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A SUFFICIENT CONDITION FOR HYPONORMAL TOEPLITZ OPERATORS ON THE BERGMAN SPACE

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Abstract. In this paper we consider the sufficient condition for hyponormal Toeplitz operators T_φ with non-harmonic symbols

$$\varphi(z) = \sum_{\ell=1}^{k} a_\ell z^{m_\ell} \overline{z}^{n_\ell}$$

with $m_\ell - n_\ell = \delta > 0$ for all $1 \leq \ell \leq k$, and $a_\ell \in \mathbb{C}$ on the Bergman spaces. In particular, we will observe the characteristics of the hyponormality of the Toeplitz operators T_φ according to the positional relationship of the coefficients a_ℓ's.

1. Introduction

In this paper, we deal with Toeplitz operators on the Bergman spaces of the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. The main results of this paper is to find sufficient conditions for the hyponormality of Toeplitz operators with non-harmonic symbols $\varphi(z)$ that are bivariate polynomials in z and $\overline{z}$ on the Bergman spaces. In particular, we will consider the hyponormality of Toeplitz operators according to the positional relationship of the coefficient of non-harmonic symbols $\varphi(z) = \sum_{\ell=1}^{k} a_\ell z^{m_\ell} \overline{z}^{n_\ell}$.

The Bergman space is one of the Hilbert spaces of analytic functions which possess a reproducing kernel. The early 1970's marked the beginning of function theoretic studies in the Bergman spaces. The analysis of Bergman spaces involves many operator-theoretic problems, and as such, it plays an important role in studying both operator theory and function theory (see [1, 2, 4]). For $1 \leq p < \infty$, $L^p(\mathbb{D})$ denotes the Banach space of Lebesgue measurable functions

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f on $\mathbb{D}$ with

$$\|f\|_p = \left(\int_{\mathbb{D}} |f(z)|^p dA(z)\right)^{\frac{1}{p}} < \infty,$$

where dA denotes the the normalized area measure on $\mathbb{D}$. The space $L^2(\mathbb{D})$ is a Hilbert space with the inner product

$$\langle f,\ g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z).$$

The Bergman space $A^2(\mathbb{D})$ is defined to be the subspace of $L^2(\mathbb{D})$ consisting of analytic functions. Let $L^\infty(\mathbb{D})$ be the space of bounded area measurable functions on $\mathbb{D}$ with

$$\|f\|_\infty = \text{ess sup}\{|f(z)| : z \in \mathbb{D}\} < \infty.$$

For $\varphi \in L^\infty(\mathbb{D})$, the *Toeplitz operator* T_φ with symbol φ on the Bergman space $A^2(\mathbb{D})$ is defined by

$$T_\varphi f = P(\varphi f) \quad \text{for } f \in A^2(\mathbb{D}),$$

where P is the orthogonal projection that maps $L^2(\mathbb{D})$ onto $A^2(\mathbb{D})$. A bounded linear operator T on a Hilbert space is said to be hyponormal if its self-commutator $[T^*, T] := T^*T - TT^*$ is positive (semidefinite). Since the hyponormality of operators is translation invariant we may assume that constant term is zero. We shall list the well-known properties of Toeplitz operators T_φ on the Bergman spaces. Let f, g be in $L^\infty(\mathbb{D})$ and $\alpha, \beta \in \mathbb{C}$. Then we can easily check that $T_{\alpha f+\beta g} = \alpha T_f + \beta T_g$, $T_f^* = T_{\overline{f}}$, and $T_{\overline{f}}T_g = T_{\overline{f}g}$ if f or g is analytic.

We need several auxiliary lemmas to prove the main theorem. We begin with:

Lemma 1.1 ([5]). *For any $s, t \in \mathbb{N} \cup \{0\}$,*

$$P(\overline{z}^t z^s) = \begin{cases} \frac{s-t+1}{s+1} z^{s-t} & \text{if } s \geq t, \\ 0 & \text{if } s < t. \end{cases}$$

The proof for Lemma 1.2 follows the proof of Lemma 2.1 in [5].

Lemma 1.2 ([5]). *For $0 \leq m \leq N$ and $c_j \in \mathbb{C}$ $(j = 0, 1, 2, \ldots)$, we deduce that*

(i) $\|\overline{z}^m \sum_{j=0}^\infty c_j z^j\|^2 = \sum_{j=0}^\infty \frac{1}{j+m+1}|c_j|^2$,

(ii) $\|P(\overline{z}^m \sum_{j=0}^\infty c_j z^j)\|^2 = \sum_{j=m}^\infty \frac{j-m+1}{(j+1)^2}|c_j|^2$.

In [13], the author characterize the hyponormality of Toeplitz operators $T_{\overline{g}+f}$ with bounded and analyitc functions f and g by $\|(I-P)(\overline{g}k)\| \leq \|(I-P)(\overline{f}k)\|$ for every k in $A^2(\mathbb{D})$. Furthermore, many authors have used the inequality to study the hyponormal Toeplitz operators (cf. [5, 6, 10]). However, we will consider the hyponormality of T_φ on $A^2(\mathbb{D})$ with the non-analytic symbol φ. So, in our case, we cannot apply that inequality to φ, since we cannot seperate

φ to analytic and coanalytic parts. Therefore we directly calculate the self-commutator of T_φ.

Recently, many authors have been studied relating to the hyponormality of Toeplitz operator with non-harmonic symbols on the (weighted) Bergman space which have been recently obtained in [3], [7], [8], [9], [11] and [14]. In [14], the author provide a necessary condition for the complex constant C for the operator $T_{z^n+C|z|^s}$ to be hyponormal on the Bergman space, and after that, in [3], the authors consider the sufficient conditions of hyponormality of T_{f+g} on the Bergman space, where $f(z) = a_{m,n}z^m\overline{z}^n$ $(m > n)$ and $g(z) = a_{j,k}z^j\overline{z}^k$ $(j > k)$ with $m - n > j - k$. In [11], hyponormality of Toeplitz operator $T_{z^n+C|z|^s}$ is characterized on the weighted Bergman spaces. Recently, the authors as in [7] characterized the necessary conditions for the hyponormality of Toeplitz operators T_φ on the Bergman space, where $\varphi(z) = az^m\overline{z}^n + bz^s\overline{z}^t$ $(m \geq n,\ t \geq s)$ with $m \neq t$ and $m - n = t - s$.

In this paper we focus on hyponormality of Toeplitz operators with non-harmonic symbols and more precisely, sufficient condition for hyponormal Toeplitz operators T_φ with symbols $\varphi(z) = \sum_{\ell=1}^k a_\ell z^{m_\ell}\overline{z}^{n_\ell}$, where $m_\ell - n_\ell = \delta > 0$ for all $1 \leq \ell \leq k$, and $a_\ell \in \mathbb{C}$ on the Bergman spaces. We will observe the hyponormality according to the positional relationship of the coefficients a_ℓ's. As a result, we give the cases where the terms of some coefficients of φ are on two orthogonal straight lines (Theorem 2.2), the coefficients a_{i_0} and a_{ℓ_0} of symbol φ satisfies the following property $|\arg(a_{i_0}) - \arg(a_{\ell_0})| > \frac{\pi}{2}$ (Theorem 2.4), and lastly some coefficients of φ are on a straight line (Corollary 2.7).

2. Sufficient condition for hyponormal Toeplitz operators

In this section, we consider the sufficient condition for hyponormal Toeplitz operators with non-harmonic symbols of the form $\varphi(z) = \sum_{\ell=1}^k a_\ell z^{m_\ell}\overline{z}^{n_\ell}$. We record here some results on the sufficient condition for the hyponormality of Toeplitz operators on the Bergman spaces with non-harmonic symbols, which have been recently developed in [3]. The authors proved the sufficient conditions for the hyponormality of Toeplitz operators with non-harmonic polynomials of fixed relative degree.

Theorem 2.1 ([3]). *Let*

$$\varphi(z) = a_1 z^{m_1}\overline{z}^{n_1} + \cdots + a_k z^{m_k}\overline{z}^{n_k}$$

with $m_1 - n_1 = \cdots = m_k - n_k = \delta \geq 0$, and a_s all lying in the same quarter-plane for $1 \leq s \leq k$ (that is, we have $\max_{1\leq s,t\leq k} |\arg(a_s) - \arg(a_t)| \leq \frac{\pi}{2}$). Then T_φ is hyponormal.

First, to overcome the assumption $\max_{1\leq s,t\leq k} |\arg(a_s) - \arg(a_t)| \leq \frac{\pi}{2}$ in Theorem 2.1, for $\varphi(z) = \sum_{\ell=1}^k a_\ell z^{m_\ell}\overline{z}^{n_\ell}$ with $m_\ell - n_\ell = \delta \geq 0$, we consider the case where some coefficients of φ are on two orthogonal straight lines.

Theorem 2.2. *Suppose that*

$$\varphi(z)=\sum_{\ell=1}^{4}a_\ell z^{m_\ell}\overline{z}^{n_\ell}$$

with $m_\ell-n_\ell=\delta>0$ *for all* $1\le \ell\le 4$, *and* $a_\ell\in\mathbb{C}$ *such that* $|\arg(a_s)-\arg(a_t)| =\frac{\pi}{2}$ *with* $s-t$ *is odd and* $|\arg(a_s)-\arg(a_t)|=\pi$ *with* $s-t$ *is even for any* $1\le t<s\le 4$. *For* $j\ge\delta$, *if*

$$\left(\frac{|a_s|}{m_s+j+1}-\frac{|a_t|}{m_t+j+1}\right)^2\ge\frac{j-\delta+1}{j+\delta+1}\left(\frac{|a_s|}{n_s+j+1}-\frac{|a_t|}{n_t+j+1}\right)^2 \tag{2.1}$$

for any s,t *such that* $s-t$ *is even, then* T_φ *is hyponormal.*

Proof. For $\varphi(z)=\sum_{\ell=1}^{4}a_\ell z^{m_\ell}\overline{z}^{n_\ell}$, T_φ is hyponormal if and only if

$$\left\langle (T_\varphi^*T_\varphi-T_\varphi T_\varphi^*)\sum_{j=0}^{\infty}c_jz^j,\sum_{j=0}^{\infty}c_jz^j\right\rangle\ge 0$$

for all $c_j\in\mathbb{C}$ with $\sum_{j\ge0}\frac{|c_j|^2}{j+1}<\infty$. Observe that

$$\begin{aligned}
&\left\|T_\varphi\sum_{j=0}^{\infty}c_jz^j\right\|^2-\left\|T_\varphi^*\sum_{j=0}^{\infty}c_jz^j\right\|^2\\
=&\sum_{\ell=1}^{4}\sum_{j=0}^{\infty}\frac{m_\ell+j-n_\ell+1}{(m_\ell+j+1)^2}|a_\ell|^2|c_j|^2-\sum_{\ell=1}^{4}\sum_{j=\delta}^{\infty}\frac{n_\ell+j-m_\ell+1}{(n_\ell+j+1)^2}|a_\ell|^2|c_j|^2\\
&+2\sum_{1\le t<s}^{4}\sum_{j=\delta}^{\infty}\left\{\frac{(j+1)\mathrm{Re}(a_s\overline{a}_t)}{(n_s+j+1)(n_t+j+1)}c_{j-m_s+n_s}\overline{c}_{j-m_t+n_t}\right\}\\
&-2\sum_{1\le t<s}^{4}\sum_{j=0}^{\infty}\left\{\frac{(j+1)\mathrm{Re}(\overline{a}_sa_t)}{(m_s+j+1)(m_t+j+1)}c_{j+m_t-n_t}\overline{c}_{j+m_s-n_s}\right\}.
\end{aligned}$$

It follows from $m_1-n_1=\cdots=m_4-n_4=\delta$ that T_φ is hyponormal if and only if

$$\begin{aligned}
&\sum_{\ell=1}^{4}\sum_{j=0}^{\infty}\frac{j+\delta+1}{(m_\ell+j+1)^2}|a_\ell|^2|c_j|^2-\sum_{\ell=1}^{4}\sum_{j=\delta}^{\infty}\frac{j-\delta+1}{(n_\ell+j+1)^2}|a_\ell|^2|c_j|^2\\
&+2\sum_{1\le t<s}^{4}\left\{\sum_{j=\delta}^{\infty}\frac{(j+1)\mathrm{Re}(a_s\overline{a}_t)}{(n_s+j+1)(n_t+j+1)}|c_{j-\delta}|^2\right.\\
&\qquad\left.-\sum_{j=0}^{\infty}\frac{(j+1)\mathrm{Re}(\overline{a}_sa_t)}{(m_s+j+1)(m_t+j+1)}|c_{j+\delta}|^2\right\}\ge 0
\end{aligned}$$

or equivalently,

$$
\begin{aligned}
&\sum_{\ell=1}^{4}\sum_{j=0}^{\infty}\frac{j+\delta+1}{(m_\ell+j+1)^2}|a_\ell|^2|c_j|^2-\sum_{\ell=1}^{4}\sum_{j=\delta}^{\infty}\frac{j-\delta+1}{(n_\ell+j+1)^2}|a_\ell|^2|c_j|^2\\
&+2\sum_{1\le t<s}^{4}\left\{\sum_{j=0}^{\infty}\frac{(j+\delta+1)\mathrm{Re}(a_s\overline{a}_t)}{(m_s+j+1)(m_t+j+1)}|c_j|^2\right.\\
&\qquad\left.-\sum_{j=\delta}^{\infty}\frac{(j-\delta+1)\mathrm{Re}(\overline{a}_s a_t)}{(n_s+j+1)(n_t+j+1)}|c_j|^2\right\}\ge 0.
\end{aligned}
\tag{2.2}
$$

Observe that $|\arg(a_s)-\arg(a_t)|=\frac{\pi}{2}$ with $s-t$ is odd and $|\arg(a_s)-\arg(a_t)|=\pi$ with $s-t$ is even for any $1\le t<s\le 4$,

$$
\begin{cases}
\mathrm{Re}(a_s\overline{a}_t)=\mathrm{Re}(\overline{a}_s a_t)=0 & \text{if } s-t \text{ is odd},\\
\mathrm{Re}(a_s\overline{a}_t)=\mathrm{Re}(\overline{a}_s a_t)=-|a_s a_t| & \text{if } s-t \text{ is even}.
\end{cases}
$$

Thus, T_φ is hyponormal if and only if

$$
\begin{aligned}
&\sum_{\ell=1}^{4}\sum_{j=0}^{\infty}\frac{j+\delta+1}{(m_\ell+j+1)^2}|a_\ell|^2|c_j|^2-\sum_{\ell=1}^{4}\sum_{j=\delta}^{\infty}\frac{j-\delta+1}{(n_\ell+j+1)^2}|a_\ell|^2|c_j|^2\\
&-2\sum_{s,t,s-t:\text{even}}^{4}|a_s a_t|\left\{\sum_{j=0}^{\infty}\frac{j+\delta+1}{(m_s+j+1)(m_t+j+1)}|c_j|^2\right.\\
&\qquad\left.-\sum_{j=\delta}^{\infty}\frac{j-\delta+1}{(n_s+j+1)(n_t+j+1)}|c_j|^2\right\}\ge 0
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
&\sum_{j=0}^{\infty}\left\{\left(\frac{|a_1|}{m_1+j+1}-\frac{|a_3|}{m_3+j+1}\right)^2\right.\\
&\qquad\left.+\left(\frac{|a_2|}{m_2+j+1}-\frac{|a_4|}{m_4+j+1}\right)^2\right\}(j+\delta+1)|c_j|^2\\
&-\sum_{j=\delta}^{\infty}\left\{\left(\frac{|a_1|}{n_1+j+1}-\frac{|a_3|}{n_3+j+1}\right)^2\right.\\
&\qquad\left.+\left(\frac{|a_2|}{n_2+j+1}-\frac{|a_4|}{n_4+j+1}\right)^2\right\}(j-\delta+1)|c_j|^2\ge 0.
\end{aligned}
$$

For $j\ge\delta$, if

$$
\left(\frac{|a_1|}{m_1+j+1}-\frac{|a_3|}{m_3+j+1}\right)^2\ge\frac{j-\delta+1}{j+\delta+1}\left(\frac{|a_1|}{n_1+j+1}-\frac{|a_3|}{n_3+j+1}\right)^2 \text{ and}
$$

$$\left(\frac{|a_2|}{m_2+j+1}-\frac{|a_4|}{m_4+j+1}\right)^2 \geq \frac{j-\delta+1}{j+\delta+1}\left(\frac{|a_2|}{n_2+j+1}-\frac{|a_4|}{n_4+j+1}\right)^2,$$

then T_φ is hyponormal. □

Example 2.3. Let $\varphi(z) = 2z^3\overline{z} + 2iz^4\overline{z}^2 - z^5\overline{z}^3 - iz^6\overline{z}^4$. Then, from the inequality (2.1), we want to show that

$$\left(\frac{2}{j+4}-\frac{1}{j+6}\right)^2 \geq \frac{j-1}{j+3}\left(\frac{2}{j+2}-\frac{1}{j+4}\right)^2$$

and

$$\left(\frac{2}{j+5}-\frac{1}{j+7}\right)^2 \geq \frac{j-1}{j+3}\left(\frac{2}{j+3}-\frac{1}{j+5}\right)^2$$

or equivalently,

$$(j+2)^2(j+3)(j+8)^2 \geq (j-1)(j+6)^4 \tag{2.3}$$

and

$$(j+3)^3(j+9)^2 \geq (j-1)(j+7)^4 \tag{2.4}$$

for any $j \geq 2$. By simple calculations, the inequalities (2.3) and (2.4) hold for any $j \geq 2$ since

$$(j+2)^2(j+3)(j+8)^2 - (j-1)(j+6)^4 = 68j^2 + 784j + 2064$$

and

$$(j+3)^3(j+9)^2 - (j-1)(j+7)^4 = j^4 + 23j^3 + 213j^2 + 585j + 778.$$

Therefore, T_φ is hyponormal.

Next, to overcome the assumption in Theorem 2.1, we consider the case where the terms of the symbol φ satisfies the following property

$$|\arg(a_{s_0}) - \arg(a_{t_0})| > \frac{\pi}{2}.$$

Theorem 2.4. *Let*

$$\varphi(z) = \sum_{\ell=1}^{k} a_\ell z^{m_\ell}\overline{z}^{n_\ell}$$

with $m_\ell, n_\ell \in \mathbb{N} \cup \{0\}$, $m_\ell - n_\ell = \delta > 0$ *for all* $1 \leq \ell \leq k$, *and* $a_\ell \in \mathbb{C}$ *such that* $\max_{1\leq s,t\leq k} |\arg(a_s) - \arg(a_t)| \leq \frac{\pi}{2}$ *except for* $|\arg(a_1) - \arg(a_2)| > \frac{\pi}{2}$. *If*

$$Re(a_1\overline{a}_2) \geq -\frac{\sqrt{(2m_1^2+n_1+m_1)(2m_2^2+n_2+m_2)}}{2m_1m_2+m_1+n_2}|a_1||a_2|,$$

then T_φ *is hyponormal.*

Proof. By the similar arguments as in the proof of Theorem 2.2, we deduce that hyponormality of the Toeplitz operator T_φ is equivalent to

$$\begin{aligned}
&\left\|T_\varphi \sum_{j=0}^{\infty} c_j z^j\right\|^2 - \left\|T_\varphi^* \sum_{j=0}^{\infty} c_j z^j\right\|^2 \\
&= \sum_{\ell=1}^{k}\sum_{j=0}^{\infty} \frac{j+\delta+1}{(m_\ell+j+1)^2}|a_\ell|^2|c_j|^2 - \sum_{\ell=1}^{k}\sum_{j=\delta}^{\infty} \frac{j-\delta+1}{(n_\ell+j+1)^2}|a_\ell|^2|c_j|^2 \\
&\quad + 2\sum_{1\le t<s}^{k}\left\{\sum_{j=0}^{\infty}\frac{(j+\delta+1)\mathrm{Re}(a_s\overline{a}_t)}{(m_s+j+1)(m_t+j+1)}|c_j|^2 \right.\\
&\qquad\left. - \sum_{j=\delta}^{\infty}\frac{(j-\delta+1)\mathrm{Re}(\overline{a}_s a_t)}{(n_s+j+1)(n_t+j+1)}|c_j|^2\right\} \ge 0.
\end{aligned}$$

Since $\max_{1\le s,t\le k}|\arg(a_s)-\arg(a_t)| \le \frac{\pi}{2}$, we have that $\mathrm{Re}(a_s\overline{a}_t) = \mathrm{Re}(\overline{a}_s a_t) \ge 0$ for any $(a_s,a_t) \ne (a_1,a_2)$. Thus, for any $\ell \ge 3$ we have

$$\sum_{j=0}^{\infty} \frac{j+\delta+1}{(m_\ell+j+1)^2}|a_\ell|^2|c_j|^2 - \sum_{j=\delta}^{\infty} \frac{j-\delta+1}{(n_\ell+j+1)^2}|a_\ell|^2|c_j|^2 > 0,$$

since $\frac{j+\delta+1}{(m_\ell+j+1)^2} - \frac{j-\delta+1}{(n_\ell+j+1)^2} > 0$ for $j \ge \delta$ and

$$2\left\{\sum_{j=0}^{\infty}\frac{(j+\delta+1)\mathrm{Re}(a_s\overline{a}_t)}{(m_s+j+1)(m_t+j+1)}|c_j|^2 - \sum_{j=\delta}^{\infty}\frac{(j-\delta+1)\mathrm{Re}(\overline{a}_s a_t)}{(n_s+j+1)(n_t+j+1)}|c_j|^2\right\} > 0$$

for any $(a_s,a_t) \ne (a_1,a_2)$. For the case of (a_1,a_2) with $|\arg(a_1)-\arg(a_2)| > \frac{\pi}{2}$ implies that $\mathrm{Re}(a_1\overline{a}_2) = \mathrm{Re}(\overline{a}_1 a_2) = -\lambda|a_1||a_2|$ for some $0 < \lambda \le 1$. Hence, if

$$\begin{aligned}
&\sum_{\ell=1}^{2}\sum_{j=0}^{\infty} \frac{j+\delta+1}{(m_\ell+j+1)^2}|a_\ell|^2|c_j|^2 - \sum_{\ell=1}^{2}\sum_{j=\delta}^{\infty} \frac{j-\delta+1}{(n_\ell+j+1)^2}|a_\ell|^2|c_j|^2 \\
&\quad - 2\lambda\left\{\sum_{j=0}^{\infty}\frac{(j+\delta+1)|a_1a_2|}{(m_1+j+1)(m_2+j+1)}|c_j|^2 \right.\\
&\qquad\left. - \sum_{j=\delta}^{\infty}\frac{(j-\delta+1)|a_1a_2|}{(n_1+j+1)(n_2+j+1)}|c_j|^2\right\} \ge 0,
\end{aligned} \tag{2.5}$$

then T_φ is hyponormal. If $j < \delta$, then

$$\sum_{\ell=1}^{2}\sum_{j=0}^{\delta-1} \frac{j+\delta+1}{(m_\ell+j+1)^2}|a_\ell|^2|c_j|^2 - 2\lambda\sum_{j=0}^{\delta-1}\frac{(j+\delta+1)|a_1a_2|}{(m_1+j+1)(m_2+j+1)}|c_j|^2 \ge 0.$$

If $m_1 > m_2$, then set

$$f(x,j,\lambda) := \left(\frac{j+\delta+1}{(m_1+j+1)^2} - \frac{j-\delta+1}{(n_1+j+1)^2}\right)\cdot\frac{|a_1|^2}{|a_2|^2}x^2 + 2\lambda\left(\frac{j-\delta+1}{(n_1+j+1)(n_2+j+1)} - \frac{j+\delta+1}{(m_1+j+1)(m_2+j+1)}\right)\cdot\frac{|a_1|}{|a_2|}x + \left(\frac{j+\delta+1}{(m_2+j+1)^2} - \frac{j-\delta+1}{(n_2+j+1)^2}\right)$$

for any $j \geq \delta$. Since $\frac{j+\delta+1}{(m_1+j+1)^2} - \frac{j-\delta+1}{(n_1+j+1)^2} > 0$, $f(x,j,\lambda)$ is concave upward for x. So that we must be satisfied that the discriminant of $f(x,j,\lambda)$ is not positive since $\frac{j-\delta+1}{(n_1+j+1)(n_2+j+1)} - \frac{j+\delta+1}{(m_1+j+1)(m_2+j+1)} < 0$ for any $j \geq \delta$. Observe that

$$\lambda^2 \leq \frac{\left(\frac{j+\delta+1}{(m_1+j+1)^2} - \frac{j-\delta+1}{(n_1+j+1)^2}\right)\left(\frac{j+\delta+1}{(m_2+j+1)^2} - \frac{j-\delta+1}{(n_2+j+1)^2}\right)}{\left(\frac{j-\delta+1}{(n_1+j+1)(n_2+j+1)} - \frac{j+\delta+1}{(m_1+j+1)(m_2+j+1)}\right)^2} \tag{2.6}$$

or equivalently,

$$\begin{aligned}\lambda^2 &\leq \frac{\{(j+\delta+1)(n_1+j+1)^2-(j-\delta+1)(m_1+j+1)^2\}\{(j+\delta+1)(n_2+j+1)^2-(j-\delta+1)(m_2+j+1)^2\}}{\{(j+\delta+1)(n_1+j+1)(n_2+j+1)-(j-\delta+1)(m_1+j+1)(m_2+j+1)\}^2}\\ &= \frac{\{(n_1+m_1)j+n_1^2+m_1^2+n_1+m_1\}\{(n_2+m_2)j+n_2^2+m_2^2+n_2+m_2\}}{\{j(m_1+m_2-\delta)+(m_1m_2+n_1n_2+m_1+m_2-\delta)\}^2}\end{aligned} \tag{2.7}$$

since

$$(j+\delta+1)(n_1+j+1)^2-(j-\delta+1)(m_1+j+1)^2 = \delta\{(n_1+m_1)j+n_1^2+m_1^2+n_1+m_1\},$$

$$(j+\delta+1)(n_2+j+1)^2-(j-\delta+1)(m_2+j+1)^2 = \delta\{(n_2+m_2)j+n_2^2+m_2^2+n_2+m_2\},$$

and

$$\begin{aligned}&(j+\delta+1)(n_1+j+1)(n_2+j+1)-(j-\delta+1)(m_1+j+1)(m_2+j+1)\\ &= \delta j(m_1+m_2-\delta)+\delta(m_1m_2+n_1n_2+m_1+m_2-\delta).\end{aligned}$$

Define g by

$$g(j) := \frac{\{(n_1+m_1)j+n_1^2+m_1^2+n_1+m_1\}\{(n_2+m_2)j+n_2^2+m_2^2+n_2+m_2\}}{\{j(m_1+m_2-\delta)+(m_1m_2+n_1n_2+m_1+m_2-\delta)\}^2}.$$

Since g is of the form

$$g(j) = \frac{\alpha(j+a)(j+c)}{(j+b)^2},$$

where $\alpha \in \mathbb{R}$, $a > b > c > 0$, g is increasing for any positive j. Thus, if $\lambda^2 \leq g(\delta)$, then T_φ is hyponormal. Since $\mathrm{Re}(a_1\overline{a}_2) = -\lambda|a_1||a_2|$, we get that if

$$\mathrm{Re}(a_1\overline{a}_2) \geq -\sqrt{g(\delta)}|a_1||a_2|,$$

then T_φ is hyponormal.

If $m_1 < m_2$, then we set

$$\begin{aligned} h(x,j,\lambda) :=& \left(\frac{j+\delta+1}{(m_2+j+1)^2} - \frac{j-\delta+1}{(n_2+j+1)^2} \right) \cdot \frac{|a_2|^2}{|a_1|^2} x^2 \\ &+ 2\lambda \left(\frac{j-\delta+1}{(n_1+j+1)(n_2+j+1)} - \frac{j+\delta+1}{(m_1+j+1)(m_2+j+1)} \right) \cdot \frac{|a_2|}{|a_1|} x \\ &+ \left(\frac{j+\delta+1}{(m_1+j+1)^2} - \frac{j-\delta+1}{(n_1+j+1)^2} \right) \end{aligned}$$

for any $j \geq \delta$. By the similar argument as above, we get that if

$$\mathrm{Re}(a_1 \overline{a}_2) \geq -\frac{\sqrt{(2m_1^2+m_1+n_1)(2m_2^2+m_2+n_2)}}{2m_1m_2+m_2+n_1} |a_1||a_2|,$$

then T_φ is hyponormal. This completes the proof. □

Corollary 2.5. *Suppose that*

$$\varphi(z) = az^m \overline{z}^n + bz^{m+1} \overline{z}^{n+1}$$

with $m > n$ *and* $|\arg a - \arg b| > \frac{\pi}{2}$. *If*

$$Re(a_1 \overline{a}_2) \geq -\frac{\sqrt{(2m^2+5m+n+4)(2m^2+m+n)}}{2m^2+3m+n+1} |a_1||a_2|,$$

then T_φ *on* $A^2(\mathbb{D})$ *is hyponormal.*

Example 2.6. Let $\varphi(z) = (-1+i)z^2\overline{z} + 3iz^3\overline{z}^2 + 4z^4\overline{z}^3$. Then $|\arg(4) - \arg(3i)| = \frac{\pi}{2}$, $|\arg(3i) - \arg(-1+i)| < \frac{\pi}{2}$ and $|\arg(4) - \arg(-1+i)| = \frac{3\pi}{4} > \frac{\pi}{2}$. By Theorem 2.4, T_φ is hyponormal since

$$-4 \geq \frac{\sqrt{(8+2+1)(32+4+3)}}{2 \cdot 8 + 4 + 1} \cdot \sqrt{2} \cdot 4 = -\frac{4\sqrt{858}}{21}.$$

Lastly, we consider the case where some coefficients of φ are on a straight line. The following result is the special case of Theorem 3.1 in [12]. The proof of the following corollary is more concise than the proof of Theorem 3.1 in [12].

Corollary 2.7. *Suppose that*

$$\varphi(z) = \sum_{\ell=1}^{k} a_\ell z^{m_\ell} \overline{z}^{n_\ell}$$

with $m_\ell, n_\ell \in \mathbb{N} \cup \{0\}$, $m_\ell - n_\ell = \delta > 0$ *for all* $1 \leq \ell \leq k$, *and* $a_\ell \in \mathbb{C}$ *such that* $|\arg(a_s) - \arg(a_t)| = \pi$ *for any* $1 \leq t < s \leq k$ *with* $s - t$ *is odd. Then,* T_φ *is hyponormal if and only if*

$$\left(\sum_{\ell=1}^{k} \frac{(-1)^{\ell+1}|a_\ell|}{m_\ell+j+1} \right)^2 \geq \frac{j-\delta+1}{j+\delta+1} \left(\sum_{\ell=1}^{k} \frac{(-1)^{\ell+1}|a_\ell|}{n_\ell+j+1} \right)^2$$

for any $j \geq \delta$.

Proof. By the similar arguments as in the proof of Theorem 2.2, T_φ is hyponormal if and only if

$$\begin{aligned}&\sum_{\ell=1}^{k}\sum_{j=0}^{\infty}\frac{j+\delta+1}{(m_\ell+j+1)^2}|a_\ell|^2|c_j|^2-\sum_{\ell=1}^{k}\sum_{j=\delta}^{\infty}\frac{j-\delta+1}{(n_\ell+j+1)^2}|a_\ell|^2|c_j|^2\\&+2\sum_{1\le t<s}^{k}\left\{\sum_{j=0}^{\infty}\frac{(j+\delta+1)\mathrm{Re}(a_s\overline{a}_t)}{(m_s+j+1)(m_t+j+1)}|c_j|^2\right.\\&\qquad\left.-\sum_{j=\delta}^{\infty}\frac{(j-\delta+1)\mathrm{Re}(\overline{a}_s a_t)}{(n_s+j+1)(n_t+j+1)}|c_j|^2\right\}\ge 0.\end{aligned}\tag{2.8}$$

Observe that $|\arg(a_s)-\arg(a_t)|=\pi$ for any $1\le t<s\le k$ with $s-t$ is odd, we have that

$$\begin{cases}\mathrm{Re}(a_s\overline{a}_t)=\mathrm{Re}(\overline{a}_s a_t)=-|a_s a_t| & \text{if } s-t \text{ is odd,}\\ \mathrm{Re}(a_s\overline{a}_t)=\mathrm{Re}(\overline{a}_s a_t)=|a_s a_t| & \text{if } s-t \text{ is even.}\end{cases}$$

It follows from relation (2.8) that T_φ is hyponormal if and only if

$$\begin{aligned}&\sum_{\ell=1}^{k}\sum_{j=0}^{\infty}\frac{j+\delta+1}{(m_\ell+j+1)^2}|a_\ell|^2|c_j|^2-\sum_{\ell=1}^{k}\sum_{j=\delta}^{\infty}\frac{j-\delta+1}{(n_\ell+j+1)^2}|a_\ell|^2|c_j|^2\\&+2\sum_{s,t,s-t:\text{even}}^{k}|a_s a_t|\left\{\sum_{j=0}^{\infty}\frac{(j+\delta+1)}{(m_s+j+1)(m_t+j+1)}|c_j|^2\right.\\&\qquad\left.-\sum_{j=\delta}^{\infty}\frac{(j-\delta+1)}{(n_s+j+1)(n_t+j+1)}|c_j|^2\right\}\\&-2\sum_{s,t,s-t:\text{odd}}^{k}|a_s a_t|\left\{\sum_{j=0}^{\infty}\frac{(j+\delta+1)}{(m_s+j+1)(m_t+j+1)}|c_j|^2\right.\\&\qquad\left.-\sum_{j=\delta}^{\infty}\frac{(j-\delta+1)}{(n_s+j+1)(n_t+j+1)}|c_j|^2\right\}\ge 0.\end{aligned}\tag{2.9}$$

By the direct calculations, we have

$$\begin{aligned}&\sum_{j=0}^{\infty}(j+\delta+1)\left(\sum_{\ell=1}^{k}\frac{(-1)^{\ell+1}}{m_\ell+j+1}|a_\ell||c_j|\right)^2\\&-\sum_{j=\delta}^{\infty}(j-\delta+1)\left(\sum_{\ell=1}^{k}\frac{(-1)^{\ell+1}}{n_\ell+j+1}|a_\ell||c_j|\right)^2\ge 0.\end{aligned}$$

Thus T_φ is hyponormal if and only if

$$\left(\sum_{\ell=1}^{k}\frac{(-1)^{\ell+1}|a_\ell|}{m_\ell+j+1}\right)^2 \geq \frac{j-\delta+1}{j+\delta+1}\left(\sum_{\ell=1}^{k}\frac{(-1)^{\ell+1}|a_\ell|}{n_\ell+j+1}\right)^2$$

for any $j \geq \delta$. This completes the proof. □

Corollary 2.8. *Suppose that*

$$\varphi(z) = a_1 z^{m_1}\overline{z}^{n_1} + a_2 z^{m_2}\overline{z}^{n_2} + a_3 z^{m_3}\overline{z}^{n_3}$$

with $m_1 - n_1 = m_2 - n_2 = m_3 - n_3 = \delta \geq 0$, *and* $a_j \in \mathbb{C}$ $(j = 1, 2, 3)$ *such that* $|\arg(a_1) - \arg(a_2)| = \pi$ *and* $|\arg(a_2) - \arg(a_3)| = \pi$. *Then,* T_φ *is hyponormal if and only if*

$$\begin{aligned}&\left(\frac{|a_1|}{m_1+j+1} - \frac{|a_2|}{m_2+j+1} + \frac{|a_3|}{m_3+j+1}\right)^2 \\ &\geq \frac{j-\delta+1}{j+\delta+1}\left(\frac{|a_1|}{n_1+j+1} - \frac{|a_2|}{n_2+j+1} + \frac{|a_3|}{n_3+j+1}\right)^2\end{aligned} \tag{2.10}$$

for any $j \geq \delta$.

Example 2.9. (i) Let $\varphi(z) = z^2\overline{z} - z^3\overline{z}^2 + z^4\overline{z}^3$. Then, from the inequality (2.10), we want to show that

$$\left(\frac{1}{j+3} - \frac{1}{j+4} + \frac{1}{j+5}\right)^2 \geq \frac{j}{j+2}\left(\frac{1}{j+2} - \frac{1}{j+3} + \frac{1}{j+4}\right)^2$$

or equivalently,

$$\frac{(j^2+8j+17)^2}{(j+3)^2(j+4)^2(j+5)^2} \geq \frac{j(j^2+6j+10)^2}{(j+2)^3(j+3)^2(j+4)^2} \tag{2.11}$$

and by a simple calculation, the inequality (2.11) is equivalent to

$$(j^2+8j+17)^2(j+2)^3 \geq j(j^2+6j+10)^2(j+5)^2$$

and holds for any $j \geq \delta = 1$ since

$$(j^2+8j+17)^2(j+2)^3 = j^7+22j^6+206j^5+1060j^4+3225j^3+5782j^2+5644j+2312$$

and

$$j(j^2+6j+10)^2(j+5)^2 = j^7+22j^6+201j^5+980j^4+2700j^3+4000j^2+2500j.$$

Therefore, T_φ is hyponormal.

(ii) Let $\psi(z) = z^2\overline{z} - 2z^3\overline{z}^2 + z^4\overline{z}^3$. Then, from the inequality (2.10), we show that

$$\left(\frac{1}{j+3} - \frac{2}{j+4} + \frac{1}{j+5}\right)^2 \geq \frac{j}{j+2}\left(\frac{1}{j+2} - \frac{2}{j+3} + \frac{1}{j+4}\right)^2$$

or equivalently,

$$\frac{4}{(j+3)^2(j+4)^2(j+5)^2} \geq \frac{4j}{(j+2)^3(j+3)^2(j+4)^2}. \tag{2.12}$$

Note that the inequality (2.12) is equivalent to

$$(j+2)^3 \geq j(j+5)^2$$

and does not hold for any $j \geq \delta = 1$ since $(j+2)^3 - j(j+5)^2 = -4j^2 - 13j + 8$. Therefore, T_ψ is not hyponormal.

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