Bull. Korean Math. Soc. **61** (2024), No. 4, pp. 897–906
https://doi.org/10.4134/BKMS.b230064
pISSN: 1015-8634 / eISSN: 2234-3016

THE CHERN SECTIONAL CURVATURE OF A HERMITIAN MANIFOLD

Pandeng Cao and Hongjun Li

Abstract. On a Hermitian manifold, the Chern connection can induce a metric connection on the background Riemannian manifold. We call the sectional curvature of the metric connection induced by the Chern connection the Chern sectional curvature of this Hermitian manifold. First, we derive expression of the Chern sectional curvature in local complex coordinates. As an application, we find that a Hermitian metric is Kähler if the Riemann sectional curvature and the Chern sectional curvature coincide. As subsequent results, Ricci curvature and scalar curvature of the metric connection induced by the Chern connection are obtained.

1. Introduction

Suppose (M, h) is an n-dimensional Hermitian manifold, and $g = \operatorname{Re} h$ is the background Riemannian metric associated to h. Let $z = (z^1, \dots, z^n)$ and $x = (x^1, \dots, x^{2n})$ be local complex and real coordinates of a point $p \in M$, where $z^\alpha = x^\alpha + \sqrt{-1}x^{n+\alpha}$, $1 \le \alpha \le n$. In this paper, we assume that lowercase Greek indices run from 1 to n and lowercase Latin indices run from 1 to $2n$. In local coordinates, we denote by $h = h_{\alpha\bar{\beta}}(z)dz^\alpha d\bar{z}^\beta$ and $g = \operatorname{Re} h = g_{ij}(x)dx^i dx^j$. Let $\omega = \sqrt{-1}h_{\alpha\bar{\beta}}(z)dz^\alpha \wedge d\bar{z}^\beta$ be the Kähler form associated to the Hermitian metric h. If the Kähler form ω is closed, i.e., $d\omega = 0$, then we call h a Kähler metric.

Let us denote by D and ∇ the Chern connection and the Levi-Civita (or Riemannian) connection, respectively. The curvature operators of the Chern connection and the Levi-Civita connection are denoted by $\mathfrak{R}$ and R, respectively. It is well known that h is Kähler if and only if the Chern connection D coincides with the Levi-Civita ∇ (refer to [4, 10, 14], etc.). Hence $\mathfrak{R}$ is the linear extension of R over $\mathbb{C}$ under Kähler hypothesis.

We define a bundle isomorphism $_o : TM \to T^{1,0}M$ by

$$u_o = \frac{1}{2}(u - \sqrt{-1}Ju), \quad \forall u \in TM,$$

Received February 7, 2023; Revised February 12, 2024; Accepted February 29, 2024.
2020 *Mathematics Subject Classification.* Primary 53C55, 53B35.
Key words and phrases. Chern connection, metric connection, Chern sectional curvature.

where J is the complex structure on M. Let $u,\ v \in TM$, we set $\xi = u_o,\ \eta = v_o \in T^{1,0}M$. If h is Kähler, then

$$R(Ju,u,v,Jv) = 2\mathfrak{R}\left(\xi,\bar{\xi},\eta,\bar{\eta}\right), \tag{1}$$

$$\begin{aligned} &R(u,v,v,u) \\ &= \frac{1}{2}\left[\mathfrak{R}\left(\xi,\bar{\eta},\eta,\bar{\xi}\right) + \mathfrak{R}\left(\eta,\bar{\xi},\xi,\bar{\eta}\right) - \mathfrak{R}\left(\xi,\bar{\eta},\xi,\bar{\eta}\right) - \mathfrak{R}\left(\eta,\bar{\xi},\eta,\bar{\xi}\right)\right]. \end{aligned} \tag{2}$$

The first formula can be referred to [14], and the second formula can be referred to [9, 11]. Especially, if we take $v = u$ in (1) or take $v = Ju$ in (2), then (1) and (2) become

$$R(Ju,u,u,Ju) = 2\mathfrak{R}(\xi,\bar{\xi},\xi,\bar{\xi}) \tag{3}$$

under Kähler hypothesis.

In this paper, we will give a geometric characterization of the right hand side in (2) when h is non-Kähler. In local coordinates, the right hand side in (2) can be written as

$$\frac{1}{2}\mathfrak{R}_{\alpha\bar{\beta}\gamma\bar{\delta}}\big(\xi^\alpha\bar{\eta}^\beta - \eta^\alpha\bar{\xi}^\beta\big)\big(\eta^\gamma\bar{\xi}^\delta - \xi^\gamma\bar{\eta}^\delta\big),$$

where $\mathfrak{R}_{\alpha\bar{\beta}\gamma\bar{\delta}} = \mathfrak{R}\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}, \frac{\partial}{\partial z^\gamma}, \frac{\partial}{\partial \bar{z}^\delta}\right)$. Under the bundle isomorphism $_o$, any Hermitian connection on $T^{1,0}M$ induces a metric connection on the Riemannian manifold (M,g) [4]. A connection is metric if it is compatible with the background Riemannian metric $g = \operatorname{Re} h$ [4]. When no confusion can rise, we still denote by D the metric connection induced by the Chern connection D under the bundle isomorphism $_o$. We find that

$$\frac{\frac{1}{2}\mathfrak{R}_{\alpha\bar{\beta}\gamma\bar{\delta}}\big(\xi^\alpha\bar{\eta}^\beta - \eta^\alpha\bar{\xi}^\beta\big)\big(\eta^\gamma\bar{\xi}^\delta - \xi^\gamma\bar{\eta}^\delta\big)}{\big(h_{\alpha\bar{\beta}}\xi^\alpha\bar{\xi}^\beta\big)\cdot\big(h_{\gamma\bar{\delta}}\eta^\gamma\bar{\eta}^\delta\big) - \frac{1}{4}\left[h_{\alpha\bar{\beta}}\left(\xi^\alpha\bar{\eta}^\beta + \eta^\alpha\bar{\xi}^\beta\right)\right]^2}$$

is just the sectional curvature of the metric connection induced by the Chern connection D under the bundle isomorphism $_o$, and we call it the Chern sectional curvature of (M,h).

Theorem 1.1. *Let (M,h) be a Hermitian manifold with the background Riemannian metric $g = \operatorname{Re} h$. Suppose D is the metric connection induced by the Chern connection under the bundle isomorphism $_o$. For arbitrary $u = u^i\frac{\partial}{\partial x^i}$, $v = v^i\frac{\partial}{\partial x^i} \in TM$, we have*

$$g\left((D^2u)(v,u),v\right) = \frac{1}{2}\mathfrak{R}_{\alpha\bar{\beta}\gamma\bar{\delta}}\big(\xi^\alpha\bar{\eta}^\beta - \eta^\alpha\bar{\xi}^\beta\big)\big(\eta^\gamma\bar{\xi}^\delta - \xi^\gamma\bar{\eta}^\delta\big), \tag{4}$$

where $\xi = u_o$ and $\eta = v_o$. Especially,

$$g\left((D^2u)(Ju,u),Ju\right) = 2\mathfrak{R}_{\alpha\bar{\beta}\gamma\bar{\delta}}\xi^\alpha\bar{\xi}^\beta\xi^\gamma\bar{\xi}^\delta. \tag{5}$$

From (4) in Theorem 1.1, we can see

$$g\left((D^2u)(v,u),v\right) = g\left((D^2v)(u,v),u\right).$$

Recently, Li and Qiu [6] proved that a Hermitian manifold such that (3) is Kähler. Hence, (5) in Theorem 1.1 and the main result in [6] can imply the following corollary.

Corollary 1.2. *Let (M,h) be a Hermitian manifold such that the Riemann sectional curvature and the Chern sectional curvature coincide. Then h is a Kähler metric.*

When h is not Kähler, there are four Ricci curvatures and two scalar curvatures of the Chern connection D, which are respectively denoted by

$$\begin{aligned}
\mathbf{Ric}_D^{(1)} &= \sqrt{-1}\mathfrak{R}^{(1)}_{\alpha\bar\beta}dz^\alpha\wedge d\bar z^\beta \quad \textit{with} \quad \mathfrak{R}^{(1)}_{\alpha\bar\beta} = h^{\bar\delta\gamma}\mathfrak{R}_{\gamma\bar\delta\alpha\bar\beta},\\
\mathbf{Ric}_D^{(2)} &= \sqrt{-1}\mathfrak{R}^{(2)}_{\alpha\bar\beta}dz^\alpha\wedge d\bar z^\beta \quad \textit{with} \quad \mathfrak{R}^{(2)}_{\alpha\bar\beta} = h^{\bar\delta\gamma}\mathfrak{R}_{\alpha\bar\beta\gamma\bar\delta},\\
\mathbf{Ric}_D^{(3)} &= \sqrt{-1}\mathfrak{R}^{(3)}_{\alpha\bar\beta}dz^\alpha\wedge d\bar z^\beta \quad \textit{with} \quad \mathfrak{R}^{(3)}_{\alpha\bar\beta} = h^{\bar\delta\gamma}\mathfrak{R}_{\gamma\bar\beta\alpha\bar\delta},\\
\mathbf{Ric}_D^{(4)} &= \sqrt{-1}\mathfrak{R}^{(4)}_{\alpha\bar\beta}dz^\alpha\wedge d\bar z^\beta \quad \textit{with} \quad \mathfrak{R}^{(4)}_{\alpha\bar\beta} = h^{\bar\delta\gamma}\mathfrak{R}_{\alpha\bar\delta\gamma\bar\beta},
\end{aligned}$$

$$\boldsymbol{s}_D^{(1)} = h^{\bar\beta\alpha}\mathfrak{R}^{(1)}_{\alpha\bar\beta}, \quad \boldsymbol{s}_D^{(2)} = h^{\bar\beta\alpha}\mathfrak{R}^{(3)}_{\alpha\bar\beta}.$$

We can also define the Ricci curvature of the induced metric connection D.

Suppose $\{e_i\}_{i=1}^{2n}$ is an orthonormal frame with respect to the background Riemannian metric g. We define the Ricci curvature $\mathbf{Ric}_D$ of the induced metric connection D by

$$\mathbf{Ric}_D(u,v) = \sum_{i=1}^{2n} g\left((D^2u)(e_i,v),e_i\right), \quad u,\, v\in TM. \tag{6}$$

We denote by

$$\mathfrak{R}_{ij} = \sum_{k=1}^{2n} g\left(\left(D^2\frac{\partial}{\partial x^i}\right)\left(e_k,\frac{\partial}{\partial x^j}\right),e_k\right) \tag{7}$$

the Ricci curvature tensor of the induced metric connection D. But we find that $\mathfrak{R}_{ij}\neq\mathfrak{R}_{ji}$, and $\mathfrak{R}_{ij}=\mathfrak{R}_{ji}$ if and only if the conformal invariant $b_{\alpha\bar\beta}=0$. The definition of $b_{\alpha\bar\beta}$ can be seen in Section 2. By using Westlake's result [12], we can see $\mathfrak{R}_{ij}=\mathfrak{R}_{ji}$ if (M,h) is a conformally Kähler manifold.

The scalar curvature $\boldsymbol{s}_D$ of the induced metric connection D is defined by

$$\boldsymbol{s}_D = \sum_{i,j=1}^{2n} g\left((D^2e_j)(e_i,e_j),e_i\right). \tag{8}$$

We also derive expression of $\boldsymbol{s}_D$ in local complex coordinates, and find that $\boldsymbol{s}_D$ is equal to four times of the second Chern scalar curvature $\boldsymbol{s}_D^{(2)}$.

2. The Chern sectional curvature

The Chern connection D is the unique connection on the holomorphic tangent bundle $T^{1,0}M$, which is compatible with the Hermitian metric h and the complex structure J. We denote by

$$\theta = (\theta^\alpha_\beta) = \left(\Gamma^\alpha_{\beta;\gamma} dz^\gamma\right) = \left(h^{\bar{\lambda}\alpha} \frac{\partial h_{\beta\bar{\lambda}}}{\partial z^\gamma} dz^\gamma\right)$$

the $n \times n$ matrix of connection 1-forms. The curvature operator $\mathfrak{R}$ of D is defined by

$$\mathfrak{R}(\xi, \bar{\eta}, \zeta, \bar{\chi}) = h\left(\left(D_\zeta D_{\bar{\chi}} - D_{\bar{\chi}} D_\zeta - D_{[\zeta, \bar{\chi}]}\right)\xi, \bar{\eta}\right), \quad \forall \xi,\ \eta,\ \zeta,\ \chi \in T^{1,0}M, \tag{9}$$

where $[\ ,\ ]$ is the Lie bracket. We call

$$\mathfrak{R}_{\alpha\bar{\beta}\gamma\bar{\delta}} = \mathfrak{R}\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}, \frac{\partial}{\partial z^\gamma}, \frac{\partial}{\partial \bar{z}^\delta}\right) = -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^\gamma \partial \bar{z}^\delta} + \frac{\partial h_{\alpha\bar{\lambda}}}{\partial z^\gamma} h^{\bar{\lambda}\kappa} \frac{\partial h_{\kappa\bar{\beta}}}{\partial \bar{z}^\delta} \tag{10}$$

the holomorphic sectional curvature tensor of D. Ulteriorly, the holomorphic sectional curvature is defined by

$$\mathrm{HSC}(\xi) = \frac{\mathfrak{R}\left(\xi, \bar{\xi}, \xi, \bar{\xi}\right)}{h\left(\xi, \xi\right)^2}, \quad \forall \xi \in T^{1,0}M. \tag{11}$$

We still denote by D the metric connection on the Riemannian manifold (M, g), which is induced by the Chern connection D under the bundle isomorphism $_o$. Then

$$\begin{aligned} D\frac{\partial}{\partial x^\alpha} &= \frac{1}{2}\left(\theta^\beta_\alpha + \bar{\theta}^\beta_\alpha\right) \otimes \frac{\partial}{\partial x^\beta} - \frac{\sqrt{-1}}{2}\left(\theta^\beta_\alpha - \bar{\theta}^\beta_\alpha\right) \otimes \frac{\partial}{\partial x^{\beta+n}}, \\ D\frac{\partial}{\partial x^{\alpha+n}} &= \frac{\sqrt{-1}}{2}\left(\theta^\beta_\alpha - \bar{\theta}^\beta_\alpha\right) \otimes \frac{\partial}{\partial x^\beta} + \frac{1}{2}\left(\theta^\beta_\alpha + \bar{\theta}^\beta_\alpha\right) \otimes \frac{\partial}{\partial x^{\beta+n}}. \end{aligned} \tag{12}$$

For a general Hermitian manifold (M, h), the induced metric connection D is compatible with the background Riemannian metric g, but not torsion free.

Definition. Let (M, h) be a Hermitian manifold. We call

$$K_D(u, v) = \frac{g((D^2 u)(v, u), v)}{g(u, u)g(v, v) - g(u, v)^2} \tag{13}$$

the Chern sectional curvature of the 2-plane $\Pi(u, v)$ spanned by two linearly independent tangent vectors $u = u^i \frac{\partial}{\partial x^i}$, $v = v^i \frac{\partial}{\partial x^i} \in TM$.

The Levi-Civita connection ∇ is the unique connection on the tangent bundle TM, which is torsion free and compatible with the background Riemannian metric $g = \mathrm{Re}\, h$. Let us denote by

$$\varphi^k_i = \gamma^k_{ij} dx^j$$

the connection 1-forms of the Levi-Civita connection ∇, where

$$\gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right)$$

is the connection coefficient of ∇. The curvature operator R of ∇ is defined by

$$R(u,v,w,y)=g\left(\left(\nabla_u\nabla_v-\nabla_v\nabla_u-\nabla_{[u,v]}\right)w,y\right),\quad \forall u,\ v,\ w,\ y\in TM. \tag{14}$$

The Riemann sectional curvature on the 2-plane $\Pi(u,v)$ is defined by

$$K_\nabla(u,v)=\frac{R(u,v,v,u)}{g(u,u)g(v,v)-g(u,v)^2}. \tag{15}$$

We call $\Pi(u,Ju)$ a holomorphic plane section [14].

If (M,h) is a Kähler manifold, then the induced metric connection D and the Levi-Civita connection ∇ coincide, thus $K_D(u,v)=K_\nabla(u,v)$.

Now if we extend the background Riemannian metric g linearly over $\mathbb{C}$ to the complexified tangent bundle $T_{\mathbb{C}}M=TM\otimes\mathbb{C}=T^{1,0}M\oplus T^{0,1}M$, then

$$g\left(\frac{\partial}{\partial z^\alpha},\frac{\partial}{\partial z^\beta}\right)=g\left(\frac{\partial}{\partial \bar z^\alpha},\frac{\partial}{\partial \bar z^\beta}\right)=0,$$

$$g\left(\frac{\partial}{\partial z^\alpha},\frac{\partial}{\partial \bar z^\beta}\right)=\overline{g\left(\frac{\partial}{\partial \bar z^\alpha},\frac{\partial}{\partial z^\beta}\right)}=\frac{1}{2}h_{\alpha\bar\beta}.$$

Hence

$$h\left(\xi,\bar\eta\right)=2g\left(\xi,\bar\eta\right),\quad \forall\xi,\ \eta\in T^{1,0}M. \tag{16}$$

Next we give an expression of the Chern sectional curvature in local complex coordinates.

Proof of Theorem 1.1 We denote by $D\frac{\partial}{\partial x^i}=\tilde\theta_i^j\frac{\partial}{\partial x^j}$ and $\tilde\theta=(\tilde\theta_i^j)$. It follows from (12) that

$$\tilde\theta=F\mathrm{diag}\left\{\theta,\bar\theta\right\}F^{-1}, \tag{17}$$

where $F=\begin{pmatrix} I & I\\ \sqrt{-1}I & -\sqrt{-1}I\end{pmatrix}$, $F^{-1}=\frac{1}{2}\begin{pmatrix} I & -\sqrt{-1}I\\ I & \sqrt{-1}I\end{pmatrix}$ is the inverse of F, I is the $n\times n$ identity matrix, $\mathrm{diag}\left\{\theta,\bar\theta\right\}=\begin{pmatrix}\theta & 0\\ 0 & \bar\theta\end{pmatrix}$. In order to simplify the calculation process, we introduce the following notations. Set

$$\frac{\partial}{\partial x}=\left(\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^{2n}}\right),\quad \frac{\partial}{\partial z}=\left(\frac{\partial}{\partial z^1},\dots,\frac{\partial}{\partial z^{n}}\right),\quad \frac{\partial}{\partial \bar z}=\left(\frac{\partial}{\partial \bar z^1},\dots,\frac{\partial}{\partial \bar z^{n}}\right),$$

then

$$\frac{\partial}{\partial x}=\left(\frac{\partial}{\partial z},\frac{\partial}{\partial \bar z}\right)F^{\mathrm{t}},$$

where F^{t} means the transpose of F. We still denote by A^{t} the transpose of a vector A. For any $u=u^i\frac{\partial}{\partial x^i}\in TM$, $\xi=u_o=\xi^\alpha\frac{\partial}{\partial z^\alpha}\in T^{1,0}M$, we set

$$\boldsymbol{u}=(u^1,\dots,u^{2n}),\quad \boldsymbol{\xi}=(\xi^1,\dots,\xi^n),$$

then

$$\boldsymbol{u}=(\boldsymbol{\xi},\bar{\boldsymbol{\xi}})F^{-1}.$$

By a straightforward computation, we have

$$\begin{aligned} D^2u &= u^j\left(d\tilde{\theta}^i_j - \tilde{\theta}^k_j \wedge \tilde{\theta}^i_k\right) \otimes \frac{\partial}{\partial x^i} \\ &= \boldsymbol{u}\left(d\tilde{\theta} - \tilde{\theta}\wedge\tilde{\theta}\right) \otimes \left(\frac{\partial}{\partial x}\right)^{\mathrm{t}} \\ &= (\boldsymbol{\xi}, \bar{\boldsymbol{\xi}})\mathrm{diag}\left\{d\theta - \theta\wedge\theta, d\bar{\theta} - \bar{\theta}\wedge\bar{\theta}\right\} \otimes \begin{pmatrix} \left(\frac{\partial}{\partial z}\right)^{\mathrm{t}} \\ \left(\frac{\partial}{\partial \bar{z}}\right)^{\mathrm{t}} \end{pmatrix} \\ &= \boldsymbol{\xi}\left(\bar{\partial}\theta\right) \otimes \left(\frac{\partial}{\partial z}\right)^{\mathrm{t}} + \bar{\boldsymbol{\xi}}\left(\partial\bar{\theta}\right) \otimes \left(\frac{\partial}{\partial \bar{z}}\right)^{\mathrm{t}} \\ &= \xi^\alpha\left(\bar{\partial}\theta^\mu_\alpha\right) \otimes \frac{\partial}{\partial z^\mu} + \bar{\xi}^\beta\left(\partial\bar{\theta}^\mu_\beta\right) \otimes \frac{\partial}{\partial \bar{z}^\mu} \\ &= \xi^\alpha\left(\mathfrak{R}^\mu_{\alpha\gamma\bar{\delta}} dz^\gamma \wedge d\bar{z}^\delta\right) \otimes \frac{\partial}{\partial z^\mu} - \bar{\xi}^\beta\left(\overline{\mathfrak{R}^\mu_{\beta\delta\bar{\gamma}}} dz^\gamma \wedge d\bar{z}^\delta\right) \otimes \frac{\partial}{\partial \bar{z}^\mu}, \end{aligned}$$

where $d = \partial + \bar{\partial}$ is the exterior differentiation. Hence

$$\left(D^2u\right)(v, w) = \left(\eta^\gamma\bar{\zeta}^\delta - \zeta^\gamma\bar{\eta}^\delta\right)\left(\xi^\alpha \mathfrak{R}^\mu_{\alpha\gamma\bar{\delta}} \frac{\partial}{\partial z^\mu} - \bar{\xi}^\beta\overline{\mathfrak{R}^\mu_{\beta\delta\bar{\gamma}}} \frac{\partial}{\partial \bar{z}^\mu}\right),$$

$$g\left(\left(D^2u\right)(v, w), y\right) = \frac{1}{2}\mathfrak{R}_{\alpha\bar{\beta}\gamma\bar{\delta}}\left(\xi^\alpha\bar{\chi}^\beta - \chi^\alpha\bar{\xi}^\beta\right)\left(\eta^\gamma\bar{\zeta}^\delta - \zeta^\gamma\bar{\eta}^\delta\right), \tag{18}$$

where $w,\, y \in TM$, $\zeta = w_o,\ \chi = y_o \in T^{1,0}M$. Especially, we can obtain (4) by taking $w = u$ and $y = v$ in (18). Note that

$$(Ju)_o = \frac{1}{2}\left(Ju - \sqrt{-1}J^2u\right) = \frac{\sqrt{-1}}{2}\left(u - \sqrt{-1}Ju\right) = J(u_o).$$

In order to prove (5), we only replace v and η in (4) with Ju and $\sqrt{-1}v$, respectively. □

According to (4) in Theorem 1.1, we have

$$K_D(u, v) = \frac{\frac{1}{2}\mathfrak{R}_{\alpha\bar{\beta}\gamma\bar{\delta}}\left(\xi^\alpha\bar{\eta}^\beta - \eta^\alpha\bar{\xi}^\beta\right)\left(\eta^\gamma\bar{\xi}^\delta - \xi^\gamma\bar{\eta}^\delta\right)}{\left(h_{\alpha\bar{\beta}}\xi^\alpha\bar{\xi}^\beta\right)\cdot\left(h_{\gamma\bar{\delta}}\eta^\gamma\bar{\eta}^\delta\right) - \frac{1}{4}\left[h_{\alpha\bar{\beta}}\left(\xi^\alpha\bar{\eta}^\beta + \eta^\alpha\bar{\xi}^\beta\right)\right]^2}. \tag{19}$$

It follows from the above formula that

$$K_D(u, v) = K_D(v, u).$$

Remark 2.1. By (18), we have

$$g\left(\left(D^2u\right)(Jw, w), Ju\right) = 2\mathfrak{R}\left(u_o, \overline{u_o}, w_o, \overline{w_o}\right). \tag{20}$$

Remark 2.2. By (18), it is clear that

$$g\left(\left(D^2u\right)(v, w), y\right) = g\left(\left(D^2u\right)(Jv, Jw), y\right) = g\left(\left(D^2Ju\right)(v, w), Jy\right),$$
$$g\left(\left(D^2u\right)(v, w), y\right) = -g\left(\left(D^2y\right)(v, w), u\right) = -g\left(\left(D^2u\right)(w, v), y\right).$$

But we can not expect the following formula

$$g\left(\left(D^2u\right)(v,w),y\right)=g\left(\left(D^2v\right)(u,y),w\right)$$

always holds for arbitrary u, v, w, $y\in TM$.

As an application, we can extend Lu's result [8] of Kähler manifolds with non-negative (or non-positive) Riemann sectional curvatures on Kähler-like manifolds with non-negative (or non-positive) Chern sectional curvature. Yang and Zheng [13] defined Kähler-like manifolds, which are classes of non-Kähler manifolds.

Definition ([13])**.** A Hermitian metric h is called Kähler-like, if its holomorphic sectional curvature tensor satisfies $\mathfrak{R}_{\alpha\bar{\beta}\gamma\bar{\delta}}=\mathfrak{R}_{\gamma\bar{\beta}\alpha\bar{\delta}}$ for all α, β, γ, $\delta=1,2,\ldots,n$.

By using the same method as that in [8], we have the following result without details.

Proposition 2.3. *Let (M,h) be a Kähler-like manifold with non-negative (resp. non-positive) Chern sectional curvature. Then*

$$\left|\mathfrak{R}\left(\xi,\bar{\xi},\eta,\bar{\eta}\right)\right|^2\leq\mathfrak{R}\left(\xi,\bar{\xi},\xi,\bar{\xi}\right)\mathfrak{R}\left(\eta,\bar{\eta},\eta,\bar{\eta}\right). \tag{21}$$

Proposition 2.4. *For arbitrary $u=u^i\frac{\partial}{\partial x^i}$, $w=w^i\frac{\partial}{\partial x^i}\in TM$, we have*

$$\mathbf{Ric}_D(u,w)=\mathfrak{R}^{(4)}_{\alpha\bar{\delta}}\xi^\alpha\bar{\zeta}^\delta+\mathfrak{R}^{(3)}_{\alpha\bar{\delta}}\zeta^\alpha\bar{\xi}^\delta, \tag{22}$$

where $\xi=u_o$, $\zeta=w_o$.

Proof. For any lowercase Greek index α, we denote by $\alpha^*=\alpha+n$. By (18), we have

$$\begin{aligned}
g\left(\left(D^2u\right)\left(\frac{\partial}{\partial x^\kappa},w\right),\frac{\partial}{\partial x^\lambda}\right)&=\mathrm{Re}\left(\mathfrak{R}_{\alpha\bar{\lambda}\kappa\bar{\delta}}\xi^\alpha\bar{\zeta}^\delta-\mathfrak{R}_{\alpha\bar{\lambda}\gamma\bar{\kappa}}\xi^\alpha\zeta^\gamma\right),\\
g\left(\left(D^2u\right)\left(\frac{\partial}{\partial x^\kappa},w\right),\frac{\partial}{\partial x^{\lambda^*}}\right)&=\mathrm{Im}\left(\mathfrak{R}_{\alpha\bar{\lambda}\kappa\bar{\delta}}\xi^\alpha\bar{\zeta}^\delta-\mathfrak{R}_{\alpha\bar{\lambda}\gamma\bar{\kappa}}\xi^\alpha\zeta^\gamma\right),\\
g\left(\left(D^2u\right)\left(\frac{\partial}{\partial x^{\kappa^*}},w\right),\frac{\partial}{\partial x^\lambda}\right)&=-\mathrm{Im}\left(\mathfrak{R}_{\alpha\bar{\lambda}\kappa\bar{\delta}}\xi^\alpha\bar{\zeta}^\delta+\mathfrak{R}_{\alpha\bar{\lambda}\gamma\bar{\kappa}}\xi^\alpha\zeta^\gamma\right),\\
g\left(\left(D^2u\right)\left(\frac{\partial}{\partial x^{\kappa^*}},w\right),\frac{\partial}{\partial x^{\lambda^*}}\right)&=\mathrm{Re}\left(\mathfrak{R}_{\alpha\bar{\lambda}\kappa\bar{\delta}}\xi^\alpha\bar{\zeta}^\delta+\mathfrak{R}_{\alpha\bar{\lambda}\gamma\bar{\kappa}}\xi^\alpha\zeta^\gamma\right).
\end{aligned}$$

Set $L=(L_{\kappa\bar{\lambda}})_{1\leq\kappa,\lambda\leq n}=L_1+\sqrt{-1}L_2$, where $L_{\kappa\bar{\lambda}}=\mathfrak{R}_{\alpha\bar{\lambda}\kappa\bar{\delta}}\xi^\alpha\bar{\zeta}^\delta$, $L_1=\mathrm{Re}\,L$ and $L_2=\mathrm{Im}\,L$. Set $K=(K_{\kappa\lambda})_{1\leq\kappa,\lambda\leq n}=K_1+\sqrt{-1}K_2$, where $L_{\kappa\lambda}=\mathfrak{R}_{\alpha\bar{\lambda}\gamma\bar{\kappa}}\xi^\alpha\zeta^\gamma$, $K_1=\mathrm{Re}\,K$ and $K_2=\mathrm{Im}\,K$. Being similar to the formula (2.18) in [7], we can write as

$$\begin{aligned}
&\begin{pmatrix} L_1-K_1 & L_2-K_2\\ -L_2-K_2 & L_1+K_1\end{pmatrix}\\
&=F\mathrm{diag}\left\{L,\bar{L}\right\}F^{-1}-\frac{1}{2}\mathrm{diag}\left\{I,-I\right\}F\mathrm{diag}\left\{K,\bar{K}\right\}F^{\mathrm{t}}\mathrm{diag}\left\{I,-I\right\}.
\end{aligned}$$

Denote by $G = (g_{ij})$ and $H = (h_{\alpha\bar{\beta}})$, then

$$G^{-1} = F\text{diag}\left\{H^{-1}, \bar{H}^{-1}\right\} F^{-1}.$$

A direct computation shows that

$$\begin{aligned} &\text{tr}\left[G^{-1}\begin{pmatrix} L_1 - K_1 & L_2 - K_2 \\ -L_2 - K_2 & L_1 + K_1 \end{pmatrix}\right] \\ =&\,\text{tr}\left(H^{-1}L\right) + \text{tr}\left(\bar{H}^{-1}\bar{L}\right) \\ =&\, h^{\bar{\lambda}\kappa}\mathfrak{R}_{\alpha\bar{\lambda}\kappa\bar{\delta}}\xi^{\alpha}\bar{\zeta}^{\delta} + \overline{h^{\bar{\lambda}\kappa}\mathfrak{R}_{\alpha\bar{\lambda}\kappa\bar{\delta}}\xi^{\alpha}\bar{\zeta}^{\delta}}, \end{aligned}$$

where tr ( ) means the trace of a square matrix. Hence

$$\textbf{Ric}_D(u, w) = g^{kl} g\left(\left(D^2 u\right)\left(\frac{\partial}{\partial x^k}, w\right), \frac{\partial}{\partial x^l}\right) = \mathfrak{R}^{(4)}_{\alpha\bar{\delta}}\xi^{\alpha}\bar{\zeta}^{\delta} + \mathfrak{R}^{(3)}_{\alpha\bar{\delta}}\zeta^{\alpha}\bar{\xi}^{\delta}.$$

This completes the proof. □

We recall that the Ricci curvature tensor $\mathfrak{R}_{ij}$ of the connection D is defined by

$$\begin{aligned} \mathfrak{R}_{ij} &= \sum_{k=1}^{2n} g\left(\left(D^2 \frac{\partial}{\partial x^i}\right)\left(e_k, \frac{\partial}{\partial x^j}\right), e_k\right) \\ &= g^{kl} g\left(\left(D^2 \frac{\partial}{\partial x^i}\right)\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j}\right), \frac{\partial}{\partial x^l}\right). \end{aligned}$$

From (22), we can see $\mathfrak{R}_{ij} \neq \mathfrak{R}_{ji}$ in general. It follows from (22) that $\mathfrak{R}_{ij} = \mathfrak{R}_{ji}$ if and only if $\mathfrak{R}^{(3)}_{\alpha\bar{\delta}} = \mathfrak{R}^{(4)}_{\alpha\bar{\delta}}$. Are there some Hermitian manifolds such that $\mathfrak{R}_{ij} = \mathfrak{R}_{ji}$?

Let

$$T^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta;\gamma} - \Gamma^{\alpha}_{\gamma;\beta} = h^{\bar{\lambda}\alpha}\left(\frac{\partial h_{\beta\bar{\lambda}}}{\partial z^{\gamma}} - \frac{\partial h_{\gamma\bar{\lambda}}}{\partial z^{\beta}}\right)$$

be the torsion tensor of the Chern connection D. Set $T_{\beta} = \sum_{\alpha=1}^{n} T^{\alpha}_{\beta\alpha}$. Lee [5] first introduced the following conformal invariant

$$b_{\alpha\bar{\beta}} = \frac{\partial T_{\alpha}}{\partial \bar{z}^{\beta}} - \frac{\partial T_{\bar{\beta}}}{\partial z^{\alpha}}, \tag{23}$$

where $T_{\bar{\beta}} = \overline{T_{\beta}}$. In fact, it is easy to see

$$b_{\alpha\bar{\beta}} = \mathfrak{R}^{(4)}_{\alpha\bar{\beta}} - \mathfrak{R}^{(3)}_{\alpha\bar{\beta}}. \tag{24}$$

If there exists a positive scalar function ρ in (M, h) such that ρh is a Kähler metric, we call (M, h) a conformally Kähler manifold. Westlake [12] proved $b_{\alpha\bar{\beta}} = 0$ on conformally Kähler manifolds. Hence $\mathfrak{R}_{ij} = \mathfrak{R}_{ji}$ on conformally Kähler manifolds [1]. Next we provide an example such that $\mathfrak{R}_{ij} = \mathfrak{R}_{ji}$.

Example 2.5. Let $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z|^2 = \sum_{\alpha=1}^n |z^\alpha|^2 < 1\}$ be the unit ball in $\mathbb{C}^n$ $(n > 1)$ endowed with a Hermitian metric $h = \frac{4|dz|^2}{(1-|z|^2)^2}$. We put the symbols as above. A direct computation shows that

$$\begin{aligned}
\mathfrak{R}_{\alpha\bar{\beta}\mu\bar{\nu}} =& \frac{4\delta_{\alpha\beta}}{(1-|z|^2)^4}\left[(1-|z|^2)\delta_{\mu\nu} + \bar{z}^\mu z^\nu\right],\\
\mathfrak{R}^{(3)}_{\alpha\bar{\beta}} =& \frac{2}{(1-|z|^2)^2}\left[(1-|z|^2)\delta_{\alpha\beta} + \bar{z}^\alpha z^\beta\right],\\
\mathfrak{R}^{(4)}_{\alpha\bar{\beta}} =& \frac{2}{(1-|z|^2)^2}\left[(1-|z|^2)\delta_{\alpha\beta} + \bar{z}^\alpha z^\beta\right],
\end{aligned}$$

where $\delta_{\alpha\beta}$ are Kronecker symbols. Hence $\mathfrak{R}^{(4)}_{\alpha\bar{\beta}} = \mathfrak{R}^{(3)}_{\alpha\bar{\beta}}$ yields $\mathfrak{R}_{ij} = \mathfrak{R}_{ji}$.

Proposition 2.6. *Let (M, h) be a Hermitian manifold with the background Riemannian metric $g = \mathrm{Re}\, h$. Suppose D is the metric connection induced by the Chern connection under the bundle isomorphism $_o$. Then $\boldsymbol{s}_D$ is equal to four times of the second Chern scalar curvature $\boldsymbol{s}^{(2)}_D$, i.e.,*

$$\boldsymbol{s}_D = 4\boldsymbol{s}^{(2)}_D. \tag{25}$$

Proof. For simplicity, we denote by

$$\mathbf{Ric}_D(u) := \mathbf{Ric}_D(u,u) = \mathfrak{R}_{ij}u^iu^j,$$

then

$$\begin{aligned}
\boldsymbol{s}_D &= \frac{1}{2}g^{ij}\left(\mathfrak{R}_{ij} + \mathfrak{R}_{ji}\right) = \frac{1}{2}g^{ij}\frac{\partial^2\mathbf{Ric}_D(u)}{\partial u^i\partial u^j} = 2h^{\bar{\delta}\alpha}\frac{\partial^2\mathbf{Ric}_D(u)}{\partial\xi^\alpha\partial\bar{\xi}^\delta}\\
&= 2h^{\bar{\delta}\alpha}h^{\bar{\lambda}\kappa}\left(\mathfrak{R}_{\alpha\bar{\lambda}\kappa\bar{\delta}} + \mathfrak{R}_{\kappa\bar{\delta}\alpha\bar{\lambda}}\right) = 4\boldsymbol{s}^{(2)}_D.
\end{aligned}$$

This completes the proof. □

There are various Hermitian connections on the holomorphic tangent bundle in [2]. We can consider the metric connection induced by any Hermitian connection under the bundle isomorphism $_o$. For example, we can consider the Levi-Civita connection ∇^{lc} on $T^{1,0}M$ (i.e. the restriction of the complexified Levi-Civita connection ∇ to $T^{1,0}M$, see [3]). We denote by $\boldsymbol{s}_{\nabla^{lc}}$ the scalar curvature of the metric connection induced by the Levi-Civita connection ∇^{lc} on $T^{1,0}M$. Gauduchon [1] showed that $\boldsymbol{s}_{\nabla^{lc}}$ and the Riemannian scalar curvature $\boldsymbol{s}_\nabla$ coincide on a compact Hermitian manifold if and only if the compact Hermitian manifold is balanced.

Acknowledgemets. The authors are very grateful to the referee for providing many helpful suggestions that substantially improved the manuscript. The authors gratefully acknowledge the financial supports by the National Science Foundation of China under Grant No. 12001165.

References

[1] P. Gauduchon, *La 1-forme de torsion d'une variété hermitienne compacte*, Math. Ann. **267** (1984), no. 4, 495–518. https://doi.org/10.1007/BF01455968

[2] P. Gauduchon, *Hermitian connections and Dirac operators*, Boll. Un. Mat. Ital. B (7) **11** (1997), no. 2, suppl., 257–288.

[3] J. He, K. Liu, and X. Yang, *Levi-Civita Ricci-flat metrics on compact complex manifolds*, J. Geom. Anal. **30** (2020), no. 1, 646–666. https://doi.org/10.1007/s12220-019-00156-9

[4] D. Huybrechts, *Complex Geometry*, Universitext, Springer, Berlin, 2005.

[5] H.-C. Lee, *A kind of even-dimensional differential geometry and its application to exterior calculus*, Amer. J. Math. **65** (1943), 433–438. https://doi.org/10.2307/2371967

[6] H. Li and C. Qiu, *A note on holomorphic sectional curvature of a Hermitian manifold*, Glasg. Math. J. **64** (2022), no. 3, 739–745. https://doi.org/10.1017/S0017089522000064

[7] H. Li, C. Qiu, and G. Zhong, *Curvatures of strongly convex Kähler-Finsler manifolds*, Differential Geom. Appl. **86** (2023), Paper No. 101957, 25 pp. https://doi.org/10.1016/j.difgeo.2022.101957

[8] Q. K. Lu, *A note on the extremum of sectional curvature of a Kähler manifold*, Sci. Sinica Ser. A **27** (1984), no. 4, 367–371.

[9] Q. K. Lu and Y. C. Xu, *A note on transitive domains*, Chinese Math. **2** (1961), 11–26; translated from Acta Math. Sinica **11**, 11–23.

[10] A. Moroianu, *Lectures on Kähler Geometry*, London Mathematical Society Student Texts, 69, Cambridge Univ. Press, Cambridge, 2007. https://doi.org/10.1017/CBO9780511618666

[11] Y.-T. Siu, *The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds*, Ann. of Math. (2) **112** (1980), no. 1, 73–111. https://doi.org/10.2307/1971321

[12] W. J. Westlake, *Conformally Kähler manifolds*, Proc. Cambridge Philos. Soc. **50** (1954), 16–19. https://doi.org/10.1017/S0305004100029029

[13] B. Yang and F. Zheng, *On curvature tensors of Hermitian manifolds*, Comm. Anal. Geom. **26** (2018), no. 5, 1195–1222. https://doi.org/10.4310/CAG.2018.v26.n5.a7

[14] F. Zheng, *Complex Differential Geometry*, AMS/IP Studies in Advanced Mathematics, 18, Amer. Math. Soc., Providence, RI, 2000. https://doi.org/10.1090/amsip/018

Pandeng Cao
School of Mathematical Sciences
Xiamen university
Xiamen 361005, P. R. China
Email address: caopandeng2023@163.com

Hongjun Li
School of Mathematics and Statistics
Henan University
Kaifeng 475004, P. R. China
Email address: lihj@vip.henu.edu.cn