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THE NEUMANN PROBLEM FOR A CLASS OF COMPLEX HESSIAN QUOTIENT EQUATIONS

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Abstract. In this paper, we study the Neumann problem for the complex Hessian quotient equation $\frac{\sigma_k(\tau\Delta uI+\partial\bar{\partial}u)}{\sigma_l(\tau\Delta uI+\partial\bar{\partial}u)}=\psi$ with $0\leq l<k\leq n$. We prove a priori estimate and global C^1 estimates, in particular, we use the double normal second derivatives on the boundary to establish the global C^2 estimates and prove the existence and the uniqueness for the Neumann problem of the above complex Hessian quotient equation.

1. Introduction

In this paper, we consider the Neumann problem for following fully nonlinear second order elliptic partial differential equations

$$\frac{\sigma_k(\tau\Delta uI+\partial\bar{\partial}u)}{\sigma_l(\tau\Delta uI+\partial\bar{\partial}u)}=\psi(z),\quad \text{in } \Omega\subset\mathbb{C}^n, \tag{1.1}$$

where Ω is a C^4 domain in $\mathbb{C}^n$ with the unit outer normal vector ν on $\partial\Omega$. ψ is a smooth positive function in $\overline{\Omega}$. Denote by $U=\tau\Delta uI+\partial\bar{\partial}u$ with $\tau\geq 0$, $\lambda[U]=(\lambda_1,\ldots,\lambda_n)$ are the eigenvalues of U. Then

$$\sigma_k(\tau\Delta uI+\partial\bar{\partial}u)=\sigma_k(\lambda[U])=\sum_{1\leq i_1<i_2<\cdots<i_k\leq n}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}.$$

Recall that the Garding's cone is defined as

$$\Gamma_k=\{\lambda\in\mathbb{R}^n:\sigma_j(\lambda)>0,\forall\ 1\leq j\leq k\}.$$

To ensure the ellipticity of (1.1), we need $\lambda[U]\in\Gamma_k$. Hence we introduce the following definition.

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Definition 1.1. A real value function $u \in C^2(\Omega)$ is called (η, k)-admissible if $\lambda[U] \in \Gamma_k$ for any $z \in \Omega$.

If $\tau = 0$, (1.1) is known as the classic Hessian quotient equation. The corresponding Dirichlet problem or Neumann problem have been studied extensively in the past four decades for the real case. For example, Caffarelli-Nirenberg-Spruck [2] and Ivochkina [19] have solved the Dirichlet problem of Monge-Ampère equation, Caffarelli-Nirenberg-Spruck [3] have solved the Dirichlet problem of k-Hessian equation. The Dirichlet problem for the general Hessian quotient equation was solved by Trudinger in [32]. In 1986, Lions-Trudinger-Urbas solved the Neumann problem of Monge-Ampère equation in [22]. The Neumann problem of k-Hessian equations was solved by Ma-Qiu [23], and Chen-Zhang [6] generalized the result to the Neumann problem of Hessian quotient equations. Chen-Wei [5] have solved the Neumann problem of Hessian quotient equations in the complex case. Some other fully nonlinear equations with Dirichlet boundary and Neumann boundary were also studied in [4, 10, 24–26].

In the complex setting, the Hessian quotient equations are related to many important problems in both Kähler and non-Kähler geometry. It is well known that the complex Monge-Ampère equation ($\tau = 0, k = n, l = 0$) was solved by Yau [33] on closed Kähler manifolds in the resolution of the Calabi conjecture. It is worth pointing out that the works of Yau connect the geometry and partial differential equations. Then Tosatti-Weinkove [30,31] have solved the analogous problem for the equation on closed Hermitian manifolds. The corresponding Dirichlet problem on strongly pseudo-convex domains was solved by Caffarelli-Kohn-Nirenberg-Spruck [1] in their milestone work. Then Cherrier-Hanani [7] and Guan-Li [14–16] extended [1]'s results to complex manifolds. After Yau and Caffarelli-Kohn-Nirenberg-Spruck's works, the fully nonlinear equations in complex domain or manifolds attracted many mathematicians in partial differential equations. For complex Hessian equation ($\tau = 0, l = 0$), Hou-Ma-Wu [18] established the second order estimates for the equation without boundary on Kähler manifold, and then Dinew-Kolodziej [9] solved the equation by combining the Liouville theorem and Hou-Ma-Wu's results. Zhang [34] and Székelyhidi [29] have solved the equation without boundary on Hermitian manifolds. Gu-Nguyen [13] were able to obtain continuous solutions to the complex Hessian equation with boundary on Hermitian manifolds, and Collins-Picard [8] solved the same problem under the existence of a subsolution. For Hessian quotient equation, the $(n, n-1)$-Hessian quotient equation have appeared in a problem proposed by Donaldson in the setting of moment maps and was solved by Song-Weinkove [27]. Then (n, l)-Hessian quotient equation was considered by Fang-Lai-Ma [11] on Kähler manifolds, and by Guan-Li [15], Guan-Sun [17] on Hermitian manifolds. The general (k, l)-Hessian quotient equation with $k < n$ without boundary on Hermitian manifolds was studied by Székelyhidi [29] and by Sun [28]. The corresponding Dirichlet problem on Hermitian manifolds was studied by Feng-Ge-Zheng [12].

Recently, Jiao-Wang [20] consider the Dirichlet problem for a class of fully nonlinear elliptic equation in Euclidean space,

$$f(\tau\Delta u + D^2 u) = \psi \quad \text{in } \Omega,$$

which is arising form conformal geometry. The development of the Neumann boundary problem for fully nonlinear equation is one of the motivations for us to study the corresponding Hessian quotient equation. In this article we want to know about the Neumann problem of the following complex Hessian quotient equations

$$\begin{cases} \dfrac{\sigma_k(\tau\Delta uI + \partial\bar{\partial}u)}{\sigma_l(\tau\Delta uI + \partial\bar{\partial}u)} = \psi(z), & \text{in } \Omega, \\ u_\nu = -\beta u + \phi(z), & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $0 \le l < k \le n$, β is a positive constant, ν is the unit outer normal vector on $\partial\Omega$. The main theorem is as follows.

Theorem 1.2. *Let $\Omega \subset \mathbb{C}^n$ be a domain with C^4 boundary, $\tau > 0$, $\phi \in C^3(\partial\Omega)$ and $0 < \psi \in C^2(\overline{\Omega})$. Then there exists a unique (η, k)-admissible solution $u \in C^{3,\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$ for the Neumann problem* (1.2).

Remark 1.3. The key proof for Theorem 1.2 is to establish the a priori estimates for equation (1.2), and then the existence result easily follows by the continuity method. The presence of $\tau > 0$ is crucial to the global estimates for second derivatives. For $\tau = 0$, Chen-Zhang [6] obtain the global C^2 estimates for equation(1.2) by some important inequalities of real Hessian quotient operator.

Following the same methods for real equation, we can obtain the existence results for the real counterpart of equation (1.2).

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^n$ be a domain with C^4 boundary, $\phi \in C^3(\partial\Omega)$ and $0 < \psi \in C^2(\overline{\Omega})$. Then there exists a unique k-admissible solution $u \in C^{3,\alpha}(\overline{\Omega})$ for $0 < \alpha < 1$ to the Neumann problem*

$$\begin{cases} \dfrac{\sigma_k(\tau\Delta uI + D^2 u)}{\sigma_l(\tau\Delta uI + D^2 u)} = \psi(x), & \text{in } \Omega, \\ u_\nu = -\beta u + \phi(x), & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

for any $1 \le l < k \le n$ and β is a positive constant, where ν is the unit outer normal vector on $\partial\Omega$.

The rest of this paper is organized as follows. In Section 2 we recall some properties of the elementary symmetric function σ_k and prove key lemma. In Section 3 we prove the C^0 estimates and gradient estimates. In Section 4 the second order estimates are derived. In Section 5, we prove the existence of a solution.

2. Preliminaries

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, we recall the definition of elementary symmetric functions for $1 \le k \le n$,

$$\sigma_k(\lambda) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}.$$

The Garding cone is defined by $\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_i(\lambda) > 0, \forall\ 1 \le i \le k\}$. We denote $\sigma_{k-1}(\lambda|i) = \frac{\partial \sigma_k}{\partial \lambda_i}$ and $\sigma_{k-2}(\lambda|ij) = \frac{\partial^2 \sigma_k}{\partial \lambda_i \partial \lambda_j}$. Next, we list some properties of σ_k which will be used later.

Lemma 2.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ with $\lambda \in \Gamma_k$ and $1 \le k \le n$. Then we have*

(1) $\sigma_{k-1}(\lambda|i) > 0$ *for* $1 \le i \le n$;
(2) $\sigma_k(\lambda) = \sigma_k(\lambda|i) + \lambda_i \sigma_{k-1}(\lambda|i)$ *for* $1 \le i \le n$;
(3) *If* $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, *then* $\sigma_{k-1}(\lambda|1) \le \sigma_{k-1}(\lambda|2) \le \cdots \le \sigma_{k-1}(\lambda|n)$;
(4) $\sum_{i=1}^n \sigma_{k-1}(\lambda|i) = (n-k+1)\sigma_{k-1}(\lambda)$;
(5) *If* $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, *then we have*

$$\sigma_{k-1}(\lambda|k) \ge C_{n,k} \sum_i \sigma_{k-1}(\lambda|i),$$

where $C_{n,k}$ is a positive constant only depending on n and k;
(6) *If* $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, *we have* $\sigma_{k-1}(\lambda|k) \ge C_{n,k}\sigma_{k-1}(\lambda)$;
(7) *If* $n \ge k > l \ge 0$, $n \ge r > s \ge 0$, $k \ge r$, *and* $l \ge s$, *we have*

$$\left[\frac{\sigma_k(\lambda)/C_n^k}{\sigma_l(\lambda)/C_n^l}\right]^{\frac{1}{k-l}} \le \left[\frac{\sigma_r(\lambda)/C_n^r}{\sigma_s(\lambda)/C_n^s}\right]^{\frac{1}{r-s}};$$

(8) *If* $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_k$, *then* $[\frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}]^{\frac{1}{k-l}}$ $(0 \le l < k \le n)$ *are concave with respect to λ. Hence, for any $(\xi_1, \ldots, \xi_n)$, we have*

$$\sum_{i,j} \frac{\partial^2 [\frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}]}{\partial \lambda_i \partial \lambda_j} \xi_i \xi_j \le (1 - \frac{1}{k-l}) \frac{\left[\sum_i \frac{\partial [\frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}]}{\partial \lambda_i} \xi_i\right]^2}{\frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}}.$$

Let $z = (z_1, \ldots, z_n)$ be a point in $\mathbb{C}^n$. Given $\xi \in \mathbb{R}^{2n}$, $D_\xi u$ denote the directional derivative of u along ξ. For the complex variables, we use the following notations:

$$\partial_k u = \frac{\partial u}{\partial z_k}, \quad \partial_{\bar{k}} u = \frac{\partial u}{\partial z_{\bar{k}}}, \quad \partial_{i\bar{j}} u = \frac{\partial^2 u}{\partial z_i \partial z_{\bar{j}}}, \quad \partial_{i\bar{j}k} = \frac{\partial^3 u}{\partial z_i \partial z_{\bar{j}} \partial z_k}.$$

For simplicity, we write $u_i = \partial_i u$, $u_{i\bar{j}} = \partial_{i\bar{j}} u$, $u_{i\bar{j}k} = \partial_{i\bar{j}k} u$, and so on. It holds that

$$|\nabla u|^2 := \sum_{j=1}^n \partial_j u \overline{\partial_j u} = \frac{1}{4}|Du|^2. \tag{2.1}$$

Then in complex coordinates, we denote by $\eta_{i\bar{j}} \equiv \tau \Delta u \delta_{ij} + u_{i\bar{j}}$. Let $\lambda \equiv \lambda(u_{i\bar{j}}) = (\lambda_1, \ldots, \lambda_n)$ be the eigenvalues of $\{u_{i\bar{j}}\}$ and $\eta \equiv \lambda(\eta(u_{i\bar{j}})) = (\eta_1, \ldots, \eta_n)$ be the eigenvalues of $\{\eta_{i\bar{j}}\}$. Then we can get $\eta_i = \tau \sum_{j=1}^{n} \lambda_j + \lambda_i$. With our notations, equation (1.1) can be written as

$$f(\lambda) = F(u_{i\bar{j}}) = G(\eta_{i\bar{j}}) = \left(\frac{\sigma_k}{\sigma_l}\right)^{\frac{1}{k-l}} (\eta_{i\bar{j}}) = \widetilde{\psi}(z, u) = \psi^{\frac{1}{k-l}}. \tag{2.2}$$

For convenience, we also can introduce the following notations:

$$F^{i\bar{j}} = \frac{\partial F}{\partial u_{i\bar{j}}}, \quad F^{i\bar{j}, k\bar{l}} = \frac{\partial^2 F}{\partial u_{i\bar{j}} \partial u_{k\bar{l}}}, \quad G^{i\bar{j}} = \frac{\partial G}{\partial \eta_{i\bar{j}}}, \quad G^{i\bar{j}, k\bar{l}} = \frac{\partial^2 G}{\partial \eta_{i\bar{j}} \partial \eta_{k\bar{l}}}.$$

Let u be an admissible solution of equation (1.1), $\{u_{i\bar{j}}\}$ is diagonal at a single point, equation (1.1) is expressed in the form

$$F^{i\bar{j}} = f_i \delta_{ij} \text{ with } f_i = \frac{\partial f}{\partial \lambda_i}.$$

In the following, we assume $\lambda_1 \geq \cdots \geq \lambda_n$, then we can get $\eta_1 \geq \cdots \geq \eta_n$, which implies $\sigma_l(\eta|1) \leq \cdots \leq \sigma_l(\eta|n)$ for $1 \leq l \leq k-1$. Therefore, at a point $z \in \Omega$ where $\{u_{i\bar{j}}(z)\}$ is diagonal, we have

$$F^{i\bar{i}} = \frac{\partial F}{\partial u_{i\bar{i}}} = \frac{\partial G}{\partial \eta_{r\bar{s}}} \cdot \frac{\partial \eta_{r\bar{s}}}{\partial u_{i\bar{i}}} = \tau \sum_{k=1}^{n} G^{k\bar{k}} + G^{i\bar{i}}.$$

Lemma 2.2. *If $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_k$, then $[\frac{\sigma_k(\eta)}{\sigma_l(\eta)}]^{\frac{1}{k-l}}$ $(0 \leq l < k \leq n)$ are concave with respect to λ.*

Proof. It is equal to prove that for any $(\xi_1, \ldots, \xi_n)$, we have

$$\sum_{i,j} \frac{\partial^2 [\frac{\sigma_k(\eta)}{\sigma_l(\eta)}]}{\partial \lambda_i \partial \lambda_j} \xi_i \xi_j \leq (1 - \frac{1}{k-l}) \frac{\left[\sum_i \frac{\partial [\frac{\sigma_k(\eta)}{\sigma_l(\eta)}]}{\partial \lambda_i} \xi_i\right]^2}{\frac{\sigma_k(\eta)}{\sigma_l(\eta)}}.$$

By Lemma 2.1, we can obtain

$$\begin{aligned}
\sum_{i,j} \frac{\partial^2 [\frac{\sigma_k(\eta)}{\sigma_l(\eta)}]}{\partial \lambda_i \partial \lambda_j} \xi_i \xi_j &= \sum_{i,j,a,b} \frac{\partial^2 [\frac{\sigma_k(\eta)}{\sigma_l(\eta)}]}{\partial \eta_a \partial \eta_b} \frac{\partial \eta_a}{\partial \lambda_i} \frac{\partial \eta_b}{\partial \lambda_j} \xi_i \xi_j \\
&\leq (1 - \frac{1}{k-l}) \frac{\left[\sum_a \frac{\partial [\frac{\sigma_k(\eta)}{\sigma_l(\eta)}]}{\partial \eta_a} (\sum_i \frac{\partial \eta_a}{\partial \lambda_i} \xi_i)\right]^2}{\frac{\sigma_k(\eta)}{\sigma_l(\eta)}} \\
&\leq (1 - \frac{1}{k-l}) \frac{\left[\sum_{a,i} \frac{\partial [\frac{\sigma_k(\eta)}{\sigma_l(\eta)}]}{\partial \eta_a} \frac{\partial \eta_a}{\partial \lambda_i} \xi_i\right]^2}{\frac{\sigma_k(\eta)}{\sigma_l(\eta)}}
\end{aligned}$$

$$\leq (1-\frac{1}{k-l})\frac{\left[\sum_i \frac{\partial[\frac{\sigma_k(\eta)}{\sigma_l(\eta)}]}{\partial\lambda_i}\xi_i\right]^2}{\frac{\sigma_k(\eta)}{\sigma_l(\eta)}}.$$

□

Lemma 2.3. *Let $0 \leq l < k \leq n$, r be an $n\times n$ Hermitian matrix, $\lambda = \lambda(r)$ be the eigenvalues of r with $\eta = \tau\sum_{j=1}^n \lambda_j \cdot \mathbf{1} + \lambda \in \Gamma_k$. Then*

$$f_1 \geq \frac{\tau}{n\tau+1}C_{n,k,l}\sum_{i=1}^n f_i,$$

where $0 \leq f_1 \leq \cdots \leq f_n$ are the eigenvalues of $\{F^{i\bar{j}}(r)\}$, $C_{n,k,l}$ is a constant depending on n, k and l.

Proof. When $l = 0$, the results follows by the same method as in [10, Lemma 9]. Hence we only need to prove the lemma for $l \geq 1$. Without loss of generality, we assume that the matrix r is diagonal with $\lambda_1(r) \geq \cdots \geq \lambda_n(r)$. Hence, $\eta_1(r) \geq \cdots \geq \eta_n(r)$. Direct calculations shows that

$$\begin{aligned} f_i &= \frac{\partial(\frac{\sigma_k}{\sigma_l})^{\frac{1}{k-l}}(\eta)}{\partial\eta_p}\frac{\partial\eta_p}{\partial\lambda_i} \\ &= \tfrac{1}{k-l}(\frac{\sigma_k}{\sigma_l})^{\frac{1}{k-l}-1}\sum_p \tfrac{\sigma_{k-1}(\eta|p)(\tau+\delta_{ip})\sigma_l-\sigma_k\sigma_{l-1}(n|p)(\tau+\delta_{ip})}{\sigma_l^2(\eta)} \\ &= \tfrac{1}{k-l}(\frac{\sigma_k}{\sigma_l})^{\frac{1}{k-l}-1}\frac{(\tau+1)\sigma_{k-1}(\eta|i)\sigma_l(\eta|i)(1-\alpha_i)+\tau\sum_{p\neq i}\sigma_{k-1}(\eta|p)\sigma_l(\eta|p)(1-\alpha_p)}{\sigma_l^2(\eta)}, \end{aligned}$$

where α_p is defined by

$$\alpha_p := \frac{\sigma_k(\eta|p)\sigma_{l-1}(\eta|p)}{\sigma_{k-1}(\eta|p)\sigma_l(\eta|p)}, \quad \forall p = 1,\ldots,n.$$

Note that

$$\sigma_{k-1}(\eta|p) \geq C_{n,k}\sum_{i=1}^n \sigma_{k-1}(\eta|i),\ \sigma_l(\eta|p) \geq C_{n,l}\sum_{i=1}^n \sigma_l(\eta|i), \quad \forall p \geq k. \tag{2.3}$$

We can divide into two cases:

Case 1: $\sigma_k(\eta|p) > 0$. We have $\eta|p \in \Gamma_k$ since $\eta \in \Gamma_k$, $\eta|p$ is the n-dimensional vector with zero on the pth slot.

By Newton-MacLaurin inequality for $1 \leq p \leq n$

$$\alpha_p \leq \frac{C_{n-1}^{l-1}C_{n-1}^k}{C_{n-1}^l C_{n-1}^{k-1}} = \frac{l(n-k)}{k(n-l)},$$

that is to say

$$1-\alpha_p \geq 1-\frac{l(n-k)}{k(n-l)} = \frac{n(k-l)}{k(n-l)}.$$

Case 2: $\sigma_k(\eta|p) \leq 0$. Then we have $\alpha_p \leq 0$, and

$$1-\alpha_p \geq 1 \geq \frac{n(k-l)}{k(n-l)}.$$

Now we can estimate by (2.3)

$$\begin{aligned} f_1 &\geq \tfrac{1}{k-l}\big(\tfrac{\sigma_k}{\sigma_l}\big)^{\frac{1}{k-l}-1}\sum_{p\geq k}\tfrac{\tau\sigma_{k-1}(\eta|p)\sigma_l(\eta|p)(1-\alpha_p)}{\sigma_l^2(\eta)} \\ &\geq \tfrac{n(k-l)}{k(n-l)}\tfrac{\tau(n-k+1)}{k-l}C_{n,k}C_{n,l}\big(\tfrac{\sigma_k}{\sigma_l}\big)^{\frac{1}{k-l}-1}\tfrac{\sigma_{k-1}(\eta)\sigma_l(\eta)}{\sigma_l^2(\eta)} \\ &= \tau C_{n,k,l}\big(\tfrac{\sigma_k}{\sigma_l}\big)^{\frac{1}{k-l}-1}\tfrac{\sigma_{k-1}(\eta)}{\sigma_l(\eta)} \\ &\geq C_{n,k,l}\tfrac{\tau}{n\tau+1}\sum_{i=1}^n f_i, \end{aligned}$$

where the last equality is using the following equation

$$\begin{aligned} \sum_{i=1}^n f_i =& \tfrac{1}{k-l}\big(\tfrac{\sigma_k}{\sigma_l}\big)^{\frac{1}{k-l}-1}\big[(\tau+1)\tfrac{\sum_{i=1}^n\sigma_{k-1}(\eta|i)\sigma_l(\eta)-\sum_{i=1}^n\sigma_{l-1}(\eta|i)\sigma_k(\eta)}{\sigma_l^2(\eta)} \\ &+\tau\tfrac{\sum_{i=1}^n\sum_{p\neq i}\sigma_{k-1}(\eta|p)\sigma_l(\eta)-\sum_{i=1}^n\sum_{p\neq i}\sigma_{l-1}(\eta|p)\sigma_k(\eta)}{\sigma_l^2(\eta)}\big] \\ =& \tfrac{n\tau+1}{k-l}\big(\tfrac{\sigma_k}{\sigma_l}\big)^{\frac{1}{k-l}-1}\tfrac{\sum_{i=1}^n\sigma_{k-1}(\eta|i)\sigma_l(\eta)-\sigma_k(\eta)\sigma_{l-1}(\eta|i)}{\sigma_l^2} \\ \leq& \tfrac{n\tau+1}{k-l}\big(\tfrac{\sigma_k}{\sigma_l}\big)^{\frac{1}{k-l}-1}\tfrac{1}{\sigma_l}\sum_{i=1}^n\sigma_{k-1}(\eta|i) \\ =& \tfrac{(n\tau+1)(n-k+1)}{(k-l)}\big(\tfrac{\sigma_k}{\sigma_l}\big)^{\frac{1}{k-l}-1}\tfrac{\sigma_{k-1}(\eta)}{\sigma_l(\eta)}. \end{aligned}$$

□

Remark 2.4. By Newton-Maclaurin inequality, if $k>l\geq 0$, we see that

$$\begin{aligned} \sum f_i(\lambda) =& \tfrac{n\tau+1}{k-l}\big(\tfrac{\sigma_k}{\sigma_l}\big)^{\frac{1}{k-l}-1}\tfrac{\sum_{i=1}^n\sigma_{k-1}(\lambda|i)\sigma_l-\sigma_k\sigma_{l-1}(\lambda|i)}{\sigma_l^2} \\ =& \tfrac{n\tau+1}{k-l}\big(\tfrac{\sigma_k}{\sigma_l}\big)^{\frac{1}{k-l}-1}\tfrac{(n-k+1)\sigma_{k-1}\sigma_l-(n-l+1)\sigma_{l-1}\sigma_k}{\sigma_l^2} \\ =& \tfrac{(n\tau+1)(n-k+1)}{k-l}\big(\tfrac{\sigma_k}{\sigma_l}\big)^{\frac{1}{k-l}}\tfrac{\sigma_{k-1}}{\sigma_k}-\tfrac{(n\tau+1)(n-l+1)}{k-l}\big(\tfrac{\sigma_k}{\sigma_l}\big)^{\frac{1}{k-l}}\tfrac{\sigma_{l-1}}{\sigma_l} \\ =& (n\tau+1)\big(\tfrac{\sigma_k}{\sigma_l}\big)^{\frac{1}{k-l}}\big[\tfrac{kC_n^k\sigma_{k-1}}{(k-l)C_n^{k-1}\sigma_k}-\tfrac{lC_n^l\sigma_{l-1}}{(k-l)C_n^{l-1}\sigma_l}\big] \\ =& \tfrac{n\tau+1}{k-l}\big(\tfrac{\sigma_k}{\sigma_l}\big)^{\frac{1}{k-l}}\big[k\tfrac{\sigma_{k-1}C_n^k}{\sigma_kC_n^{k-1}}-l\tfrac{\sigma_{l-1}C_n^l}{\sigma_lC_n^{l-1}}\big] \\ \geq& (n\tau+1)\big(\tfrac{\sigma_k}{\sigma_l}\big)^{\frac{1}{k-l}}\tfrac{\sigma_{k-1}/C_n^{k-1}}{\sigma_k/C_n^k} \\ =& (n\tau+1)\big[\tfrac{\sigma_k/C_n^k}{\sigma_l/C_n^l}\big]^{\frac{1}{k-l}}\tfrac{\sigma_{k-1}/C_n^{k-1}}{\sigma_k/C_n^k}\big[\tfrac{C_n^k}{C_n^l}\big]^{\frac{1}{k-l}} \\ \geq& (n\tau+1)\big(\tfrac{C_n^k}{C_n^l}\big)^{\frac{1}{k-l}} := (n\tau+1)C_{n,k,l} \end{aligned} \tag{2.4}$$

for $\lambda\in\mathbb{R}^n$ with $\eta\in\Gamma_k$.

Proposition 2.5. *If $\lambda\in\mathbb{R}^n$ with $\eta\in\Gamma_k$, the operator $\frac{\sigma_k}{\sigma_l}(\eta)$ is strictly elliptic with respect to λ.*

3. C^1 estimates

In this section, we prove the C^1 estimates for the equation (1.2). We always assume that the conditions in Theorem 1.2 hold.

3.1. C^0 estimates

Theorem 3.1. *Suppose $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is an (η, k)-admissible solution to* (1.2)*. Then we have*

$$|u|_{C^0} \leq \frac{n\tau + 2}{n\tau + 1} C,$$

where C depends on $n, k, l, \Omega, \beta, \max|\phi|, \max|\psi|$.

Proof. On the one hand, suppose that u attains its maximum at $z_0 \in \partial\Omega$. Hence,

$$0 \leq D_\nu u(z_0) = -\beta u(z_0) + \phi(z_0).$$

Then we have

$$\max_{\partial\Omega} u = u(z_0) \leq \frac{1}{\beta}\phi(z_0) \leq \frac{1}{\beta}\max_{\partial\Omega}|\phi(z)|.$$

On the other hand, define a constant

$$B = \frac{1}{n\tau + 1}\left[\frac{C_n^l}{C_n^k}\max_{\overline{\Omega}}|\psi|\right]^{\frac{1}{k-l}}.$$

Note that

$$\begin{aligned}\frac{\sigma_k(\tau\Delta uI + \partial\overline{\partial}u)}{\sigma_l(\tau\Delta uI + \partial\overline{\partial}u)} &= \psi(z)\\ &\leq \max_{\overline{\Omega}}\psi = \frac{\sigma_k(\tau\Delta(B|z|^2)I + \partial\overline{\partial}(B|z|^2))}{\sigma_l(\tau\Delta(B|z|^2)I + \partial\overline{\partial}(B|z|^2))}.\end{aligned}$$

Without loss of generality, we suppose $0 \in \Omega$, hence we know that the function $u - B|z|^2$ attains its minimum at a point $z_1 \in \partial\Omega$. It is immediate to see that

$$0 \geq D_\nu(u - B|z|^2)|_{z=z_1} \geq -\beta u(z_1) - \max_{\partial\Omega}|\phi| - 2B\text{diam}(\Omega).$$

Therefore,

$$\begin{aligned}\min_{\partial\Omega} u &\geq \min_{\partial\Omega}(u - B|z|^2)\\ &\geq -\frac{1}{\beta}\max_{\partial\Omega}|\phi| - \frac{2B}{\beta}\text{diam}(\Omega) - B\text{diam}^2(\Omega)\\ &\geq -\frac{n\tau + 2}{n\tau + 1}C.\end{aligned}$$

□

3.2. Global gradient estimates

Theorem 3.2. *Suppose $u \in C^3(\Omega)$ is an (η, k)-admissible solution to* (1.2). *Assume $0 \in \Omega$ and $B_r(0) \subset \Omega$. Then, we have*

$$|\nabla u|(0) \leq \frac{C}{r}\sqrt{\frac{n\tau+1}{\tau}},$$

where C depends on $n, k, l, \Omega, |u|_{C^0}$, $\inf \psi$ and $|\psi|_{C^1}$.

Proof. Consider the following test function on $B_r(0) \subset \Omega$,

$$H(z) = \ln|\nabla u| + h(u) + \ln\zeta(z),$$

where $\zeta(z) = r^2 - |z|^2$ and $|\nabla u| = \sqrt{\sum_{k=1}^n u_k u_{\bar{k}}}$. Define

$$h(s) = \delta(s+L)^2, \tag{3.1}$$

here we set $L := |u|_{C^0} + 1$, δ is a sufficiently small constant such that

$$h'' - 4(h')^2 = 2\delta - 16\delta^2(u+L)^2 \geq \delta.$$

Assume H attains its maximum at $z_0 \in B_r(0)$. At z_0, we have

$$0 = H_i = \frac{|\nabla u|_i^2}{2|\nabla u|^2} + h'u_i + \frac{\zeta_i}{\zeta} \tag{3.2}$$

and

$$\begin{aligned} 0 &\geq F^{i\bar{j}}H_{i\bar{j}} \\ &= F^{i\bar{j}}\frac{|\nabla u|_{i\bar{j}}^2}{2|\nabla u|^2} - F^{i\bar{j}}\frac{|\nabla u|_i^2|\nabla u|_{\bar{j}}^2}{2|\nabla u|^4} + h'F^{i\bar{j}}u_{i\bar{j}} + h''F^{i\bar{j}}u_iu_{\bar{j}} + F^{i\bar{j}}\frac{\zeta_{i\bar{j}}}{\zeta} - F^{i\bar{j}}\frac{\zeta_i\zeta_{\bar{j}}}{\zeta^2} \\ &= F^{i\bar{j}}\frac{|\nabla u|_{i\bar{j}}^2}{2|\nabla u|^2} - F^{i\bar{j}}\frac{|\nabla u|_i^2|\nabla u|_{\bar{j}}^2}{2|\nabla u|^4} + h'F^{i\bar{j}}u_{i\bar{j}} + h''F^{i\bar{j}}u_iu_{\bar{j}} - \frac{\sum F^{i\bar{i}}}{\zeta} - \frac{F^{i\bar{j}}z_{\bar{i}}z_j}{\zeta^2}. \end{aligned} \tag{3.3}$$

Assume that $|\nabla u(z_0)|^2 \geq 1$. By Cauchy-Schwartz inequality and (3.2), we obtain

$$F^{i\bar{j}}\frac{|\nabla u|_i^2|\nabla u|_{\bar{j}}^2}{2|\nabla u|^4} \leq 4(h')^2F^{i\bar{j}}u_iu_{\bar{j}} + \frac{4}{\zeta^2}F^{i\bar{j}}\zeta_i\zeta_{\bar{j}}. \tag{3.4}$$

Direct calculation shows

$$F^{i\bar{j}}|\nabla u|_{i\bar{j}}^2 = F^{i\bar{j}}(u_{ki}u_{\bar{k}\bar{j}} + u_{\bar{k}i}u_{k\bar{j}}) + u_k\widetilde{\psi}_{\bar{k}} + u_{\bar{k}}\widetilde{\psi}_k. \tag{3.5}$$

Combining with (3.3), (3.4), (3.5), Lemma 2.3 and Remark 2.4, we have

$$\begin{aligned} 0 &\geq -C + \delta F^{i\bar{j}}u_iu_{\bar{j}} - \frac{1}{\zeta}\sum_{i=1}^n F^{i\bar{i}} - \frac{5r^2}{\zeta^2}\sum_{i=1}^n F^{i\bar{i}} \\ &\geq -C + \frac{\tau}{n\tau+1}\delta C_{n,k,l}|\nabla u|^2\sum_{i=1}^n F^{i\bar{i}} - \frac{6r^2}{\zeta^2}\sum_{i=1}^n F^{i\bar{i}}. \end{aligned}$$

Assume $|\nabla u(z_0)|^2 \geq \max\left\{\frac{C}{\frac{\tau}{n\tau+1}\frac{\delta}{2}C_{n,k,l}(n\tau+1)C_{n,k,l}}, 1\right\}$. We arrive at

$$0 \geq \left(\frac{\tau}{n\tau+1}\frac{\delta C_{n,k,l}}{2}|\nabla u|^2 - \frac{6r^2}{\zeta^2}\right)\sum_{i=1}^n F^{i\bar{i}},$$

which implies

$$\zeta^2(z_0)|\nabla u|^2(z_0) \leq \frac{12r^2}{\frac{\tau}{n\tau+1}\delta C_{n,k,l}}.$$

Therefore, by $H(0) \leq H(z_0)$ and an easy calculation, we obtain

$$|\nabla u|(0) \leq \frac{1}{r}\sqrt{\frac{12}{\delta C_{n,k,l}}\frac{n\tau+1}{\tau}}.$$

□

Now we prove the global gradient estimates by the following proposition.

Theorem 3.3. *Suppose u is a C^3 (η, k)-admissible solution to* (1.2)*. Then we have*

$$\sup_{\Omega}|\nabla u| \leq \sqrt{\frac{n\tau+1}{\tau}}C,$$

where C depends on $n, k, l, \beta, \Omega, |u|_{C^0}, |\phi|_{C^3}$, $\inf\psi$ and $|\psi|_{C^1}$.

Proof. Consider the following auxiliary function

$$T(z) = \ln|\nabla w| + Ad + h(u),$$

where $w = u + (-\beta u + \varphi(z))d$, $d(z) = \operatorname{dist}(z, \partial\Omega)$, h is a smooth function defined by (3.1), A is a positive constant to be determined later. Suppose that G attains its maximum at $z_0 \in \overline{\Omega}$, we divide the proof into three cases.

Case 1: $z_0 \in \Omega_\mu := \{x \,|\, d(z, \partial\Omega) \geq \mu\}$. We can bound $|\nabla u|(z_0)$ by Theorem 3.2.

Case 2: $z_0 \in \partial\Omega$. We denote by $\tilde{\varphi} = -\beta u + \varphi(z)$. Notice that

$$w_\nu = u_\nu + \tilde{\varphi}_\nu d + \tilde{\varphi} d_\nu = 0;\ u_\nu = \tilde{\varphi} \quad \text{on } \partial\Omega.$$

Hence, at z_0,

$$\begin{aligned} 0 \leq \frac{\partial}{\partial\nu}T &= \frac{|\nabla w|^2_\nu}{2|\nabla w|^2} + Ad_\nu + h' u_\nu \\ &= \frac{\frac{1}{2}(\sum_{k=1}^{2n-1} D_k w D_{k\nu} w + D_\nu w D_{\nu\nu} w)}{2|\nabla w|^2} + Ad_\nu + h' u_\nu \\ &\leq \sup_{\partial\Omega}\{|\Pi_{ij}|\} - A + h'(u)\tilde{\varphi}(z_0, u), \end{aligned}$$

where Π_{ij} is the second fundamental form of $\partial\Omega$. We choose

$$A = \sup_{\partial\Omega}\{|\Pi_{ij}|\} + \sup_{\Omega}|h'||\tilde{\varphi}| + 1,$$

which yields a contradiction to $\frac{\partial T}{\partial\nu} < 0$.

Case 3: $z_0 \in \Omega \setminus \Omega_\mu$. Hence at z_0, we have

$$0 = T_i(z_0) = \frac{|\nabla w|_i^2}{2|\nabla w|^2} + Ad_i + h'u_i, \tag{3.6}$$

and

$$\begin{aligned} 0 &\geq F^{i\bar{j}} T_{i\bar{j}} \\ &= F^{i\bar{j}} \tfrac{|\nabla w|^2_{i\bar{j}}}{2|\nabla w|^2} - 2F^{i\bar{j}} \tfrac{|\nabla w|_i^2 |\nabla w|_{\bar{j}}^2}{4|\nabla w|^4} + AF^{i\bar{j}} d_{i\bar{j}} + h' F^{i\bar{j}} u_{i\bar{j}} + h'' F^{i\bar{j}} u_i u_{\bar{j}} \\ &\geq F^{i\bar{j}} \tfrac{|\nabla w|^2_{i\bar{j}}}{2|\nabla w|^2} - 4A^2 F^{i\bar{j}} d_i d_{\bar{j}} + (h'' - 4(h')^2) F^{i\bar{j}} u_i u_{\bar{j}} + AF^{i\bar{j}} d_{i\bar{j}} + h' F^{i\bar{j}} u_{i\bar{j}}, \end{aligned} \tag{3.7}$$

where in the last inequality we used (3.6) and Cauchy-Schwarz inequality.

It is immediate to see that

$$F^{i\bar{j}} |\nabla w|^2_{i\bar{j}} = w_k F^{i\bar{j}} w_{\bar{k}i\bar{j}} + F^{i\bar{j}} w_{ki\bar{j}} w_{\bar{k}} + F^{i\bar{j}} (w_{ki} w_{\bar{k}\bar{j}} + w_{k\bar{j}} w_{\bar{k}i}). \tag{3.8}$$

Recall that $\varphi(z,u) = -\beta u + \phi(z)$. We have

$$\begin{aligned} F^{i\bar{j}} w_{\bar{k}i\bar{j}} &= F^{i\bar{j}} u_{\bar{k}i\bar{j}} + F^{i\bar{j}} (\varphi d)_{\bar{k}i\bar{j}} \\ &= F^{i\bar{j}} (-\beta u_{\bar{k}i} d_{\bar{j}} - \beta u_{\bar{k}\bar{j}} d_i - \beta u_{i\bar{j}} d_{\bar{k}} - \beta u_{\bar{k}} d_{i\bar{j}} - \beta u_i d_{\bar{k}\bar{j}} \\ &\quad - \beta u_{\bar{j}} d_{\bar{k}i} - \beta u d_{\bar{k}i\bar{j}}) + F^{i\bar{j}} (\phi(x) d)_{\bar{k}i\bar{j}} + \widetilde{\psi}_{\bar{k}} (1 - \beta d). \end{aligned}$$

By Cauchy-Schwarz inequality we get

$$\begin{aligned} F^{i\bar{j}} w_{ki\bar{j}} w_{\bar{k}} + w_k F^{i\bar{j}} w_{\bar{k}i\bar{j}} &\geq (\widetilde{\psi}_{\bar{k}} w_k + \widetilde{\psi}_k w_{\bar{k}})(1 - \beta d) - \varepsilon F^{i\bar{j}} (u_{ki} u_{\bar{k}\bar{j}} + u_{k\bar{j}} u_{\bar{k}i}) \\ &\quad - C_\varepsilon \sum_{i=1}^n F^{i\bar{i}} (|\nabla u|^2 + |\nabla u| + 1), \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} F^{i\bar{j}} (w_{ki} w_{\bar{k}\bar{j}} + w_{k\bar{j}} w_{\bar{k}i}) &\geq (1 - \beta d)^2 F^{i\bar{j}} (u_{ki} u_{\bar{k}\bar{j}} + u_{k\bar{j}} u_{\bar{k}i})(1 - \varepsilon) \\ &\quad - C_\varepsilon \sum_{i=1}^n F^{i\bar{i}} (|\nabla u|^2 + |\nabla u| + 1). \end{aligned} \tag{3.10}$$

Combining with (3.8), (3.9) and (3.10), we obtain

$$F^{i\bar{j}} |\nabla w|^2_{i\bar{j}} \geq -C|\nabla u|^2 - C \sum_{i=1}^n F^{i\bar{i}} (|\nabla u|^2 + |\nabla u| + 1), \tag{3.11}$$

if we choose μ chosen sufficiently small. Substituting (3.11) into (3.7), we have

$$\begin{aligned} 0 \geq\ & -C - C \sum_{i=1}^n F^{i\bar{i}} - 4A^2 F^{i\bar{j}} d_i d_{\bar{j}} \\ & + \left(h'' - 4(h')^2\right) F^{i\bar{j}} u_i u_{\bar{j}} + AF^{i\bar{j}} d_{i\bar{j}} + h' F^{i\bar{j}} u_{i\bar{j}}. \end{aligned} \tag{3.12}$$

Note that $h' > 2\delta$, $h'' - 4(h')^2 > \delta$ and

$$F^{i\bar{j}}u_i u_{\bar{j}} \geq \frac{\tau}{n\tau+1} C \sum_i F^{i\bar{i}} |\nabla u|^2, \quad \sum_i F^{i\bar{i}} \geq (n\tau+1)C,$$

(3.12) yields that

$$\frac{\tau}{n\tau+1}\delta C \sum_{i=1}^n F^{i\bar{i}} |\nabla u|^2 \leq C + C(1+A^2)\sum_i F^{i\bar{i}},$$

then, we have

$$|\nabla u|^2 \leq \frac{n\tau+1}{\tau} C.$$

Combining with Case 1, Case 2 and Case 3, the global gradient estimates is completed. □

4. Global estimates for second derivatives

In this section, we prove the a priori estimates of global second-order derivatives.

4.1. Reduce global second derivatives to double normal second derivatives on the boundary

Theorem 4.1. *Suppose u is a C^4 (η, k)-admissible solution to* (1.2). *Then we have*

$$\sup_{(z,\zeta)\in\Omega\times S^{2n-1}} D_{\zeta\zeta}u(z) \leq C\sqrt{\frac{n\tau+1}{\tau}}(1+\sup_{\partial\Omega}|D_{\nu\nu}u|),$$

where C depends on $n, k, l, \beta, \Omega, |u|_{C^1}, \inf\psi, |\psi|_{C^2}$ and $|\phi|_{C^3}$.

Proof. We consider the function

$$\Phi(z,\zeta) = h(r)(D_{\zeta\zeta}u - v(z,\zeta)) + |\nabla u|^2,$$

where $v(z,\zeta) = a^l D_l u + b$, $a^l = -2\langle\zeta,\nu\rangle\langle\zeta', D\nu^l\rangle - 2\beta\langle\zeta,\nu\rangle(\zeta')^l$, $b = 2\langle\zeta,\nu\rangle\langle\zeta', D\phi\rangle$ and $\zeta' = \zeta - \langle\zeta,\nu\rangle\nu$. The function h is defined by

$$h = e^{-Ar}.$$

Here $r \in C^2(\overline{\Omega})$ with $r|_{\partial\Omega} = 0$ and $D_\nu r = 1$ on $\partial\Omega$, $A = 1 + 2\max_{\partial\Omega}\{|\Pi_{ij}|\} + |\beta|$ and Π_{ij} is the second fundamental form of the boundary.

Denote

$$\max_{(z,\zeta)\in\overline{\Omega}\times S^{2n-1}} \Phi(z,\zeta) = \Phi(z_0,\zeta_0).$$

Then for $\zeta_0 \in S^{2n-1}$, $\max_{z\in\overline{\Omega}}\Phi(z,\zeta_0)$ is attained at $z_0 \in \overline{\Omega}$.

Case 1: $z_0 \in \Omega$.

Differentiating Φ at z_0, we obtain

$$0 = \Phi_i = h' r_i(D_{\zeta_0\zeta_0}u - v(z_0,\zeta_0)) + h(r)(D_{\zeta_0\zeta_0}u - v(z_0,\zeta_0))_i + u_k u_{\bar{k}i} + u_{\bar{k}}u_{ki},$$

and

$$
\begin{aligned}
0 &\geq F^{i\bar{j}}\Phi_{i\bar{j}} \\
&= h'F^{i\bar{j}}r_{i\bar{j}}(D_{\zeta_0\zeta_0}u - v(z_0,\zeta_0)) \\
&\quad + F^{i\bar{j}}h''r_i r_{\bar{j}}(D_{\zeta_0\zeta_0}u - v(z_0,\zeta_0)) + h'r_iF^{i\bar{j}}(D_{\zeta_0\zeta_0}u - v(z_0,\zeta_0))_{\bar{j}} \\
&\quad + F^{i\bar{j}}h'r_{\bar{j}}(D_{\zeta_0\zeta_0}u - v(z_0,\zeta_0))_i + hF^{i\bar{j}}(D_{\zeta_0\zeta_0}u - v(z_0,\zeta_0))_{i\bar{j}} \\
&\quad + F^{i\bar{j}}(u_{k\bar{j}}u_{\bar{k}i} + u_{\bar{k}\bar{j}}u_{ki} + u_k u_{\bar{k}i\bar{j}} + u_{\bar{k}}u_{ki\bar{j}}) \\
&= 2h'F^{i\bar{j}}r_i\left(-\frac{h'r_{\bar{j}}(D_{\zeta_0\zeta_0}u - v(z_0,\zeta_0)) + u_k u_{\bar{k}\bar{j}} + u_{\bar{k}}u_{k\bar{j}}}{h}\right) \\
&\quad + (h'F^{i\bar{j}}r_{i\bar{j}} + F^{i\bar{j}}h''r_ir_{\bar{j}})(D_{\zeta_0\zeta_0}u - v(z_0,\zeta_0)) + F^{i\bar{j}}(u_{k\bar{j}}u_{\bar{k}i} + u_{\bar{k}\bar{j}}u_{ki}) \\
&\quad + F^{i\bar{j}}(u_k u_{\bar{k}i\bar{j}} + u_{\bar{k}}u_{ki\bar{j}}) + hF^{i\bar{j}}(D_{\zeta_0\zeta_0}u - v(z_0,\zeta_0))_{i\bar{j}}.
\end{aligned}
$$

Note that

$$
\widetilde{\psi}_{\zeta_0\zeta_0} = G^{i\bar{j},k\bar{l}}\eta_{i\bar{j}\zeta_0}\eta_{k\bar{l}\zeta_0} + G^{i\bar{j}}\eta_{i\bar{j}\zeta_0\zeta_0} \leq G^{i\bar{j}}\eta_{i\bar{j}\zeta_0\zeta_0} = F^{i\bar{j}}u_{i\bar{j}\zeta_0\zeta_0},
$$

which implies that

$$
\begin{aligned}
0 \geq\ & -A^2hF^{i\bar{j}}r_ir_{\bar{j}}(D_{\zeta_0\zeta_0}u - v(z_0,\zeta_0)) - 2AF^{i\bar{j}}r_i(u_ku_{\bar{k}\bar{j}} + u_{\bar{k}}u_{k\bar{j}}) \\
& - AhF^{i\bar{j}}r_{i\bar{j}}(D_{\zeta_0\zeta_0}u - v(z_0,\zeta_0)) + F^{i\bar{j}}(u_{k\bar{j}}u_{\bar{k}i} + u_{\bar{k}\bar{j}}u_{ki}) + u_k\widetilde{\psi}_{\bar{k}} + u_{\bar{k}}\widetilde{\psi}_k \\
& + h\widetilde{\psi}_{\zeta_0\zeta_0} - hF^{i\bar{j}}(a^l_{i\bar{j}}D_lu + 2a^l_i(D_lu)_{\bar{j}} + a^l(D_lu)_{i\bar{j}} + b_{i\bar{j}}).
\end{aligned}
$$

By Cauchy-Schwarz inequality, we see

$$
\begin{aligned}
& -2AF^{i\bar{j}}r_i(u_ku_{\bar{k}\bar{j}} + u_{\bar{k}}u_{k\bar{j}}) - hF^{i\bar{j}}(a^l_{i\bar{j}}D_lu + 2a^l_i(D_lu)_{\bar{j}} + a^l(D_lu)_{i\bar{j}} + b_{i\bar{j}}) \\
\geq\ & -\frac{1}{4}F^{i\bar{j}}(u_{\bar{k}\bar{j}}u_{ki} + u_{k\bar{j}}u_{\bar{k}i}) - 32A^2|\nabla u|^2F^{i\bar{j}}r_ir_{\bar{j}} \\
& -\frac{1}{4}F^{i\bar{j}}(D_lu)_{\bar{j}}(D_lu)_i - C_1(h+h^2)\sum F^{i\bar{i}} - ha^lD_l\widetilde{\psi}.
\end{aligned}
$$

Note that $\sum_{i=1}^{2n} D_{l\bar{j}}uD_{li}u = \sum_{p=1}^{n} 2u_{p\bar{j}}u_{\bar{p}i} + 2u_{\bar{p}\bar{j}}u_{pi}$. Above all, we then arrive at

$$
\begin{aligned}
0 \geq\ & -A^2hF^{i\bar{j}}r_ir_{\bar{j}}(D_{\zeta_0\zeta_0}u - v(z_0,\zeta_0)) - AhF^{i\bar{j}}r_{i\bar{j}}(D_{\zeta_0\zeta_0}u - v(z_0,\zeta_0)) \\
& + \frac{1}{4}F^{i\bar{j}}(u_{k\bar{j}}u_{\bar{k}i} + u_{\bar{k}\bar{j}}u_{ki}) - 32A^2|\nabla u|^2F^{i\bar{j}}r_ir_{\bar{j}} + h\widetilde{\psi}_uu_{\zeta_0\zeta_0} \\
& - C_3h - C_2 - C_1(h+h^2)\sum F^{i\bar{i}}.
\end{aligned}
$$

By Lemma 2.2, we obtain that

$$
\begin{aligned}
0 \geq\ & D_{\zeta_0\zeta_0}u\left[-A^2hF^{i\bar{j}}r_ir_{\bar{j}} - AhF^{i\bar{j}}r_{i\bar{j}}\right] + h\widetilde{\psi}_uu_{\zeta_0\zeta_0} \\
& + v(z_0,\zeta_0)[A^2hF^{i\bar{j}}r_ir_{\bar{j}} + AhF^{i\bar{j}}r_{i\bar{j}}] + \frac{1}{8}\frac{\tau}{n\tau+1}C_{n,k,l}\sum F^{i\bar{i}}(D_{\zeta_0\zeta_0}u)^2 \\
& - 32A^2|\nabla u|^2F^{i\bar{j}}r_ir_{\bar{j}} - C_1(h+h^2)\sum F^{i\bar{i}} - C_3h - C_2
\end{aligned}
$$

$$\geq -C_4|D_{\zeta_0\zeta_0}u|\sum F^{i\bar{i}} + \frac{1}{8}\frac{\tau}{n\tau+1}C_{n,k,l}(D_{\zeta_0\zeta_0}u)^2\sum F^{i\bar{i}} - C_5\sum F^{i\bar{i}},$$

where C_4, C_5 depends on $|u|_{C^1}, \inf u, |\phi|_{C^3}, |r|_{C^2}, \partial\Omega$ and β. It implies that

$$D_{\zeta_0\zeta_0}u(z_0) \leq C\sqrt{\frac{n\tau+1}{\tau}},$$

and hence the result is proved.

Case 2: $z_0 \in \partial\Omega$. We further divide this case into two subcases according to whether the direction ζ_0 is tangential or non-tangential to the boundary.

Case 2.1: If ζ_0 is non-tangential at $z_0 \in \partial\Omega$, then we can write

$$\zeta_0 = \langle\zeta_0,\gamma\rangle\gamma + \langle\zeta_0,\nu\rangle\nu,$$

where $\langle\gamma,\nu\rangle = 0$. Then, we have

$$\begin{aligned} D_{\zeta_0\zeta_0}u(z_0) = &\langle\zeta_0,\gamma\rangle^2 D_{\gamma\gamma}u(z_0) + \langle\zeta_0,\nu\rangle^2 D_{\nu\nu}u(z_0) \\ &+ 2\langle\zeta_0,\nu\rangle[\zeta_0 - \langle\zeta_0,\nu\rangle\nu][D\phi - \beta Du - D_l u D\nu^l], \end{aligned}$$

which implies that

$$\Phi(z_0,\zeta_0) = \langle\zeta_0,\gamma\rangle^2\Phi(z_0,\gamma) + \langle\zeta_0,\nu\rangle^2\Phi(z_0,\nu).$$

By the definition of $\Phi(z_0,\zeta_0)$, we know

$$\Phi(z_0,\zeta_0) = \Phi(z_0,\nu) \leq C(1+\max_{\partial\Omega}|D_{\nu\nu}u|).$$

Case 2.2: If ζ_0 is tangential at $z_0 \in \partial\Omega$, then by (2.1) we have

$$\begin{aligned} 0 &\leq D_\nu\Phi(z_0,\zeta_0) \\ &= -A(D_{\zeta_0\zeta_0}u - a^l D_l u - b) + D_\nu D_{\zeta_0\zeta_0}u \\ &\quad - D_\nu a^l D_l u - a^l D_\nu D_l u - D_\nu b + \frac{1}{2}D_k u D_\nu D_k u \\ &\leq -AD_{\zeta_0\zeta_0}u + D_\nu D_{\zeta_0\zeta_0}u + C|D_\nu D_k u| + C \\ &= D_{\zeta_0\zeta_0}(-\beta u + \phi) - (D_{\zeta_0\zeta_0}\nu^k)D_k u - 2(D_{\zeta_0}\nu^k)D_{\zeta_0}D_k u \\ &\quad - AD_{\zeta_0\zeta_0}u + C|D_\nu D_k u| + C \\ &\leq (-A-\beta)D_{\zeta_0\zeta_0}u - 2(D_{\zeta_0}\nu^k)D_{\zeta_0}D_k u + C|D_\nu D_k u| + C, \end{aligned} \tag{4.1}$$

where C depends on $\inf\psi, |u|_{C^1}, |b|_{C^1}, |a|_{C^1}, |\phi|_{C^3}$ and $\partial\Omega$. By the same argument in [21, 23], we see

$$\max\left\{-2(D_{\zeta_0}\nu^k)D_{\zeta_0}D_k u, |D_\nu D_k u|\right\} \leq C\left(1 + |D_{\nu\nu}u| + D_{\zeta_0\zeta_0}u\right).$$

Therefore, we have

$$0 \leq (-A+C-\beta)D_{\zeta_0\zeta_0}u + C(1+|D_{\nu\nu}u|) + C.$$

Choosing A sufficiently large such that $-A+C-\beta > 1$, then we get

$$\Phi(z_0,\zeta_0) \leq C(1+\max_{\partial\Omega}|D_{\nu\nu}u|),$$

where C depends on $\beta, \inf\psi, |u|_{C^1}, |b|_{C^1}, |a|_{C^1}, |\phi|_{C^3}, \max|r|$ and $\partial\Omega$. □

Now we estimate the double normal derivative on the boundary.

4.2. Estimate of double normal second derivatives on boundary

Theorem 4.2. *Let $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ be an (η, k)-admissible solution to equation* (1.2). *Then, we have*

$$\max_{\partial\Omega}|D_{\nu\nu}u| \le C\left(\frac{n\tau+1}{\tau}\right)^2,$$

where C depends on $n, k, l, \Omega, \beta, |u|_{C^1}, \inf\psi, |\psi|_{C^1}$ and $|\phi|_{C^3}$.

Proof. Denote $M = \max_{\partial\Omega}|D_{\nu\nu}u|$ and $\varphi(z,u) = -\beta u + \phi$. We divide our proof into two cases.

Case 1: $\sup_{\partial\Omega}|u_{\nu\nu}| = -\inf_{\partial\Omega}u_{\nu\nu} = -u_{\nu\nu}(z_1) = M$.

We construct the following auxiliary function

$$\Phi = \langle Du, Dr\rangle - \varphi(z,u) + M^{-\frac{1}{2}}(\langle Du, Dr\rangle - \varphi(z,u))^2 + \frac{1}{2}Mr,$$

where r is a smooth function such that $r|_\Omega < 0$, $r|_{\partial\Omega} = 0$ and $\frac{\partial r}{\partial\nu}|_{\partial\Omega} = 1$. Define

$$\Omega_\mu := \{z \in \Omega : d(z, \partial\Omega) < \mu\}.$$

It is obvious that $\Phi|_{\partial\Omega} = 0$. Take a small positive constant μ such that $r = -\text{dist}(\cdot, \partial\Omega)$ on Ω_μ. Note that there exists a constant C depending on $\mu, \beta, |u|_{C_1}, |r|_{C^1}, \partial\Omega, |\phi|_{C^0}$ such that

$$\Phi < 0 \quad \text{on } \partial\Omega_\mu \setminus \partial\Omega, \quad |M^{-\frac{1}{2}}(\langle Du, Dr\rangle - \varphi(z,u))| \le \frac{1}{8} \tag{4.2}$$

when $M \ge C_1$. Without loss of generality we assume that $M \ge C_1$, otherwise the proof is completed.

Case 1.1: $\max_{\overline{\Omega_\mu}}\Phi = \Phi(z_0)$ with $z_0 \in \Omega_\mu$.

Then have

$$0 = \Phi_i(z_0) = [\langle Du, Dr\rangle - \varphi(z,u)]_i(1 + 2M^{-\frac{1}{2}}(\langle Du, Dr\rangle - \varphi(z,u))) + \frac{1}{2}Mr_i$$

and

$$\begin{aligned}
0 &\ge F^{i\bar{j}}\Phi_{i\bar{j}}(z_0) \\
&= F^{i\bar{j}}[\langle Du, Dr\rangle - \varphi(z,u)]_{i\bar{j}}(1 + 2M^{-\frac{1}{2}}(\langle Du, Dr\rangle - \varphi(z,u))) \\
&\quad + 2M^{-\frac{1}{2}}F^{i\bar{j}}[\langle Du, Dr\rangle - \varphi(z,u)]_i[\langle Du, Dr\rangle - \varphi(z,u)]_{\bar{j}} + F^{i\bar{j}}\frac{1}{2}Mr_{i\bar{j}} \\
&= F^{i\bar{j}}[\langle Du, Dr\rangle - \varphi(z,u)]_{i\bar{j}}(1 + 2M^{-\frac{1}{2}}(\langle Du, Dr\rangle - \varphi(z,u))) \\
&\quad + \frac{M^{\frac{3}{2}}F^{i\bar{j}}r_ir_{\bar{j}}}{2(1 + 2M^{-\frac{1}{2}}(\langle Du, Dr\rangle - \varphi(z,u)))^2} + \frac{1}{2}MF^{i\bar{j}}r_{i\bar{j}}.
\end{aligned} \tag{4.3}$$

An easy computation shows that

$$F^{i\bar{j}}[\langle Du, Dr\rangle - \varphi(z,u)]_{i\bar{j}} \geq -C_2(1+M)\sum_{i=1}^{n} F^{i\bar{i}}, \tag{4.4}$$

where C_2 is a constant depending on $|u|_{C^1}, \inf\psi, |\psi|_{C^1}, |\phi|_{C^3}, \beta$ and $|r|_{C^3}$. By (4.2), we obtain

$$\frac{3}{4} \leq 1 + 2M^{-\frac{1}{2}}(\langle Du, Dr\rangle - \varphi(z,u)) \leq \frac{5}{4}. \tag{4.5}$$

Combining with (4.3), (4.4), (4.5), Lemma 2.3 and the fact that $|Dr|^2 = 1$ on Ω_μ, we obtain

$$0 \geq -C(1+M)\sum_{i=1}^{n} F^{i\bar{i}} + \frac{8}{25}\frac{\tau}{n\tau+1}C_{n,k,l}\sum_{i=1}^{n} F^{i\bar{i}} M^{\frac{3}{2}},$$

which implies that

$$M \leq C\left(\frac{n\tau+1}{\tau}\right)^2,$$

where C depends on $|u|_{C^1}, \inf\psi, |\psi|_{C^1}, |\phi|_{C^3}, \beta, n, k, l$ and $|r|_{C^3}$.

Case 1.2: $\max\limits_{\overline{\Omega_\mu}}\Phi = \Phi(z_0)$ with $z_0 \in \partial\Omega_\mu$.

Combining with (4.2), we know that $\max\limits_{\overline{\Omega}}\Phi = \Phi(z_2)$ with $z_2 \in \partial\Omega$. By Hopf Lemma, we have on $\partial\Omega$

$$0 \leq \frac{\partial\Phi}{\partial\nu} = (r_l D_\nu u_l + u_l D_\nu r_l - D_\nu\varphi)(1 + 2M^{-\frac{1}{2}}(\langle Du, Dr\rangle - \varphi(z,u))) + \frac{1}{2}M.$$

Then from the above inequality we have

$$0 \leq -\frac{3}{4}M - \frac{3}{4}\inf_{\partial\Omega}\varphi_\nu + \frac{1}{2}M,$$

which implies that

$$\sup_{\partial\Omega}|u_{\nu\nu}| \leq C.$$

Case 2: $\sup\limits_{\partial\Omega}|u_{\nu\nu}| = \sup\limits_{\partial\Omega}u_{\nu\nu} = u_{\nu\nu}(z_3) = M$.

Similarly, we can construct an auxiliary function

$$\bar{\Phi} = \langle Du, Dr\rangle - \varphi(z,u) - M^{-\frac{1}{2}}(\langle Du, Dr\rangle - \varphi)^2 - \frac{1}{2}Mr.$$

The similar argument works for $\bar{\Phi}$, we also obtain the conclusion. □

5. Proof of the main theorem

Proof of Theorem 1.2. Now we can give the proof of Theorem 1.2. After establishing a priori estimates in Theorem 3.1, Theorem 3.3, Theorem 4.1, Theorem 4.2 and Evans-Krylov Theorem, we obtain

$$|u|_{C^{2,\alpha}(\bar{\Omega})} \leq C$$

for some uniform C independent of $\inf \psi, \beta, |\phi|_{C^3}$ and $0 < \alpha < 1$. Applying the method of continuity, we complete the proof of Theorem 1.2. □

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