right.\\
\left.-(-1)^{k_2}H_n^{(k_2+m-1)}\right](z-n)^{2k_1+k_2+m-1}\\
+\left[(-1)^p\zeta(p)+H_n^{(p)}\right]\left[(-1)^m\zeta(m)+H_n^{(m)}\right](z-n)^{p+m}\\
+o((z-n)^{p+m})
\end{array}\right.
\end{aligned}$$

and the residue is

$$\begin{aligned}
&\operatorname{Res}[H(z),n]\\
&=\frac{1}{(p+m)!}\lim_{z\to n}\frac{d^{p+m}}{dz^{p+m}}\left\{(z-n)^{p+m+1}H(z)\right\}\\
&=\frac{(-1)^{p+m}}{b-a}\left\{\frac{1}{(n+a)^{p+m+1}}-\frac{1}{(n+b)^{p+m+1}}\right\}\\
&\quad-2\frac{(-1)^{p+m}}{b-a}\sum_{k=1}^{[(p+m)/2]}\zeta(2k)\left\{\frac{1}{(n+a)^{p+m-2k+1}}-\frac{1}{(n+b)^{p+m-2k+1}}\right\}\\
&\quad-\frac{(-1)^{p+m}}{b-a}\sum_{k=1}^{m+1}\binom{k+p-2}{p-1}\left[(-1)^k\zeta(k+p-1)-(-1)^pH_n^{(k+p-1)}\right]
\end{aligned}$$

$$\times \left[\frac{1}{(n+a)^{m-k+2}} - \frac{1}{(n+b)^{m-k+2}}\right]$$
$$+ 2\frac{(-1)^{p+m}}{b-a} \sum_{\substack{2k_1+k_2\le m+1,\\ k_1,k_2\ge 1}} \binom{k_2+p-2}{p-1}(-1)^{2k_1+k_2}\zeta(2k_1)$$
$$\times \left[\zeta(k_2+p-1) - (-1)^{p+k_2} H_n^{(k_2+p-1)}\right]$$
$$\times \left[\frac{1}{(n+a)^{m-2k_1-k_2+2}} - \frac{1}{(n+b)^{m-2k_1-k_2+2}}\right]$$
$$- \frac{(-1)^{p+m}}{b-a} \sum_{k=1}^{p+1} \binom{k+m-2}{m-1} \left[(-1)^k \zeta(k+m-1) - (-1)^m H_n^{(k+m-1)}\right]$$
$$\times \left[\frac{1}{(n+a)^{p-k+2}} - \frac{1}{(n+b)^{p-k+2}}\right]$$
$$+ 2\frac{(-1)^{p+m}}{b-a} \sum_{\substack{2k_1+k_2\le m+1,\\ k_1,k_2\ge 1}} \binom{k_2+m-2}{m-1}(-1)^{2k_1+k_2}\zeta(2k_1)$$
$$\times \left[\zeta(k_2+m-1) - (-1)^{m+k_2} H_n^{(k_2+m-1)}\right]$$
$$\times \left[\frac{1}{(n+a)^{p-2k_1-k_2+2}} - \frac{1}{(n+b)^{p-2k_1-k_2+2}}\right]$$
$$+ \frac{(-1)^{p+m}\zeta(p)\zeta(m) + (-1)^p\zeta(p)H_n^{(m)} + (-1)^m\zeta(m)H_n^{(p)} + H_n^{(p)}H_n^{(m)}}{(n+a)(n+b)}.$$

At a negative integer $-n$ and reals $-a, -b$, the poles are simple and residues are

$$\mathrm{Res}[H(z), -n] = (-1)^{p+m}\frac{\zeta(p)\zeta(m) - \zeta(p)H_{n-1}^{(m)} - \zeta(m)H_{n-1}^{(p)} + H_{n-1}^{(p)}H_{n-1}^{(m)}}{(n-a)(n-b)},$$
$$\mathrm{Res}[H(z), -a] = -(-1)^{p+m}\frac{\pi\cot(\pi a)}{b-a}\zeta(p,a)\zeta(m,a),$$
$$\mathrm{Res}[H(z), -b] = (-1)^{p+m}\frac{\pi\cot(\pi b)}{b-a}\zeta(p,b)\zeta(m,b).$$

Hence, using Lemma 2.1, we have

$$\sum_{n=0}^{\infty}\mathrm{Res}(H(z), n) + \sum_{n=1}^{\infty}\mathrm{Res}(H(z), -n) + \mathrm{Res}(H(z), -a) + \mathrm{Res}(H(z), -b) = 0.$$

Summing these four contributions yields the statement of the theorem. □

Letting $(p, m) = (1, 1)$ and $(2, 2)$ in Theorem 2.12, we can get the following corollaries.

Corollary 2.13. Let $a, b \in \mathbb{C} \setminus \mathbb{Z}$ with $a \neq b$. Then

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{H_n^2}{(n+a)(n+b)} + \sum_{n=1}^{\infty} \frac{H_{n-1}^2}{(n-a)(n-b)} + \frac{\pi \cot(\pi b)}{b-a} (\psi(b)+\gamma)^2 \\
&- \frac{\pi \cot(\pi a)}{b-a} (\psi(a)+\gamma)^2 \\
&- \frac{2}{b-a} \left\{ -\frac{\zeta(3,a)-\zeta(3,b)}{2} + \zeta(2) (\psi(b)-\psi(a)) \right\} \\
&- \frac{2}{b-a} \left\{ \sum_{n=1}^{\infty} \left[\frac{H_n}{(n+a)^2} - \frac{H_n}{(n+b)^2} \right] + \zeta(2) (\psi(b)-\psi(a)) \right. \\
&\left. + \sum_{n=0}^{\infty} \frac{H_n^{(2)}}{(n+a)(n+b)} (b-a) \right\} \\
&= 0. \qquad (24)
\end{aligned}$$

Corollary 2.14. Let $a, b \in \mathbb{C} \setminus \mathbb{Z}$ with $a \neq b$. Then

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{[H_n^{(2)}]^2}{(n+a)(n+b)} + \sum_{n=1}^{\infty} \frac{[H_{n-1}^{(2)}]^2}{(n-a)(n-b)} \\
&+ 2\zeta(2) \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{(n+a)(n+b)} - 2\zeta(2) \sum_{n=1}^{\infty} \frac{H_{n-1}^{(2)}}{(n-a)(n-b)} \\
&+ \frac{\pi \cot(\pi b)}{b-a} \left\{ \zeta(2,b)^2 - \zeta(2)^2 \right\} - \frac{\pi \cot(\pi a)}{b-a} \left\{ \zeta(2,a)^2 - \zeta(2)^2 \right\} \\
&- \frac{2}{b-a} \left\{ \begin{array}{l} -\frac{\zeta(5,a)-\zeta(5,b)}{2} + \zeta(2) (\zeta(3,a)-\zeta(3,b)) \\ +\zeta(4) (\psi(b)-\psi(a)) \end{array} \right\} \\
&- \frac{2}{b-a} \left\{ \begin{array}{l} -\zeta(2) [\zeta(3,a)-\zeta(3,b)] - \sum_{n=0}^{\infty} \left[\frac{H_n^{(2)}}{(n+a)^3} - \frac{H_n^{(2)}}{(n+b)^3} \right] \\ +2\zeta(3) [\zeta(2,a)-\zeta(2,b)] - 2 \sum_{n=0}^{\infty} \left[\frac{H_n^{(3)}}{(n+a)^2} - \frac{H_n^{(3)}}{(n+b)^2} \right] \\ -3\zeta(4) (\psi(b)-\psi(a)) - 3(b-a) \sum_{n=0}^{\infty} \frac{H_n^{(4)}}{(n+a)(n+b)} \end{array} \right\} \\
&- \frac{4}{b-a} \zeta(2) \left\{ \zeta(2) (\psi(b)-\psi(a)) + (b-a) \sum_{n=0}^{\infty} \frac{H_n^{(2)}}{(n+a)(n+b)} \right\} \\
&= 0. \qquad (25)
\end{aligned}$$

Remark 2.15. From [19, Eq. (2.24)] and [20, Thm. 3.2], we know that for integer $p \geq 2$, the parametric Euler sums

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+a)^p} \quad (a \in \mathbb{C} \setminus \mathbb{N}^-)$$

can be expressed in terms of a combination of products of (Hurwitz) zeta function and digamma function. Moreover, if considering the contour integral

$$\oint\limits_{(\infty)} \frac{\pi\cot(\pi z)\psi^{(p-1)}(-z)\psi^{(m-1)}(-z)}{(p-1)!(m-1)!} r(z)dz = 0,$$

by a similar argument as in the proof of Theorem 2.12, we also obtain a similar evaluation of Theorem 2.6. More general, we can consider the general contour integral

$$\oint\limits_{(\infty)} \frac{\pi\cot(\pi z)\psi^{(p_1-1)}(-z)\cdots\psi^{(p_m-1)}(-z)}{(p_1-1)!\cdots(p_m-1)!} r(z)dz = 0$$

and

$$\oint\limits_{(\infty)} \frac{\pi\psi^{(p_1-1)}(-z)\cdots\psi^{(p_m-1)}(-z)}{\sin(\pi z)(p_1-1)!\cdots(p_m-1)!} r(z)dz = 0$$

to establish some quite general evaluations of (alternating) parametric Euler sums. We leave the detail to the interested reader.

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