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RIGIDITY RESULTS FOR COMPACT V-STATIC SPACE

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Abstract. For ($n \geq 5$)-dimensional compact V-static spaces with zero radial Weyl curvature, we prove that ∇f is an eigenvector of Ricci tensor. Furthermore, we also achieve that (M^n, g, f) is T-flat provided $K\frac{|\nabla f|^2}{f} > 0$.

1. Introduction

A V-static space (M^n, g, f) is a Riemannian manifold (M^n, g) which admits a smooth function $f \in C^\infty(M)$ satisfying

$$f_{ij} = fR_{ij} - \frac{1}{n-1}(fR + K)g_{ij} \tag{1.1}$$

with a constant K. Here f_{ij}, R_{ij} and R denote components of the Hessian of f, components of the Ricci curvature tensor and the scalar curvature, respectively. It is worth noting that the existence of a nonzero solution to (1.1) guarantees that the scalar curvature R must be constant. The geometrical significance for this type of space has been extensively studied, and interested readers can consult the references [4, 13, 14](for harmonic Weyl curvature case, see [8]).

When $K = 0$, (1.1) becomes

$$f_{ij} = fR_{ij} - \frac{R}{n-1}fg_{ij},$$

which is called the Vacuum static space. In fact, this space has been well studied and many well known facts have been obtained, see [5, 7–12, 15, 16, 18] and the references therein.

Taking $f = \phi + 1$ and constant $K = -\frac{R}{n}$, (1.1) becomes

$$\phi_{ij} = \phi\Big(R_{ij} - \frac{R}{n-1}g_{ij}\Big) + R_{ij} - \frac{R}{n}g_{ij}.$$

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When M^n is compact, then the metric g is exactly a critical point of the total scalar curvature functional defined on the space of Riemannian metrics with unit volume. For the research in this direction, see [1, 2, 8, 17].

Throughout the article, inspired by [18], we consider rigidity results for ($n \geq 5$)-dimensional compact V-static space with $K \neq 0$ and obtain the following result:

Theorem 1.1. *Let (M^n, g, f) be an $(n \geq 5)$-dimensional compact V-static space. If $f_l W_{lijk} = 0$ (that is, zero radial Weyl curvature), then ∇f is an eigenvector of Ricci tensor at each point in the set $\Omega = \{x \in M^n; \nabla f(x) \neq 0\}$.*

Furthermore, we achieve the following result:

Theorem 1.2. *Let (M^n, g, f) be an $(n \geq 5)$-dimensional compact V-static space with zero radial Weyl curvature. If $K\frac{|\nabla f|^2}{f} > 0$, then the metric is T-flat (that is, the T tensor defined by (2.7) is zero).*

When $n = 4$, the classical identity

$$W_{ijkl}W_{pjkl} = \frac{1}{4}|W|g_{ip}$$

shows that the metric has zero radial Weyl curvature if and only if the metric is locally conformally flat.

Remark 1.3. Ye [18] has studied the Vacuum static spaces with zero radial Weyl curvature and gave some rigidity results. Our above theorems can be seen as a generalization.

Remark 1.4. By virtue of the flat T tensor (see [3, 6, 16]) and constant scalar curvature, we achieve the following local splitting result: If f is a smooth solution f to equation (1.1), then

$$g = ds^2 + (r(s))^2 g_E$$

near the level set $f^{-1}(c)$, where $ds = \frac{df}{|df|}$, $(r(s))^2 g_E = g|_{f^{-1}(c)}$ and g_E is an Einstein metric.

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2. Preliminaries

Taking $\mathring{R}_{ij} = R_{ij} - \frac{R}{n}g_{ij}$, then (1.1) can be written as

$$f_{ij} = f\mathring{R}_{ij} - \frac{1}{n(n-1)}(fR + nK)g_{ij}. \tag{2.1}$$

It is well known that the Weyl curvature tensor and the Cotton tensor are defined respectively as follows:

$$R_{ijkl} = W_{ijkl} + \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il})$$

$$
\begin{aligned}
&- \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}) \\
&= W_{ijkl} + \frac{1}{n-2}(\mathring{R}_{ik}g_{jl} - \mathring{R}_{il}g_{jk} + \mathring{R}_{jl}g_{ik} - \mathring{R}_{jk}g_{il}) \\
&+ \frac{R}{n(n-1)}(g_{ik}g_{jl} - g_{il}g_{jk}) \qquad (2.2)
\end{aligned}
$$

and

$$C_{ijk} = \mathring{R}_{ij,k} - \mathring{R}_{ik,j} + \frac{n-2}{2n(n-1)}(R_{,k}g_{ij} - R_{,j}g_{ki}). \tag{2.3}$$

From (2.3), it is easy to see that C_{ijk} is skew-symmetric with respect to the last two indices, that is $C_{ijk} = -C_{ikj}$ and trace-free in any two indices:

$$C_{iik} = 0 = C_{iji}. \tag{2.4}$$

In addition,

$$C_{ijk} + C_{jki} + C_{kij} = 0 \tag{2.5}$$

and using the Ricci identity, one has

$$C_{ilk,l} = C_{kli,l}, \quad C_{ijl,l} = C_{jil,l}, \quad C_{lij,l} = 0. \tag{2.6}$$

Associated to (1.1), there is a (0.3)-tensor T_{ijk} which can be written as

$$T_{ijk} = \frac{n-1}{n-2}(\mathring{R}_{ik}f_j - \mathring{R}_{ij}f_k) + \frac{1}{n-2}(g_{ik}\mathring{R}_{jl} - g_{ij}\mathring{R}_{kl})f_l. \tag{2.7}$$

By calculation, we enable to observe that T satisfies the following properties:

$$T_{ijk} = -T_{ikj}, \quad T_{iik} = 0 = T_{iji},$$
$$T_{ijk} + T_{jki} + T_{kij} = 0.$$

Take divergence on both sides of (2.2), we have

$$W_{ijkl,i} = -\frac{n-3}{n-2}C_{jkl}. \tag{2.8}$$

Moreover, the Bach tensor is defined by

$$B_{ik} = \frac{1}{n-3}W_{ijkl,jl} + \frac{1}{n-2}W_{ijkl}R_{jl}.$$

Combining (2.8), the above equation can also be written

$$B_{ik} = \frac{1}{n-2}(-C_{ijk,j} + W_{ijkl}R_{jl}). \tag{2.9}$$

On the other hand, we also give a few commonly used lemmas:

Lemma 2.1. *Let (M^n, g, f) be an $(n \geq 3)$-dimensional compact Riemannian manifold satisfying* (1.1). *Then the Cotton tensor, T-tensor and the Weyl curvature tensor are related by*

$$fC_{ijk} = T_{ijk} + f_l W_{lijk}. \tag{2.10}$$

Proof. The reader interested in the specific proof can refer [6] and we will not repeat it here. □

Multiplying both sides of (2.10) by f_i and utilizing the definition of T, one has

$$\begin{aligned} fC_{ijk}f_i &= T_{ijk}f_i + f_l f_i W_{lijk} \\ &= (\mathring{R}_{kl}f_j - \mathring{R}_{jl}f_k)f_l \\ &= \mathcal{P}_{jk}, \end{aligned} \tag{2.11}$$

where

$$\mathcal{P}_{jk} := (\mathring{R}_{kl}f_j - \mathring{R}_{jl}f_k)f_l. \tag{2.12}$$

Lemma 2.2 (see Lemma 5 of [18]). *Let f be a smooth solution satisfying equation* (1.1). *Then*

$$\begin{aligned} R_{ik}T_{ijk}f_j &= \frac{n-2}{2(n-1)}|T|^2 \\ &= \frac{n-1}{n-2}\Big(|\nabla f|^2|\mathring{Ric}|^2 - \frac{n}{n-1}\mathring{Ric}^2(\nabla f,\nabla f)\Big). \end{aligned} \tag{2.13}$$

Lemma 2.3. *Let (M^n,g,f) be an $(n\geq 3)$-dimensional compact Riemannian manifold satisfying* (1.1). *Then, we have*

$$\begin{aligned} \mathring{R}_{ik,j}f_j = &-\frac{f}{n-2}|\mathring{Ric}|^2 g_{ik} + \frac{nf}{n-2}\mathring{R}_{ij}\mathring{R}_{kj} + \frac{1}{n-1}(fR+nK)\mathring{R}_{ik} \\ &-(n-2)fB_{ik} + f_l C_{ilk} + C_{kli}f_l. \end{aligned} \tag{2.14}$$

Proof. From (2.10), one has

$$\begin{aligned} fC_{ijk,j} &= -f_jC_{ijk} + T_{ijk,j} + W_{lijk,j}f_l + W_{lijk}f_{lj} \\ &= -f_jC_{ijk} + T_{ijk,j} - \frac{n-3}{n-2}C_{kli}f_l + fW_{lijk}\mathring{R}_{lj}. \end{aligned} \tag{2.15}$$

Further, taking the divergence of the tensor T, we derive

$$\begin{aligned} T_{ijk,j} &= \frac{n-1}{n-2}(\mathring{R}_{ik,j}f_j + \mathring{R}_{ik}\Delta f - \mathring{R}_{ij,j}f_k - \mathring{R}_{ij}f_{kj}) \\ &\quad + \frac{1}{n-2}(g_{ik}\mathring{R}_{jl,j} - \mathring{R}_{kl,i})f_l + \frac{1}{n-2}(g_{ik}\mathring{R}_{jl} - g_{ij}\mathring{R}_{kl})f_{lj} \\ &= \frac{n-1}{n-2}\Big\{\mathring{R}_{ik,j}f_j - \frac{1}{n-1}\mathring{R}_{ik}(fR+nK) - \mathring{R}_{ij}\Big[f\mathring{R}_{jk} - \frac{fR+nK}{n(n-1)}g_{kj}\Big]\Big\} \\ &\quad - \frac{1}{n-2}\mathring{R}_{kl,i}f_l + \frac{1}{n-2}(g_{ik}\mathring{R}_{jl} - g_{ij}\mathring{R}_{kl})\Big[f\mathring{R}_{lj} - \frac{fR+nK}{n(n-1)}g_{lj}\Big] \\ &= \frac{n-1}{n-2}\mathring{R}_{ik,j}f_j - \frac{fR+nK}{n-1}\mathring{R}_{ik} - \frac{nf}{n-2}\mathring{R}_{ij}\mathring{R}_{kj} - \frac{1}{n-2}\mathring{R}_{kl,i}f_l \\ &\quad + \frac{f}{n-2}|\mathring{Ric}|^2 g_{ik} \end{aligned}$$

$$= \mathring{R}_{ik,j}f_j - \frac{1}{n-2}C_{kli}f_l - \frac{1}{n-1}(fR+nK)\mathring{R}_{ik} - \frac{nf}{n-2}\mathring{R}_{ij}\mathring{R}_{kj}$$

$$+ \frac{f}{n-2}|\mathring{Ric}|^2 g_{ik}, \tag{2.16}$$

where we used $\mathring{R}_{ij,j} = \frac{n-2}{2n}R_{,i} = 0$ and

$$\mathring{R}_{ik,j}f_j - \mathring{R}_{kl,i}f_l = -C_{kli}f_l.$$

Substituting (2.16) into (2.15) yields the desired estimate (2.14). □

3. Proof of results

3.1. Proof of Theorem 1.1

Under the condition of $f_l W_{lijk} = 0$, (2.10) becomes

$$fC_{ijk} = T_{ijk}. \tag{3.1}$$

Taking the covariant derivative for (3.1) and using (2.6), we deduce that

$$\begin{aligned} f_i C_{ijk} &= T_{ijk,i} \\ &= \frac{n-1}{n-2}(\mathring{R}_{ik}f_j - \mathring{R}_{ij}f_k)_{,i} + \frac{1}{n-2}(g_{ik}\mathring{R}_{jl} - g_{ij}\mathring{R}_{kl})_{,i}f_l \\ &\quad + \frac{1}{n-2}(g_{ik}\mathring{R}_{jl} - g_{ij}\mathring{R}_{kl})f_{li} \\ &= \frac{n-1}{n-2}(\mathring{R}_{ik,i}f_j - \mathring{R}_{ij,i}f_k) + \frac{n-1}{n-2}(\mathring{R}_{ik}f_{ji} - \mathring{R}_{ij}f_{ki}) \\ &\quad + \frac{1}{n-2}\Big[(\mathring{R}_{jl,k} - \mathring{R}_{kl,j})f_l + (g_{ik}\mathring{R}_{jl} - g_{ij}\mathring{R}_{kl})f_{li}\Big] \\ &= \frac{n-1}{2n}(R_{,k}f_j - R_{,j}f_k) + \frac{n-1}{n-2}\Big[f\mathring{R}_{ik}\mathring{R}_{ij} - \frac{1}{n(n-1)}(fR+nK)\mathring{R}_{jk} \\ &\quad - f\mathring{R}_{ij}\mathring{R}_{ik} + \frac{1}{n(n-1)}(fR+nK)\mathring{R}_{kj}\Big] + \frac{1}{n-2}\Big[C_{ljk}f_l \\ &\quad - \frac{n-2}{2n(n-1)}(R_{,k}f_j - R_{,j}f_k) - \frac{1}{n(n-1)}(fR+nK)(\mathring{R}_{jk} - \mathring{R}_{kj}) \\ &\quad + f(\mathring{R}_{jl}\mathring{R}_{lk} - \mathring{R}_{kl}\mathring{R}_{lj})\Big] \\ &= \frac{n-2}{2(n-1)}(R_{,k}f_j - R_{,j}f_k) + \frac{1}{n-2}C_{ljk}f_l. \end{aligned} \tag{3.2}$$

Multiply both sides of (3.2) by f, one has

$$\frac{n-3}{n-2}\mathcal{P}_{jk} = \frac{n-2}{2(n-1)}(fR_{,k}f_j - fR_{,j}f_k) = 0, \tag{3.3}$$

where we used the fact that R is a constant. From (3.3) we notice that $\mathcal{P}_{jk} = 0$ when $n \geq 5$. Without loss of generalization, at any fixed point $p \in \Omega$, we can choose a local frame $\{e_i\}_{i=1}^n$ such that $\nabla f \parallel e_1$, then $f_1 = |\nabla f|$ and $f_2 = f_3 = \cdots = f_n = 0$. Therefore, by (2.12), we obtain

$$\begin{aligned} 0 &= \mathcal{P}_{jk} f_j \\ &= |\nabla f|^2 \mathring{R}_{kl} f_l - \mathring{R}_{jl} f_k f_l f_j \\ &= |\nabla f|^2 (\mathring{R}_{k1} f_1 - \mathring{R}_{11} f_k). \end{aligned} \tag{3.4}$$

Obviously, (3.4) shows that

$$\mathring{R}_{1k} = 0,$$

where $k \in \{2 \cdots n\}$. Thus, we have that ∇f is an eigenvector of $\mathring{Ric}$ and the proof is finished.

3.2. Proof of Theorem 1.2

First we give a couple of lemmas, which will be useful in the subsequent proof process.

Lemma 3.1. *Let (M^n, g, f) be an $(n \geq 5)$-dimensional compact Riemannian manifold satisfying* (1.1). *Then for $\mathcal{P}_{jk}$ given by* (2.12),

$$\begin{aligned} \mathcal{P}_{jk,i} = {}& \frac{n-2}{n(n-1)}(fR + nK)T_{ijk} - \frac{1}{n-2} f|\mathring{Ric}|^2 (g_{ik} f_j - g_{ij} f_k) \\ & + \frac{2n-2}{n-2} f \mathring{R}_{il} (\mathring{R}_{lk} f_j - \mathring{R}_{jl} f_k) - (n-2) f (B_{ik} f_j - B_{ij} f_k) \\ & - f(\mathring{R}_{jl}\mathring{R}_{ik} - \mathring{R}_{kl}\mathring{R}_{ij}) f_l + (C_{ilk} f_j - C_{ilj} f_k) f_l + 2(C_{kli} f_j - C_{jli} f_k) f_l. \end{aligned} \tag{3.5}$$

Proof. From (2.12) we directly calculate

$$\begin{aligned} \mathcal{P}_{jk,i} = {}& (f_{ij}\mathring{R}_{kl} + f_j \mathring{R}_{kl,i} - f_{ik}\mathring{R}_{jl} - f_k \mathring{R}_{jl,i}) f_l + (f_j \mathring{R}_{kl} - f_k \mathring{R}_{jl}) f_{il} \\ = {}& -\frac{1}{n(n-1)}(fR + nK)\mathring{R}_{kl} g_{ij} f_l + f\mathring{R}_{kl}\mathring{R}_{ij} f_l + (\mathring{R}_{kl,i} f_j - \mathring{R}_{jl,i} f_k) f_l \\ & + \frac{1}{n(n-1)}(fR + nK)\mathring{R}_{jl} g_{ik} f_l - f\mathring{R}_{jl}\mathring{R}_{ik} f_l \\ & + (f_j \mathring{R}_{kl} - f_k \mathring{R}_{jl})[f\mathring{R}_{li} - \frac{1}{n(n-1)}(fR + nK) g_{li}] \\ = {}& -\frac{1}{n(n-1)}(fR + nK)(\mathring{R}_{kl} g_{ij} - \mathring{R}_{jl} g_{ik}) f_l + f(\mathring{R}_{kl}\mathring{R}_{ij} - \mathring{R}_{jl}\mathring{R}_{ik}) f_l \\ & + (\mathring{R}_{kl,i} f_j - \mathring{R}_{jl,i} f_k) f_l - \frac{1}{n(n-1)}(fR + nK)(f_j \mathring{R}_{ki} - f_k \mathring{R}_{ji}) \\ & + f\mathring{R}_{li}(f_j \mathring{R}_{kl} - f_k \mathring{R}_{jl}). \end{aligned} \tag{3.6}$$

Applying (2.9) and (2.14), we derive that

$$\begin{aligned} & (\mathring{R}_{kl,i} f_j - \mathring{R}_{jl,i} f_k) f_l \\ = {}& C_{kli} f_l f_j - C_{jli} f_l f_k + \mathring{R}_{ki,l} f_l f_j - \mathring{R}_{ji,l} f_l f_k \end{aligned}$$

$$
\begin{aligned}
&= C_{kli} f_l f_j - C_{jli} f_l f_k + f_j \Big\{ -\frac{1}{n-2} f|\mathring{Ric}|^2 g_{ik} + \frac{n}{n-2} f \mathring{R}_{il}\mathring{R}_{lk} \\
&\quad + \frac{1}{n-1}(fR+nK)\mathring{R}_{ik} - (n-2) f B_{ik} + C_{ilk} f_l + C_{kli} f_l \Big\} \\
&\quad - f_k \Big\{ -\frac{1}{n-2} f|\mathring{Ric}|^2 g_{ij} + \frac{n}{n-2} f \mathring{R}_{il}\mathring{R}_{jl} + \frac{1}{n-1}(fR+nK)\mathring{R}_{ij} \\
&\quad - (n-2) f B_{ij} + C_{ilj} f_l + C_{jli} f_l \Big\} \\
&= -\frac{1}{n-2} f|\mathring{Ric}|^2 (f_j g_{ik} - f_k g_{ij}) + \frac{n}{n-2} f(\mathring{R}_{kl} f_j - \mathring{R}_{jl} f_k)\mathring{R}_{il} \\
&\quad + \frac{1}{n-1}(fR+nK)(\mathring{R}_{ik} f_j - \mathring{R}_{ij} f_k) - (n-2) f (B_{ik} f_j - B_{ij} f_k) \\
&\quad + 2(C_{kli} f_j - C_{jli} f_k) f_l + (C_{ilk} f_j - C_{ilj} f_k) f_l. \qquad (3.7)
\end{aligned}
$$

Putting (3.7) into (3.6) gives the equation (3.5). □

Corollary 3.2. *Let* (M^n, g, f) *be an* $(n \geq 5)$*-dimensional compact Riemannian manifold satisfying* (1.1). *Assume* $\mathring{Ric}(\nabla f) = \mu_1 \nabla f$ *in the set* $\Omega = \{x \in M^n; \nabla f(x) \neq 0\}$, *then*

$$
\begin{aligned}
&(n-2) f B_{ik} \\
&= -\Big[\frac{1}{n-1}\mu_1 (fR+nK) + f|\mathring{Ric}|^2\Big] \frac{f_i}{|\nabla f|}\frac{f_k}{|\nabla f|} + \frac{2n-2}{n-2} f \mathring{R}_{il}\mathring{R}_{lk} \\
&\quad + 3C_{kli} f_l - \Big[f\mu_1 - \frac{1}{n}(fR+nK)\Big]\mathring{R}_{ik} \\
&\quad + \Big[\frac{1}{n(n-1)}(fR+nK)\mu_1 - \frac{1}{n-2} f|\mathring{Ric}|^2\Big] g_{ik}. \qquad (3.8)
\end{aligned}
$$

Proof. Multiply both sides of (2.5) by f_i, one has

$$C_{ijk} f_i + C_{kij} f_i + C_{jki} f_i = 0, \qquad (3.9)$$

By (2.11) and the set $f^{-1}(0)$ has the measure zero, we obtain

$$C_{ijk} f_i = 0.$$

Thus, (3.9) becomes

$$C_{kij} f_i = C_{jik} f_i. \qquad (3.10)$$

From (3.5) and the fact that $\mathcal{P}$ disappears we get the following

$$
\begin{aligned}
0 &= \frac{n-2}{n(n-1)}(fR+nK) T_{ijk} - \frac{1}{n-2} f|\mathring{Ric}|^2 (g_{ik} f_j - g_{ij} f_k) \\
&\quad + \frac{2n-2}{n-2} f \mathring{R}_{il}(\mathring{R}_{lk} f_j - \mathring{R}_{jl} f_k) - f(\mathring{R}_{jl}\mathring{R}_{ik} - \mathring{R}_{kl}\mathring{R}_{ij}) f_l \\
&\quad - (n-2) f (B_{ik} f_j - B_{ij} f_k) + (C_{ilk} f_j - C_{ilj} f_k) f_l + 2(C_{kli} f_j - C_{jli} f_k) f_l.
\end{aligned}
$$

Contract the above formula with respect to i and j, and combining with the assumption $\mathring{Ric}(\nabla f) = \mu_1 \nabla f$, we have

$$(n-2)B_{ik}f_i = \frac{1}{n-2}(n\mu_1^2 - (n-1)|\mathring{Ric}|^2)f_k. \tag{3.11}$$

Moreover, according to the definition of T_{ijk} in (2.7), it holds that

$$\begin{aligned} T_{ijk}f_j &= \frac{n-1}{n-2}(|\nabla f|^2 \mathring{R}_{ik} - \mathring{R}_{ij}f_kf_j) + \frac{1}{n-2}(g_{ik}\mathring{R}_{jl}f_j - f_i\mathring{R}_{kl})f_l \\ &= \frac{n-1}{n-2}|\nabla f|^2\mathring{R}_{ik} - \frac{n}{n-2}\mu_1 f_i f_k + \frac{1}{n-2}\mu_1|\nabla f|^2 g_{ik}. \end{aligned} \tag{3.12}$$

Thus, it follows from (3.5) that

$$\begin{aligned} 0 &= \mathcal{P}_{jk,i}f_j \\ &= \frac{n-2}{n(n-1)}(fR+nK)T_{ijk}f_j - \frac{1}{n-2}|\nabla f|^2 f|\mathring{Ric}|^2 g_{ik} + \frac{1}{n-2}f|\mathring{Ric}|^2 f_i f_k \\ &\quad + \frac{2n-2}{n-2}|\nabla f|^2 f\mathring{R}_{il}\mathring{R}_{lk} - f\mu_1|\nabla f|^2\mathring{R}_{ik} + (n-2)fB_{ij}f_kf_j \\ &\quad - (n-2)|\nabla f|^2 fB_{ik} + 3|\nabla f|^2 C_{kli}f_l - \frac{n}{n-2}\mu_1^2 f f_i f_k, \end{aligned}$$

which combines with (3.11) and (3.12) to give

$$\begin{aligned} 0 = &- (n-2)|\nabla f|^2 fB_{ik} + \frac{1}{n(n-1)}(fR+nK)\Big[(n-1)|\nabla f|^2\mathring{R}_{ik} \\ &- n\mu_1 f_i f_k + \mu_1|\nabla f|^2 g_{ik}\Big] - \frac{1}{n-2}|\nabla f|^2 f|\mathring{Ric}|^2 g_{ik} \\ &+ \frac{1}{n-2}f|\mathring{Ric}|^2 f_i f_k - f\mu_1|\nabla f|^2\mathring{R}_{ik} + \frac{2n-2}{n-2}|\nabla f|^2 f\mathring{R}_{il}\mathring{R}_{lk} \\ &+ \frac{n}{n-2}\mu_1^2 f f_i f_k + 3|\nabla f|^2 C_{kli}f_l - \frac{n-1}{n-2}f|\mathring{Ric}|^2 f_i f_k \\ &- \frac{n}{n-2}\mu_1^2 f f_i f_k \\ = &- (n-2)|\nabla f|^2 fB_{ik} + 3|\nabla f|^2 C_{kli}f_l \\ &- \Big[f\mu_1 - \frac{1}{n}(fR+nK)\Big]|\nabla f|^2\mathring{R}_{ik} \\ &- \Big[\frac{1}{n-1}\mu_1(fR+nK) + f|\mathring{Ric}|^2\Big]f_i f_k + \frac{2n-2}{n-2}|\nabla f|^2 f\mathring{R}_{il}\mathring{R}_{lk} \\ &+ \Big[\frac{1}{n(n-1)}(fR+nK)\mu_1 - \frac{1}{n-2}f|\mathring{Ric}|^2\Big]|\nabla f|^2 g_{ik}, \end{aligned} \tag{3.13}$$

and the estimate (3.8) follows. □

Substituting (3.8) into (2.14) gives directly the following

Corollary 3.3. *Let (M^n, g, f) be an $(n \geq 5)$-dimensional compact Riemannian manifold satisfying* (1.1). *Assume $\mathring{Ric}(\nabla f) = \mu_1 \nabla f$ in the set $\Omega = \{x \in M^n; \nabla f(x) \neq 0\}$, then*

$$\begin{aligned} \mathring{R}_{ik,s} f_s = & -C_{kli} f_l + \Big[\frac{1}{n-1}(fR+nK)\mu_1 + f|\mathring{Ric}|^2\Big]\frac{f_i}{|\nabla f|}\frac{f_k}{|\nabla f|} - f\mathring{R}_{il}\mathring{R}_{lk} \\ & - \frac{1}{n(n-1)}(fR+nK)\mu_1 g_{ik} + \Big[f\mu_1 + \frac{1}{n(n-1)}(fR+nK)\Big]\mathring{R}_{ik}. \end{aligned} \tag{3.14}$$

On the other hand, by virtue of (2.7), it holds that

$$\begin{aligned} T_{ijk,s} f_s = & \frac{n-1}{n-2}(\mathring{R}_{ik,s} f_j + \mathring{R}_{ik} f_{js} - \mathring{R}_{ij,s} f_k - \mathring{R}_{ij} f_{ks}) f_s \\ & + \frac{1}{n-2}(g_{ik}\mathring{R}_{jl,s} - g_{ij}\mathring{R}_{kl,s}) f_s f_l + \frac{1}{n-2}(g_{ik}\mathring{R}_{jl} - g_{ij}\mathring{R}_{kl}) f_{ls} f_s. \end{aligned} \tag{3.15}$$

From (2.1) and $\mathring{R}_{ij} f_j = \mu_1 f_i$, we deduce

$$\begin{aligned} T_{ijk,s} f_s = & \frac{n-1}{n-2}(\mathring{R}_{ik,s} f_s f_j - \mathring{R}_{ij,s} f_s f_k) + \frac{1}{n-2}(g_{ik}\mathring{R}_{jl,s} - g_{ij}\mathring{R}_{kl,s}) f_s f_l \\ & + \Big(f\mu_1 - \frac{fR+nK}{n(n-1)}\Big) T_{ijk} \\ = & \frac{n-1}{n-2}\Big\{ f_j \Big[-C_{kli} f_l - \frac{1}{n(n-1)}(fR+nK)\mu_1 g_{ik} - f\mathring{R}_{il}\mathring{R}_{lk} \\ & + \Big(f\mu_1 + \frac{1}{n(n-1)}(fR+nK)\Big)\mathring{R}_{ik}\Big] - f_k \Big[-C_{jli} f_l - f\mathring{R}_{il}\mathring{R}_{lj} \\ & - \frac{1}{n(n-1)}(fR+nK)\mu_1 g_{ij} + \Big(f\mu_1 + \frac{1}{n(n-1)}(fR+nK)\Big)\mathring{R}_{ij}\Big]\Big\} \\ & + \frac{1}{n-2}\Big\{ g_{ik} f_l \Big[-C_{jpl} f_p + \Big(f\mu_1 + \frac{1}{n(n-1)}(fR+nK)\Big)\mathring{R}_{lj} \\ & + \Big(\frac{1}{n-1}(fR+nK)\mu_1 + f|\mathring{Ric}|^2\Big)\frac{f_l}{|\nabla f|}\frac{f_j}{|\nabla f|} - f\mathring{R}_{lp}\mathring{R}_{pj} \\ & - \frac{1}{n(n-1)}(fR+nK)\mu_1 g_{lj}\Big] - g_{ij} f_l \Big[-C_{kpl} f_p - f\mathring{R}_{kp}\mathring{R}_{pl} \\ & - \frac{1}{n(n-1)}(fR+nK)\mu_1 g_{kl} + \Big(f\mu_1 + \frac{1}{n(n-1)}(fR+nK)\Big)\mathring{R}_{kl} \\ & + \Big(\frac{(fR+nK)\mu_1}{n-1} + f|\mathring{Ric}|^2\Big)\frac{f_k}{|\nabla f|}\frac{f_l}{|\nabla f|}\Big]\Big\} + \Big(f\mu_1 - \frac{fR+nK}{n(n-1)}\Big) T_{ijk} \\ = & -\frac{n-1}{n-2}(C_{kli} f_j - C_{jli} f_k) f_l - \frac{\mu_1}{n(n-2)}(fR+nK)(f_j g_{ik} - f_k g_{ij}) \\ & + \Big(f\mu_1 - \frac{fR+nK}{n(n-1)}\Big) T_{ijk} + \frac{n-1}{n-2}\Big(f\mu_1 + \frac{1}{n(n-1)}(fR+nK)\Big) \\ & \times (f_j\mathring{R}_{ik} - f_k\mathring{R}_{ij}) - \frac{n-1}{n-2} f\mathring{R}_{il}(f_j\mathring{R}_{lk} - f_k\mathring{R}_{lj}) \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{n-2}\Big[\frac{1}{n-1}(fR+nK)\mu_1 + f|\mathring{Ric}|^2\Big](f_jg_{ik}-f_kg_{ij})\\
&- \frac{f}{n-2}\mu_1^2(f_jg_{ik}-f_kg_{ij}) + \frac{\mu_1}{n-2}\Big[f\mu_1 + \frac{1}{n(n-1)}(fR+nK)\Big]\\
&\times (f_jg_{ik}-f_kg_{ij}) - \frac{\mu_1}{n(n-1)(n-2)}(fR+nK)(f_jg_{ik}-f_kg_{ij})\\
=\ & 2\mu_1 fT_{ijk} - \frac{n-1}{n-2}(C_{kli}f_j - C_{jli}f_k)f_l - \frac{n-1}{n-2}f\mathring{R}_{il}(f_j\mathring{R}_{lk} - f_k\mathring{R}_{lj})\\
&+ \frac{f}{n-2}(|\mathring{Ric}|^2 - \mu_1^2)(f_jg_{ik}-f_kg_{ij}),
\end{aligned} \tag{3.16}
$$

where the second equality follows from (3.14).

As a result, we get the following.

Corollary 3.4. *Let (M^n, g, f) be an $(n \geq 5)$-dimensional compact Riemannian manifold satisfying* (1.1). *Assume $\mathring{Ric}(\nabla f) = \mu_1 \nabla f$ in the set $\Omega = \{x \in M^n, \nabla f(x) \neq 0\}$, then*

$$
\begin{aligned}
T_{ijk,s}f_s =\ & 2\mu_1 fT_{ijk} - \frac{n-1}{n-2}(C_{kli}f_j - C_{jli}f_k)f_l - \frac{n-1}{n-2}f\mathring{R}_{il}(f_j\mathring{R}_{lk} - f_k\mathring{R}_{lj})\\
&+ \frac{f}{n-2}(|\mathring{Ric}|^2 - \mu_1^2)(f_jg_{ik}-f_kg_{ij}).
\end{aligned} \tag{3.17}
$$

In the following we give two basic facts (see Lemma 21 of [18]):

Lemma 3.5. *Let (M^n, g) be a Riemannian manifold. Then*

$$
C_{ijk,l} + C_{ikl,j} + C_{ilj,k} = R_{jp}W_{pikl} + R_{kp}W_{pilj} + R_{lp}W_{pijk}. \tag{3.18}
$$

Using (3.18), a direct calculation yields

$$
\begin{aligned}
C_{jli,k} - C_{kli,j} =\ & C_{ljk,i} - C_{ijk,l} - R_{ip}W_{pljk} + R_{jp}W_{pkil}\\
&- R_{kp}W_{pjil} + R_{lp}W_{pijk}.
\end{aligned} \tag{3.19}
$$

Furthermore, we also derive the following:

Lemma 3.6. *Let (M^n, g, f) be an $(n \geq 5)$-dimensional compact V-static space with zero radial Weyl curvature. Then, we have*

$$
\mathring{R}_{ik}C_{ijk}f_j = \frac{n-2}{2(n-1)}f|C|^2, \tag{3.20}
$$

$$
fW_{ijkl}\mathring{R}_{lj} = \frac{n-3}{n-2}C_{kpi}f_p, \tag{3.21}
$$

$$
C_{ijk}f_jC_{ipk}f_p = \frac{1}{2}|\nabla f|^2|C|^2, \tag{3.22}
$$

$$
(n-2)B_{ik}T_{ijk}f_j = \frac{3}{2}|\nabla f|^2|C|^2 + \frac{2(n-1)}{n-2}\mathring{R}_{il}\mathring{R}_{lk}T_{ijk}f_j
$$

$$-\frac{n-2}{2(n-1)}\Big[f\mu_1-\frac{1}{n}(fR+nK)\Big]f|C|^2. \tag{3.23}$$

Proof. From (2.7), (2.13) and (3.1), we have

$$\begin{aligned}f\mathring{R}_{ik}C_{ijk}f_j=\mathring{R}_{ik}T_{ijk}f_j&=\frac{n-2}{2(n-1)}|T|^2\\&=\frac{n-2}{2(n-1)}f^2|C|^2,\end{aligned}$$

which combines the fact that the lever set $f^{-1}(0)$ has measure zero infers (3.20).

Applying $f_lW_{lijk}=0$, it holds that

$$\begin{aligned}0&=W_{lijk,j}f_l+W_{lijk}f_{lj}\\&=-\frac{n-3}{n-2}C_{kpi}f_p+W_{lijk}\Big[f\mathring{R}_{lj}-\frac{1}{n(n-1)}(fR+nK)g_{lj}\Big]\\&=-\frac{n-3}{n-2}C_{kpi}f_p+fW_{lijk}\mathring{R}_{lj},\end{aligned}$$

and this leads to (3.21). From (2.7), (3.20) and the fact that the lever set $f^{-1}(0)$ has measure zero, we deduce (3.22) from

$$\begin{aligned}fC_{ijk}f_jC_{ipk}f_p&=T_{ijk}f_jC_{ipk}f_p\\&=\frac{n-1}{n-2}|\nabla f|^2\mathring{R}_{ik}C_{ipk}f_p\\&=\frac{1}{2}f|\nabla f|^2|C|^2.\end{aligned}$$

Multiply both sides of (3.8) by $C_{ijk}f_j$, we obtain

$$\begin{aligned}&(n-2)B_{ik}T_{ijk}f_j\\&=(n-2)fB_{ik}C_{ijk}f_j\\&=3C_{kli}f_lC_{ijk}f_j+\frac{2(n-1)}{n-2}\mathring{R}_{il}\mathring{R}_{lk}T_{ijk}f_j-\Big[f\mu_1-\frac{1}{n}(fR+nK)\Big]\mathring{R}_{ik}C_{ijk}f_j\\&=\frac{3}{2}|\nabla f|^2|C|^2+\frac{2(n-1)}{n-2}\mathring{R}_{il}\mathring{R}_{lk}T_{ijk}f_j-\frac{n-2}{2(n-1)}\Big[f\mu_1-\frac{1}{n}(fR+nK)\Big]f|C|^2.\end{aligned}$$

This completes the proof of Lemma 3.6. □

To prove $T=0$, motivated by [18], we need to establish a point to point formula under the condition of $f_lW_{lijk}=0$ and the equation (1.1):

Proposition 3.7. *Let (M^n,g,f) be an $(n\geq 5)$-dimensional compact V-static space with zero radial Weyl curvature. Then,*

$$\begin{aligned}&\frac{2(n-1)}{n-2}\mathring{R}_{il}\mathring{R}_{lk}T_{ijk}f_j+\mu_1|T|^2\\&=\frac{n-3}{n-2}\Big[|\nabla f|^2|C|^2+\frac{2(R+nKf^{-1})}{n(n-1)}|T|^2\Big].\end{aligned} \tag{3.24}$$

Proof. Using the method of [18], we first calculate $\Delta(f_l W_{lijk})$ as follows:

$$\begin{aligned}\Delta(f_l W_{lijk}) &= f_l \Delta W_{lijk} + 2 f_{ls} W_{lijk,s} + f_{,lss} W_{lijk}\\ &= f_l \Delta W_{lijk} + 2 f \mathring{R}_{ls} W_{lijk,s} - \frac{2}{n(n-1)}(fR + nK) W_{lijk,l}\\ &\quad - \frac{R}{n-1} f_l W_{lijk} + f_p \mathring{R}_{pl} W_{lijk} + \frac{R}{n} f_l W_{lijk}. \end{aligned} \tag{3.25}$$

From $f_l W_{lijk} = 0$, (3.1), $\mathring{Ric}(\nabla f) = \mu_1 \nabla f$ and (28) of [18], (3.25) becomes

$$\begin{aligned} 0 &= f_l (C_{jli,k} - C_{kli,j}) - f_l (B \owedge g)_{lijk} + 2 f \mathring{R}_{ls} W_{lijk,s}\\ &\quad + \frac{2(n-3)}{n(n-1)(n-2)}\Big(fR + nK\Big) C_{ijk}\\ &= f_l (C_{jli,k} - C_{kli,j}) + B_{ij} f_k - B_{ik} f_j + (B_{lk} g_{ij} - B_{lj} g_{ik}) f_l\\ &\quad + 2 f \mathring{R}_{ls} W_{lijk,s} + \frac{2(n-3)}{n(n-1)(n-2)}\Big(fR + nK\Big) C_{ijk}. \end{aligned} \tag{3.26}$$

Applying (1.1) and (3.19), we have

$$\begin{aligned} f_l (C_{jli,k} - C_{kli,j}) &= (C_{ljk,i} - C_{ijk,l}) f_l\\ &= (C_{ljk} f_l)_i - C_{ljk} f_{li} - C_{ijk,l} f_l\\ &= -C_{ljk}\Big[f \mathring{R}_{li} - \frac{1}{n(n-1)}(fR + nK) g_{li}\Big] - C_{ijk,l} f_l\\ &= -T_{ljk} \mathring{R}_{li} + \frac{1}{n(n-1)}(fR + nK) C_{ijk} - C_{ijk,l} f_l, \end{aligned} \tag{3.27}$$

where the first equality follows from $f_l W_{lijk} = 0$ and the last equality from (3.1). Substituting (3.27) into (3.26), we obtain

$$\begin{aligned} 0 = & -C_{ijk,l} f_l - \frac{n-1}{n-2} \mathring{R}_{li} (\mathring{R}_{lk} f_j - \mathring{R}_{lj} f_k) + B_{ij} f_k - B_{ik} f_j\\ & + (g_{ij} B_{lk} - g_{ik} B_{lj}) f_l + 2 f \mathring{R}_{ls} W_{lijk,s} + \frac{3n-8}{n(n-1)(n-2)}\Big(fR + nK\Big) C_{ijk}\\ & - \frac{1}{n-2} \mathring{R}_{li} (g_{lk} \mathring{R}_{jp} - g_{lj} \mathring{R}_{kp}) f_p. \end{aligned}$$

By contracting with T_{ijk} and combining (3.23) derive that

$$\begin{aligned} 0 = & -\frac{1}{2} f \langle \nabla f, \nabla |C|^2 \rangle - \frac{2(n-1)}{n-2} \mathring{R}_{li} \mathring{R}_{lk} T_{ijk} f_j - 2 B_{ik} T_{ijk} f_j\\ & + 2 f \mathring{R}_{ls} T_{ijk} W_{lijk,s} - \frac{\mu_1}{n-1} |T|^2 + \frac{3n-8}{n(n-1)(n-2)}\Big(fR + nK\Big) f |C|^2\\ = & -\frac{1}{2} f \langle \nabla f, \nabla |C|^2 \rangle - \frac{3}{n-2} |\nabla f|^2 |C|^2\\ & - \frac{2n(n-1)}{(n-2)^2} \mathring{R}_{il} \mathring{R}_{lk} T_{ijk} f_j + \frac{2n-6}{n(n-1)(n-2)}\Big(fR + nK\Big) f |C|^2 \end{aligned}$$

$$+\,2f\mathring{R}_{ls}T_{ijk}W_{lijk,s}. \tag{3.28}$$

Let ϕ be a C^1 smooth real function with compact support on M. Multiplying both sides of (3.28) by ϕ and integrating over M, we have

$$\begin{aligned}0 =\ & -\frac{1}{2}\int_M f\langle\nabla f,\nabla|C|^2\rangle\phi + 2\int_M f\mathring{R}_{ls}T_{ijk}W_{lijk,s}\phi\\ &+\frac{2n-6}{n(n-1)(n-2)}\int_M \Big(fR+nK\Big)f|C|^2\phi\\ &-\frac{2n(n-1)}{(n-2)^2}\int_M \mathring{R}_{il}\mathring{R}_{lk}T_{ijk}f_j\phi - \frac{3}{n-2}\int_M |\nabla f|^2|C|^2\phi.\end{aligned} \tag{3.29}$$

On the other hand, using the divergence theorem and (1.1), we deduce that

$$\begin{aligned}-\frac{1}{2}\int_M f\langle\nabla f,\nabla|C|^2\rangle\phi =\ & \frac{1}{2}\int_M |\nabla f|^2|C|^2\phi + \frac{1}{2}\int_M \langle\nabla f,\nabla\phi\rangle f|C|^2\\ &-\frac{1}{2(n-1)}\int_M (fR+nK)f|C|^2\phi,\end{aligned} \tag{3.30}$$

and from (2.1), (2.7), (3.1) and (3.21), we also obtain

$$\begin{aligned}&2\int_M f\mathring{R}_{ls}T_{ijk}W_{lijk,s}\phi\\ =\ & \frac{4(n-1)}{n-2}\int_M f\mathring{R}_{ik}\mathring{R}_{ls}f_jW_{lijk,s}\phi\\ =\ & -\frac{4(n-1)}{n-2}\int_M \mathring{R}_{ik}\Big[f\mathring{R}_{js}-\frac{1}{n(n-1)}(fR+nK)g_{js}\Big]fW_{lijk}\mathring{R}_{ls}\phi\\ =\ & -\frac{4(n-1)}{n-2}\int_M \mathring{R}_{ik}\Big[f\mathring{R}_{js}\mathring{R}_{ls}-\frac{1}{n(n-1)}(fR+nK)\mathring{R}_{lj}\Big]fW_{lijk}\phi\\ =\ & -\frac{4(n-1)(n-3)}{(n-2)^2}\int_M \Big[f\mathring{R}_{js}\mathring{R}_{ls}-\frac{1}{n(n-1)}(fR+nK)\mathring{R}_{lj}\Big]C_{lpj}f_p\phi\\ =\ & \frac{4(n-3)}{n(n-2)^2}\int_M (fR+nK)\mathring{R}_{lj}C_{lpj}f_p\phi\\ & -\frac{4(n-1)(n-3)}{(n-2)^2}\int_M \mathring{R}_{js}\mathring{R}_{ls}T_{lpj}f_p\phi\\ =\ & \frac{2(n-3)}{n(n-1)(n-2)}\int_M (fR+nK)f|C|^2\phi\\ & -\frac{4(n-1)(n-3)}{(n-2)^2}\int_M \mathring{R}_{kl}\mathring{R}_{il}T_{ijk}f_j\phi.\end{aligned} \tag{3.31}$$

Inserting (3.30) and (3.31) into (3.29), it is easy to get

$$\begin{aligned}0 =\ & \frac{n-8}{2(n-2)}\int_M |\nabla f|^2|C|^2\phi - \frac{6(n-1)}{n-2}\int_M \mathring{R}_{il}\mathring{R}_{lk}T_{ijk}f_j\phi\\ & -\frac{(n-4)(n-6)}{2n(n-1)(n-2)}\int_M (fR+nK)f|C|^2\phi + \frac{1}{2}\int_M \langle\nabla f,\nabla\phi\rangle f|C|^2.\end{aligned} \tag{3.32}$$

In addition, by contracting with T_{ijk} in (3.17) and combining with (3.22), one has

$$\begin{aligned}\frac{1}{2}\langle\nabla f,\nabla|T|^2\rangle &= 2\mu_1 f|T|^2-\frac{2(n-1)}{n-2}T_{ijk}f_jC_{ilk}f_l-\frac{2(n-1)}{n-2}f\mathring{R}_{il}\mathring{R}_{lk}T_{ijk}f_j\\ &= 2\mu_1 f|T|^2-\frac{n-1}{n-2}f|\nabla f|^2|C|^2-\frac{2(n-1)}{n-2}f\mathring{R}_{il}\mathring{R}_{lk}T_{ijk}f_j, \qquad (3.33)\end{aligned}$$

which implies that

$$\frac{1}{2}f\langle\nabla f,\nabla|C|^2\rangle = 2\mu_1|T|^2-\frac{2(n-1)}{n-2}\mathring{R}_{il}\mathring{R}_{lk}T_{ijk}f_j-\frac{2n-3}{n-2}|\nabla f|^2|C|^2. \qquad (3.34)$$

Hence,

$$\begin{aligned}&\frac{1}{2}\int_M f\langle\nabla f,\nabla|C|^2\rangle\phi\\ &=2\int_M\mu_1|T|^2\phi-\frac{2(n-1)}{n-2}\int_M\mathring{R}_{il}\mathring{R}_{lk}T_{ijk}f_j\phi\\ &\quad-\frac{2n-3}{n-2}\int_M|\nabla f|^2|C|^2\phi. \qquad (3.35)\end{aligned}$$

Applying the divergence theorem, (3.35) becomes

$$\begin{aligned}0 &= 2\int_M\mu_1|T|^2\phi+\frac{1}{2}\int_M f\langle\nabla f,\nabla\phi\rangle|C|^2-\frac{1}{2(n-1)}\int_M(fR+nK)f|C|^2\phi\\ &\quad-\frac{3n-4}{2(n-2)}\int_M|\nabla f|^2|C|^2\phi-\frac{2(n-1)}{n-2}\int_M\mathring{R}_{il}\mathring{R}_{lk}T_{ijk}f_j\phi,\end{aligned}$$

which combines (3.32) to derive

$$\begin{aligned}0 = &-\frac{4(n-1)}{n-2}\int_M\mathring{R}_{il}\mathring{R}_{lk}T_{ijk}f_j\phi+\frac{4(n-3)}{n(n-1)(n-2)}\int_M(fR+nK)f|C|^2\phi\\ &-2\int_M\mu_1|T|^2\phi+\frac{2(n-3)}{n-2}\int_M|\nabla f|^2|C|^2\phi.\end{aligned}$$

According to the arbitrariness of ϕ, we complete the proof of the Proposition 3.7. □

We will use the Proposition 3.7 to prove that $T=0$. Inserting (3.24) into (3.33), we have

$$\frac{1}{2}\langle\nabla f,\nabla|T|^2\rangle = 3\mu_1 f|T|^2-2f|\nabla f|^2|C|^2-\frac{2(n-3)}{n(n-1)(n-2)}(fR+nK)|T|^2,$$

which combines (3.1) infers that

$$\begin{aligned}&\frac{1}{2}f\langle\nabla f,\nabla|T|^2\rangle\\ &=\Big[3\mu_1 f^2-2|\nabla f|^2-\frac{2(n-3)}{n(n-1)(n-2)}(f^2R+nKf)\Big]|T|^2. \qquad (3.36)\end{aligned}$$

Taking $h = f^4|T|^2$, we deduce

$$\langle \nabla f, \nabla h\rangle = 2\Big[3\mu_1 - \frac{2(n-3)}{n(n-1)(n-2)}(R + nKf^{-1})\Big]fh. \tag{3.37}$$

To go further, we take divergence on both sides of $\mathring{R}_{ij}f_j = \mu_1 f_i$ and using (2.1) to derive that

$$f|\mathring{Ric}|^2 = \langle \nabla\mu_1, \nabla f\rangle - \frac{1}{n-1}\mu_1(fR + nK). \tag{3.38}$$

Differentiating along ∇f for both sides of (3.37), we have

$$\begin{aligned}\nabla^2 h(\nabla f, \nabla f) - \Big[5\mu_1 + \frac{3n-10}{n(n-1)(n-2)}(R + nKf^{-1})\Big]f\langle \nabla f, \nabla h\rangle \\ = 2\Big[3\mu_1 - \frac{2(n-3)}{n(n-1)(n-2)}(R + nKf^{-1})\Big]|\nabla f|^2 h + 6\langle \nabla\mu_1, \nabla f\rangle fh \\ + \frac{4(n-3)}{(n-1)(n-2)}Kf^{-1}|\nabla f|^2 h.\end{aligned} \tag{3.39}$$

Next, we will prove $T \equiv 0$ by a contradiction. Otherwise, h attains its maximum at a point $x_0 \in M$ and $h(x_0) > 0$. Thus, we observe from (3.37) that

$$3\mu_1(x_0) - \frac{2(n-3)}{n(n-1)(n-2)}(R + nKf^{-1})(x_0) = 0. \tag{3.40}$$

From (3.38) and (3.39), we observe

$$\begin{aligned}0 &\geq \Big\{6\Big[|\mathring{Ric}|^2 + \frac{2(n-3)}{3n(n-1)^2(n-2)}(R + nKf^{-1})^2\Big]f^2 h \\ &\quad + \frac{4(n-3)}{(n-1)(n-2)}Kf^{-1}|\nabla f|^2 h\Big\}(x_0) \\ &\geq \frac{4(n-3)}{(n-1)(n-2)}(Kf^{-1}|\nabla f|^2 h)(x_0),\end{aligned} \tag{3.41}$$

which combined with $Kf^{-1}|\nabla f|^2 > 0$ shows that

$$\Big[\frac{4(n-3)}{(n-1)(n-2)}Kf^{-1}|\nabla f|^2 h\Big](x_0) = 0. \tag{3.42}$$

This is impossible. Therefore, $T \equiv 0$. This completes the proof of Theorem 1.2.

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