Bull. Korean Math. Soc. **61** (2024), No. 4, pp. 959–968
https://doi.org/10.4134/BKMS.b230535
pISSN: 1015-8634 / eISSN: 2234-3016

THE GROWTH OF BLOCH FUNCTIONS IN SOME SPACES

Wenwan Yang and Junming Zhugeliu

Abstract. Suppose f belongs to the Bloch space with $f(0) = 0$. For $0 < r < 1$ and $0 < p < \infty$, we show that

$$\begin{aligned} M_p(r,f) &= \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p dt\right)^{1/p} \\ &\leq \left(\frac{\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+1-k)}\right)^{1/p} \rho_{\mathcal{B}} \left(\log\frac{1}{1-r^2}\right)^{1/2}, \end{aligned}$$

where $\rho_{\mathcal{B}}(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|$ and k is the integer satisfying $0 < p - 2k \leq 2$. Moreover, we prove that for $0 < r < 1$ and $p > 1$,

$$\|f_r\|_{B_q} \leq r\,\rho_{\mathcal{B}}(f)\left(\frac{1}{(1-r^2)(q-1)}\right)^{1/q},$$

where $f_r(z) = f(rz)$ and $\|\cdot\|_{B_q}$ is the Besov seminorm given by

$$\|f\|_{B_q} = \left(\int_{\mathbb{D}} |f'(z)|^q (1-|z|^2)^{q-2} dA(z)\right).$$

These results improve previous results of Clunie and MacGregor.

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the space of holomorphic functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. If $0 < r < 1$ and $0 < p < \infty$, for $f \in \mathcal{H}(\mathbb{D})$, we set

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p dt\right)^{1/p}.$$

The Hardy space H^p consists of those functions $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r,f) < \infty.$$

The Bloch space $\mathcal{B}$ consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\rho_{\mathcal{B}}(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

Received September 12, 2023; Revised December 15, 2023; Accepted December 21, 2023.
2020 *Mathematics Subject Classification.* Primary 30H30.
Key words and phrases. Bloch spaces, integral mean, Besov space.

The Bloch space is a Banach space with the norm $\|\cdot\|_{\mathcal{B}}$ defined by

$$\|f\|_{\mathcal{B}} = |f(0)| + \rho_{\mathcal{B}}(f), \quad f \in \mathcal{B}.$$

Notation: Throughout this paper, we write $U \lesssim V$ *(or* $V \gtrsim U$*) for* $U \leq cV$ *for a positive constant* c*, and moreover* $U \approx V$ *for both* $U \lesssim V$ *and* $V \lesssim U$*.*

The well-known Hardy convexity theorem [5] says that $M_p(r, f)$ is an increasing function of r and $\log M_p(r, f)$ is a convex function of $\log r$. In [9], Mashreghi showed that $\frac{d}{dr} M_p(r, f) = o(\log r)$. Analogous to the Hardy convexity theorem, area integral means of analytic functions were studied in [3], [13] and [15].

Clunie and MacGregor [2] and Makarov [8] proved that if $f \in \mathcal{B}$, then for $0 < p < \infty$,

$$M_p(r, f) = O\left(\left(\log \frac{1}{1-r}\right)^{1/2}\right) \quad \text{as } r \to 1. \tag{1.1}$$

Let $0 < r < 1$. If we write $f_r(z) = f(rz)$, then $M_p(r, f) = \|f_r\|_{H^p}$. It is shown in [4] that the order $\frac{1}{2}$ in the right of (1.1) is sharp in the sense that there is a function $f \in \mathcal{B}$ such that

$$M_p(r, f) \approx \left(\log \frac{1}{1-r}\right)^{1/2} \quad \text{as } r \to 1.$$

The fact that the right hand side of (1.1) is unbounded implies that the Bloch space is not contained in the Hardy space.

In [11, Theorem 8.9], it is proved that if $f \in \mathcal{B}$ with $f(0) = 0$, then

$$M_{2n}^{2n}(r, f) \leq n! \rho_{\mathcal{B}}(f) \left(\log \frac{1}{1-r^2}\right)^n \tag{1.2}$$

for $0 < r < 1$ and $n = 0, 1, 2, \ldots$. In this paper, we intend to improve (1.1) and generalize (1.2) to the following form.

Theorem 1. *Let* $0 < r < 1$ *and* $f \in \mathcal{B}$ *with* $f(0) = 0$*. For* $0 < p < \infty$*, let* k *be the integer such that* $0 < p - 2k \leq 2$*. Then*

$$M_p^p(r, f) \leq \frac{\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+1-k)} \rho_{\mathcal{B}}^p \left(\log \frac{1}{1-r^2}\right)^{p/2}. \tag{1.3}$$

Notice that

$$\log \frac{1}{1-r^2} = r^2 + \frac{r^4}{2} + O(r^6) \quad \text{as } r \to 0.$$

We have the following corollary.

Corollary 2. *In the conditions of Theorem 1, we have*

$$M_p(r, f) \leq \left(\frac{\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+1-k)}\right)^{1/p} \rho_{\mathcal{B}}(f)\, r \quad \textit{as } r \to 0. \tag{1.4}$$

It is well known that $\mathcal{B}$ is the maximal Möbius invariant function space. There are several other Möbius invariant function spaces contained in $\mathcal{B}$. One is the $\mathcal{Q}_p$ space defined by

$$\mathcal{Q}_p = \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{\mathcal{Q}_p}^2 = \sup_{a\in} \int_{\mathbb{D}} |f'(z)|^2 \left(1 - \left|\frac{a-z}{1-\bar{a}z}\right|^2\right)^p \mathrm{d}A(z) < \infty \right\},$$

where $0 \leq p < \infty$ and $\mathrm{d}A(w) = \pi^{-1}\mathrm{d}x\,\mathrm{d}y$ is the normalized Lebesgue measure on $\mathbb{D}$. It is known that $\mathcal{Q}_0$ is the Dirichlet space $\mathcal{D}$, $\mathcal{Q}_1$ is the so-called $\mathcal{BMOA}$, and when $p > 1$, $\mathcal{Q}_p = \mathcal{B}$. The growth of $\|f_r\|_{\mathcal{Q}_p}$ is characterized in [1], see also [14, Theorem 7.1.4]. It is proved that

$$\|f_r\|_{\mathcal{Q}_p} \leq \left(\int_0^r \left(\log\frac{r}{t}\right)^p \frac{\mathrm{d}t^2}{(1-t^2)^2}\right)^{1/2} \rho_{\mathcal{B}}(f), \tag{1.5}$$

where $0 < r < 1$ and $p > 0$. When $p = 1$, this inequality is given by Korenblum in [7]. Here we intend to characterize the case $p = 0$, that is, the growth of f_r in the seminorm of the Dirichlet space. More generally, for $1 < q < \infty$, the Besov space B_q is defined as

$$B_q = \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{B_q}^q = \int_{\mathbb{D}} |f'(z)|^q \left(1 - |z|^2\right)^{q-2} \mathrm{d}A(z) < \infty \right\}.$$

Trivially, $B_2 = \mathcal{Q}_0 = \mathcal{D}$. Moreover, B_1 is the minimal Möbius invariant space given by

$$B_1 = \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{B_1} = \int_{\mathbb{D}} |f''(z)|\,\mathrm{d}A(z) < \infty \right\}.$$

The Besov space is another Möbius invariant space on $\mathbb{D}$.

We have the following theorem.

Theorem 3. *Let $0 < r < 1$ and $f \in \mathcal{B}$ with $f(0) = 0$. We have*

(1) *For $1 < q < \infty$,*

$$\|f_r\|_{B_q} \leq r\rho_{\mathcal{B}}(f) \left(\frac{1}{(1-r^2)(q-1)}\right)^{1/q}. \tag{1.6}$$

(2) *If $f'(0) = 0$, then*

$$\|f_r\|_{B_1} \leq \frac{r^2 \tilde{\rho}_{\mathcal{B}}(f)}{(1-r^2)}, \tag{1.7}$$

where $\tilde{\rho}_{\mathcal{B}}(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)^2|f''(z)|$.

In particular, let $q = 2$ in Theorem 3, then

$$\|f_r\|_{\mathcal{D}} \leq r\rho_{\mathcal{B}}(f) \left(\frac{1}{(1-r^2)}\right)^{1/2}.$$

This is just the case $p = 0$ in (1.5).

2. Preliminaries

For $0 < r < 1$, $0 < p < \infty$ and $f \in \mathcal{H}(\mathbb{D})$, the following equality is contained in [12].

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p dt = |f(0)|^p + \frac{p^2}{2}\int_{r\mathbb{D}} |f(w)|^{p-2}|f'(w)|^2 \log\frac{r}{|w|}\, dA(w). \tag{2.1}$$

Let $w = rz$ in the right-hand side in (2.1), then we have

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p dt = |f(0)|^p + \frac{p^2r^2}{2}\int_{\mathbb{D}} |f(rz)|^{p-2}|f'(rz)|^2 \log\frac{1}{|z|}\, dA(z). \tag{2.2}$$

The following identity belongs to Hardy. We quoted it from [11, p. 174].

$$\frac{d}{dr}\left(r\frac{d}{dr}\int_0^{2\pi} |f(re^{it})|^p dt\right) = rp^2\int_0^{2\pi} |f(re^{it})|^{p-2}|f'(re^{it})|^2 dt \tag{2.3}$$

for $f \in \mathcal{H}(\mathbb{D})$.

The following lemma is quoted from [16, Theorem 5.4].

Lemma 4. *If $g \in \mathcal{H}(\mathbb{D})$, then $g \in \mathcal{B}$ if and only if the function $(1-|z|^2)^2|g''(z)|$ is bounded. Moreover, if $g(0) = g'(0) = 0$, then*

$$\tilde{\rho}_{\mathcal{B}}(g) = \sup_{z\in\mathbb{D}}(1-|z|^2)^2|g''(z)| \approx \rho_{\mathcal{B}}(g).$$

The following lemma is useful in our verification.

Lemma 5. *If $0 < r < 1$, $1 < p < \infty$, then*

(1) $\displaystyle\int_0^1 \frac{s}{(1-r^2s^2)^2}\log\frac{1}{s}\, ds = \frac{1}{4r^2}\log\frac{1}{1-r^2}$.

(2) $\displaystyle\int_0^1 \frac{(1-s^2)^{p-2}s}{(1-r^2s^2)^p}\, ds = \frac{1}{2(1-r^2)(p-1)}$.

(3) $\displaystyle\int_0^1 \frac{s}{(1-r^2s^2)^2}\, ds = \frac{1}{2(1-r^2)}$.

(4) $\displaystyle\frac{d}{dr}\left(r\frac{d}{dr}\right)\left(\log\frac{1}{1-r^2}\right)^p = \frac{4pr\left(\left(\log\frac{1}{1-r^2}\right)^{p-1} + r^2(p-1)\left(\log\frac{1}{1-r^2}\right)^{p-2}\right)}{(1-r^2)^2}$.

Proof. Straightforward computation gives this lemma. □

3. Proof of Theorems

In order of give the proof of Theorem 1, we prove the following theorem, which is weaker that Theorem 1.

Theorem 6. *Let $0 < r < 1$ and $f \in \mathcal{B}$ with $f(0) = 0$. For $0 < p < \infty$, we have*

$$M_p(r,f) \le \max\left\{1, \frac{p}{2}\right\}\rho_{\mathcal{B}}(f)\left(\log\frac{1}{1-r^2}\right)^{1/2}. \tag{3.1}$$

Proof. When $p = 2$, (2.2) gives that

$$
\begin{aligned}
(M_2(r,f))^2 &= 2r^2 \int_{\mathbb{D}} |f'(rz)|^2 \log \frac{1}{|z|} \,\mathrm{d}A(z) \\
&\le 2r^2 \int_{\mathbb{D}} \left(\frac{\rho_{\mathcal{B}}(f)}{1 - r^2|z|^2} \right)^2 \log \frac{1}{|z|} \,\mathrm{d}A(z) \\
&= 2r^2 \rho_{\mathcal{B}}^2(f) \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \frac{s}{(1 - r^2 s^2)^2} \log \frac{1}{s} \,\mathrm{d}s \,\mathrm{d}t \\
&= \rho_{\mathcal{B}}^2(f) \log \frac{1}{1 - r^2}.
\end{aligned}
$$

The last equality follows from Lemma 5(1).

Similarly, when $p > 2$, we can use Hölder's inequality to get that

$$
\begin{aligned}
(M_p(r,f))^p &= \frac{p^2 r^2}{2} \int_{\mathbb{D}} |f(rz)|^{p-2} |f'(rz)|^2 \log \frac{1}{|z|} \,\mathrm{d}A(z) \\
&\le \frac{p^2 r^2}{2} \int_{\mathbb{D}} |f(rz)|^{p-2} \left(\frac{\rho_{\mathcal{B}}(f)}{1 - r^2|z|^2} \right)^2 \log \frac{1}{|z|} \,\mathrm{d}A(z) \\
&= \frac{p^2 r^2 \rho_{\mathcal{B}}^2(f)}{2} \int_0^1 \frac{1}{\pi} \int_0^{2\pi} |f(rs\mathrm{e}^{\mathrm{i}t})|^{p-2} \,\mathrm{d}t \, \frac{s}{(1 - r^2 s^2)^2} \log \frac{1}{s} \,\mathrm{d}s \\
&\le \frac{p^2 r^2 \rho_{\mathcal{B}}^2(f)}{2\pi} \int_0^1 \left(\int_0^{2\pi} |f(rs\mathrm{e}^{\mathrm{i}t})|^p \,\mathrm{d}t \right)^{\frac{p-2}{p}} (2\pi)^{\frac{2}{p}} \, \frac{s \log \frac{1}{s} \,\mathrm{d}s}{(1 - r^2 s^2)^2} \\
&= p^2 r^2 \rho_{\mathcal{B}}^2(f) \int_0^1 \left(\frac{1}{2\pi} \int_0^{2\pi} |f(rs\mathrm{e}^{\mathrm{i}t})|^p \,\mathrm{d}t \right)^{\frac{p-2}{p}} \frac{s \log \frac{1}{s} \,\mathrm{d}s}{(1 - r^2 s^2)^2} \\
&= p^2 r^2 \rho_{\mathcal{B}}^2(f) \int_0^1 (M_p(rs,f))^{p-2} \, \frac{s \log \frac{1}{s} \,\mathrm{d}s}{(1 - r^2 s^2)^2}.
\end{aligned}
$$

Since $M_p(r,f)$ is a positive increasing function of r, we have

$$
M_p(rs,f) \le M_p(r,f), \quad 0 < s < 1.
$$

This implies that

$$
(M_p(r,f))^2 \le p^2 r^2 \rho_{\mathcal{B}}^2(f) \int_0^1 \frac{s \log \frac{1}{s} \,\mathrm{d}s}{(1 - r^2 s^2)^2} = \frac{p^2 \rho_{\mathcal{B}}^2(f)}{4} \log \frac{1}{1 - r^2}.
$$

When $0 < p < 2$, it follows from the Hölder's inequality that

$$
\begin{aligned}
(M_p(r,f))^p &= \frac{1}{2\pi} \int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}t})|^p \mathrm{d}t \\
&\le \frac{1}{2\pi} \left(\int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}t})|^2 \mathrm{d}t \right)^{p/2} \cdot (2\pi)^{\frac{2-p}{2}} \\
&= (M_2(r,f))^p
\end{aligned}
$$

$$\leq \rho_{\mathcal{B}}^p(f)\left(\log\frac{1}{1-r^2}\right)^{p/2}.$$

This completes the proof. □

Now we are ready to prove Theorem 1. It is inspired by the proof of [11, Theorem 8.9].

Proof of Theorem 1. When $0 < p \leq 2$, it is contained in Theorem 6. Thus (1.3) holds when $2k-2 < p \leq 2k$ with $k = 1$. We suppose that it also holds for some $k \in \mathbb{N}$. That is,

$$\frac{1}{2\pi}\int_0^{2\pi}|f(re^{it})|^p dt \leq \frac{\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+1-k)}\,\rho_{\mathcal{B}}^p\left(\log\frac{1}{1-r^2}\right)^{p/2} \tag{3.2}$$

for $2k-2 < p \leq 2k$.

When $2k < p \leq 2k+2$, (2.3) gives that

$$\begin{aligned}\frac{1}{2\pi}\frac{d}{dr}\left(r\frac{d}{dr}\int_0^{2\pi}|f(re^{it})|^p dt\right) &= \frac{rp^2}{2\pi}\int_0^{2\pi}|f(re^{it})|^{p-2}|f'(re^{it})|^2 dt\\ &\leq \frac{rp^2\rho_{\mathcal{B}}^2}{2\pi(1-r^2)^2}\int_0^{2\pi}|f(re^{it})|^{p-2}dt\\ &\leq \frac{rp^2\rho_{\mathcal{B}}^p\Gamma(\frac{p}{2})}{(1-r^2)^2\Gamma(\frac{p}{2}-k)}\left(\log\frac{1}{1-r^2}\right)^{\frac{p}{2}-1},\end{aligned}$$

where the last inequality follows from (3.2). Lemma 5(4) gives that

$$\frac{r\left(\log\frac{1}{1-r^2}\right)^{q-1}}{(1-r^2)^2} \leq \frac{1}{4q}\frac{d}{dr}\left(r\frac{d}{dr}\right)\left(\log\frac{1}{1-r^2}\right)^q$$

for $q > 1$. Let $q = \frac{p}{2}$, then we have

$$\begin{aligned}&\frac{rp^2\rho_{\mathcal{B}}^p\Gamma(\frac{p}{2})}{(1-r^2)^2\Gamma(\frac{p}{2}-k)}\left(\log\frac{1}{1-r^2}\right)^{\frac{p}{2}-1}\\ &\leq \frac{p\rho_{\mathcal{B}}^p\Gamma(\frac{p}{2})}{2\Gamma(\frac{p}{2}-k)}\frac{d}{dr}\left(r\frac{d}{dr}\right)\left(\log\frac{1}{1-r^2}\right)^{\frac{p}{2}}\\ &= \frac{\rho_{\mathcal{B}}^p\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+1-(k+1))}\frac{d}{dr}\left(r\frac{d}{dr}\right)\left(\log\frac{1}{1-r^2}\right)^{\frac{p}{2}}.\end{aligned}$$

Thus we arrive at

$$\frac{d}{dr}\left(r\frac{d}{dr}M_p^p(r,f)\right) \leq \frac{\rho_{\mathcal{B}}^p\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+1-(k+1))}\frac{d}{dr}\left(r\frac{d}{dr}\right)\left(\log\frac{1}{1-r^2}\right)^{\frac{p}{2}}.$$

Integrating twice gives the desired result since both sides vanish for $r \to 0$. □

Remark 7. Note that Theorems 1 and 6 coincide when $0 < p \leq 4$. However, Stirling's formula gives that

$$\frac{\left(\dfrac{\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+1-k)}\right)^{1/p}}{\left(\dfrac{p}{2\mathrm{e}}\right)^{1/2}} = 1 \quad \text{as} \;\; p \to \infty.$$

Then from Theorem 1 we deduce

$$M_p(r,f) \leq \left(\frac{p}{2\mathrm{e}}\right)^{1/2} \rho_{\mathcal{B}} \left(\log \frac{1}{1-r^2}\right)^{1/2} \qquad \text{as} \;\; p \to \infty.$$

This upper bound is better than the upper bound in Theorem 6 for large p.

Now we give a proof of Theorem 3.

Proof of Theorem 3. Similar to the proof of Theorem 6, for $q > 1$, we have

$$\begin{aligned}
\|f_r\|_{B_q}^q &= \int_{\mathbb{D}} r^q |f'(rz)|^q (1-|z|^2)^{q-2} \mathrm{d}A(z) \\
&\leq r^q \rho_{\mathcal{B}}^q(f) \int_{\mathbb{D}} \frac{(1-|z|^2)^{q-2}}{(1-|rz|^2)^q} \,\mathrm{d}A(z) \\
&= r^q \rho_{\mathcal{B}}^q(f) \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \frac{(1-s^2)^{q-2} s}{(1-r^2 s^2)^q} \,\mathrm{d}s \,\mathrm{d}t \\
&= r^q \rho_{\mathcal{B}}^q(f) \frac{1}{(1-r^2)(q-1)},
\end{aligned}$$

where the last equality follows from Lemma 5(2). Moreover, Lemma 4 implies that

$$\begin{aligned}
\|f_r\|_{B_1} &= \int_{\mathbb{D}} r^2 |f''(rz)| \mathrm{d}A(z) \\
&\leq r^2 \tilde{\rho}_{\mathcal{B}}(f) \int_{\mathbb{D}} \frac{1}{(1-|rz|^2)^2} \,\mathrm{d}A(z) \\
&= r^2 \tilde{\rho}_{\mathcal{B}}(f) \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \frac{s}{(1-r^2 s^2)^2} \,\mathrm{d}s \,\mathrm{d}t \\
&= r^2 \tilde{\rho}_{\mathcal{B}}(f) \frac{1}{(1-r^2)},
\end{aligned}$$

where the last equality follows from Lemma 5(3). This completes the proof. □

More generally, let $\alpha \geq 0$ and $p > 0$, the Dirichlet-type space $\mathcal{D}_\alpha^p$ is defined as

$$\mathcal{D}_\alpha^p = \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{\mathcal{D}_\alpha^p}^p = \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^\alpha \mathrm{d}A(z) < \infty \right\}.$$

For $\gamma > 0$, the Bloch type space $\mathcal{B}_\gamma$ is given by

$$\mathcal{B}_\gamma = \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{\mathcal{B}_\gamma} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2)^\gamma < \infty \right\}.$$

Similar to ([6, Theorem 1.7] and [10, Proposition 2.7]), we obtained a related estimation formula. If $f \in \mathcal{B}_\gamma$ with $f(0) = 0$, then similar to the proof of Theorem 3, we have

$$\begin{aligned}
\|f_r\|^p_{\mathcal{D}^p_\alpha} &= \int_{\mathbb{D}} r^p |f'(rz)|^p (1-|z|^2)^\alpha \mathrm{d}A(z) \\
&\le r^p \|f\|^p_{\mathcal{B}_\gamma} \int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|rz|^2)^{p\gamma}} \mathrm{d}A(z) \\
&= r^p \|f\|^p_{\mathcal{B}_\gamma} \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \frac{(1-s^2)^\alpha s}{(1-r^2s^2)^{p\gamma}} \mathrm{d}s\, \mathrm{d}t \\
&= 2r^p \|f\|^p_{\mathcal{B}_\gamma} \int_0^1 \frac{(1-s^2)^\alpha s}{(1-r^2s^2)^{p\gamma}} \mathrm{d}s.
\end{aligned}$$

It can be checked that

$$\int_0^1 \frac{(1-s^2)^\alpha s}{(1-r^2s^2)^{p\gamma}} \mathrm{d}s = \frac{{}_2F_1([1,p\gamma];[2+\alpha];r^2)}{2(\alpha+1)},$$

where ${}_2F_1([1,p\gamma];[2+\alpha];r^2)$ is the hypergeometric function given by

$${}_2F_1([1,p\gamma];[2+\alpha];r^2) = \sum_{k=0}^\infty \frac{r^{2k}\cdot(1)_k\cdot(p\gamma)_k}{k!\cdot(2+\alpha)_k},$$

and $(\alpha)_k$ is the Pochhammer symbol defined by

$$(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}.$$

Notice that

$$(1)_k = k!,$$

we have

$${}_2F_1([1,p\gamma];[2+\alpha];r^2) = \sum_{k=0}^\infty \frac{r^{2k}\cdot(p\gamma)_k}{(2+\alpha)_k} = \sum_{k=0}^\infty \frac{r^{2k}\cdot\Gamma(p\gamma+k)\cdot\Gamma(2+\alpha)}{\Gamma(p\gamma)\cdot\Gamma(2+\alpha+k)}.$$

For fixed p, α and γ, it follows from Stirling's formula that

$$\frac{\Gamma(p\gamma+k)\cdot\Gamma(2+\alpha)}{\Gamma(p\gamma)\cdot\Gamma(2+\alpha+k)} \approx k^{p\gamma-\alpha-2} \quad \text{as} \quad k\to\infty.$$

This implies that

$${}_2F_1([1,p\gamma];[2+\alpha];r^2) \begin{cases} \lesssim (1-r^2)^{\alpha+1-p\gamma} & \text{if } p\gamma-\alpha-1>0; \\ \lesssim \log\dfrac{1}{1-r^2} & \text{if } p\gamma-\alpha-1=0; \\ \text{is bounded} & \text{if } p\gamma-\alpha-1<0, \end{cases}$$

as $r \to 1$. Thus we have the following corollary.

Corollary 8. *Let $p > 0$, $\alpha \geq 0$ and $\gamma > 0$. For $f \in \mathcal{B}_\gamma$ with $f(0) = 0$, we have*:

(1) *If $p\gamma - \alpha - 1 < 0$, then $\mathcal{B}_\gamma \subset \mathcal{D}^p_\alpha$ with $\|f\|_{\mathcal{D}^p_\alpha} \lesssim \|f\|_{\mathcal{B}_\gamma}$.*

(2) *If $p\gamma - \alpha - 1 = 0$, then*

$$\|f_r\|_{\mathcal{D}^p_\alpha} \lesssim r\|f\|_{\mathcal{B}_\gamma} \left(\log \frac{1}{1-r^2}\right)^{1/p}.$$

(3) *If $p\gamma - \alpha - 1 > 0$, then*

$$\|f_r\|_{\mathcal{D}^p_\alpha} \lesssim r\|f\|_{\mathcal{B}_\gamma} \left(\frac{1}{1-r^2}\right)^{\frac{p\gamma-\alpha-1}{p}}.$$

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Wenwan Yang
School of Mathematics and Statistics
Guangdong University of Technology
Guangzhou 510520, P. R. China
Email address: wenwan_yang@163.com

Junming Zhugeliu
School of Mathematics and Statistics
Guangdong University of Technology
Guangzhou 510520, P. R. China
Email address: jmliu@gdut.edu.cn