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SCHATTEN CLASSES OF COMPOSITION OPERATORS ON DIRICHLET TYPE SPACES WITH SUPERHARMONIC WEIGHTS

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Abstract. In this paper, we completely characterize the Schatten classes of composition operators on the Dirichlet type spaces with superharmonic weights. Our investigation is basced on building a bridge between the Schatten classes of composition operators on the weighted Dirichlet type spaces and Toeplitz operators on weighted Bergman spaces.

1. Introduction

For an analytic self-map φ of the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$, the composition operator C_φ on $H(\mathbb{D})$, the space of analytic functions in $\mathbb{D}$, is given by

$$C_\varphi(f)(z) = f(\varphi(z)),\ f \in H(\mathbb{D}),\ z \in \mathbb{D}.$$

A weight function $\omega : \mathbb{D} \to (0, +\infty)$ is defined as integrable over $\mathbb{D}$ and bounded below by a positive constant on every compact subset of $\mathbb{D}$. The weighted Bergman space A^2_ω is defined as the set of functions $f \in H(\mathbb{D})$ with weight ω, such that

$$\|f\|_{A^2_\omega} = \Big(\int_{\mathbb{D}} |f(z)|^2 \omega(z) \mathrm{d}A(z)\Big)^{\frac{1}{2}} < \infty,$$

where $\mathrm{d}A(z)$ represents the area measure on $\mathbb{D}$ with $A(\mathbb{D}) = 1$. If $\omega(z) = (\alpha+1)(1-|z|^2)^\alpha,\ \alpha > -1$, the space is a standard Bergman space, denoted by A^2_α. Note that the weighted Bergman space A^2_ω is a reproducing kernel Hilbert space. This means that, by the Riesz representation theorem, there exists a

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function $K_\zeta \in A^2_\omega$ for every $\zeta \in \mathbb{D}$ such that $f(\zeta) = \langle f, K_\zeta\rangle_{A^2_\omega}$, where

$$\langle f, h\rangle_{A^2_\omega} = \int_{\mathbb{D}} f(\zeta)\overline{h(\zeta)}\omega(\zeta)\mathrm{d}A(\zeta)$$

is an inner product in $L^2(\mathbb{D}, \omega\mathrm{d}A)$. Moreover, K_ζ is named the reproducing kernel of A^2_ω at $\zeta \in \mathbb{D}$ and $K_\zeta(\xi) = K(\xi, \zeta) = \overline{K(\zeta, \xi)} = \overline{K_\xi(\zeta)}$; see [22] for more results about the Bergman spaces.

Let μ be a positive Borel measure on $\mathbb{D}$, and let υ be a positive Borel measure on the boundary of the unit disk, denoted by $\mathbb{T}$. Set

$$W(z) = \int_{\mathbb{D}} \log\Big|\frac{1-\bar{\zeta}z}{z-\zeta}\Big|\mathrm{d}\mu(\zeta) + \int_{\mathbb{T}} \frac{1-|z|^2}{|\xi - z|^2}\mathrm{d}\upsilon(\xi) := U_\mu(z) + P_\upsilon(z).$$

Weighted Dirichlet spaces associated with the weight W have been studied in [1, 10]. Note that $U_\mu(z)$ is a superharmonic function in $\mathbb{D}$ and $P_\upsilon(z)$ is a harmonic function in $\mathbb{D}$. In [5], the authors considered the Dirichlet-type space $\mathcal{D}_\mu$, which consists of functions $f \in H(\mathbb{D})$ satisfying

$$\int_{\mathbb{D}} |f'(z)|^2 U_\mu(z)\mathrm{d}A(z) < \infty, \tag{1.1}$$

where it is assumed that

$$\int_{\mathbb{D}} (1-|z|^2)\mathrm{d}\mu(z) < \infty.$$

We know by [5, Lemma 5.11] that (1.1) is equivalent to

$$\|f\|^2_{\mathcal{D}_\mu} = \int_{\mathbb{D}} |f'(z)|^2 V_\mu(z)\mathrm{d}A(z) < \infty, \tag{1.2}$$

where

$$V_\mu(z) = \int_{\mathbb{D}} (1-|\varphi_z(w)|^2)\mathrm{d}\mu(w),\ \varphi_z(w) = \frac{z-w}{1-\bar{z}w}.$$

In recent years, many authors pay more attention to Dirichlet type spaces. See [3, 4, 6, 7, 10]. Pau and Pérez [19] studied the Schatten classes of composition operators on the standard weighted Dirichlet spaces. Sarason and Silva [21] investigated the boundedness and compactness of composition operators acting on Dirichlet spaces with harmonic weight P_υ. The authors of [12] characterized the Schatten classes of composition operators on Dirichlet spaces with P_υ. In [5], the authors considered the boundedness and compactness of composition operators on Dirichlet spaces $\mathcal{D}_\mu$ with superharmonic weight U_μ. There are many results on the Schatten classes of Toeplitz operators acting on different weighted Bergman spaces; see [2, 11, 14, 15, 20].

In this paper, we characterize the Schatten classes of composition operators on $\mathcal{D}_\mu$ with superharmonic weight U_μ. In order to prove our main results, we build a bridge between the Schatten classes of Toeplitz operators on weighted Bergman space A^2_μ with superharmonic weight U_μ and the Schatten classes of composition operators on $\mathcal{D}_\mu$ with superharmonic weight U_μ.

Finally, we use the notation $a \lesssim b$ to denote that there is a constant $C > 0$ such that $a \leq Cb$. We write $a \approx b$ if both $a \lesssim b$ and $b \lesssim a$ hold.

2. The Bergman reproducing kernel estimates

In this section, we provide two different approximations for the reproducing kernel of A^2_μ, and our methods are in principle close to those of [12].

Theorem 1. *Assume that μ is a positive Borel measure on $\mathbb{D}$ and K is the reproducing kernel of A^2_μ. There exist positive constants C and C' such that*

$$\frac{C}{(1-|z|^2)^2 V_\mu(z)} \leq \|K_z\|^2_{A^2_\mu} \leq \frac{C'}{(1-|z|^2)^2 V_\mu(z)},\ z \in \mathbb{D}.$$

To establish Theorem 1, we shall employ the subsequent lemma.

Lemma A ([18, Lemma 2.5])**.** *Let $r, t > 0$ and $s > -1$ such that $r + t - s > 2$. suppose $t < s + 2 < r$. Then, there exists a constant $C > 0$ such that*

$$\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^s}{|1-\bar{\zeta}z|^r |1-\bar{\zeta}\xi|^t} \mathrm{d}A(\zeta) \leq C \frac{(1-|z|^2)^{2+s-r}}{|1-\bar{\xi}z|^t}$$

holds for all $z, \xi \in \mathbb{D}$.

For $z \in \mathbb{D}$, set

$$\mathbb{D}(\rho(z)) := \{\xi \in \mathbb{D}, |\xi - z| \leq \rho(z)\},\ \rho(z) = \frac{1}{2}(1-|z|^2).$$

Proof of Theorem 1. It is easy to known that

$$1-|z|^2 \approx 1-|\zeta|^2,\ |1-\bar{z}w|^2 \approx |1-\bar{\zeta}w|^2,\ \zeta \in \mathbb{D}(\rho(z)).$$

Thus $V_\mu(\zeta) \approx V_\mu(z)$. By the subharmonicity of $|K_w|^2$, there exists a constant $C_1 > 0$ such that

$$\begin{aligned} |K_w(z)|^2 V_\mu(z) &\leq \frac{C_1}{(1-|z|^2)^2} \int_{\mathbb{D}(\rho(z))} |K_w(\zeta)|^2 V_\mu(\zeta) \mathrm{d}A(\zeta) \\ &\leq \frac{C_1}{(1-|z|^2)^2} \|K_w\|^2_{A^2_\mu}. \end{aligned}$$

Hence,

$$|K_w(z)|^2 \leq \frac{C_1}{(1-|z|^2)^2 V_\mu(z)} \|K_w\|^2_{A^2_\mu}.$$

Setting $w = z$, we have that $|K_z(z)|^2 = \|K_z\|^2_{A^2_\mu} \|K_z\|^2_{A^2_\mu}$. Therefore,

$$\|K_z\|^2_{A^2_\mu} \leq \frac{C_1}{(1-|z|^2)^2 V_\mu(z)}. \tag{2.1}$$

Conversely, choose a function $h_z(\eta) = \frac{1}{(1-\bar{z}\eta)^2} \in A^2_\mu$ and then

$$\|K_z\|^2_{A^2_\mu} \geq \frac{|h_z(z)|^2}{\|h_z\|^2_{A^2_\mu}},\ z \in \mathbb{D}, \tag{2.2}$$

since

$$\|K_z\|_{A^2_\mu}^2 = \sup\{|f(z)|^2;\ f \in A^2_\mu,\ \|f\|_{A^2_\mu} \le 1\}.$$

By using Fubini's theorem and Lemma A, we can conclude that there exists a constant $C_2 > 0$ such that

$$\begin{aligned}\|h_z\|_{A^2_\mu}^2 &= \int_{\mathbb{D}} |h_z(\eta)|^2 V_\mu(\eta)\mathrm{d}A(\eta)\\ &= \int_{\mathbb{D}} \frac{1}{|1-\bar{z}\eta|^4}\int_{\mathbb{D}}(1-|\varphi_\eta(\zeta)|^2)\mathrm{d}\mu(\zeta)\mathrm{d}A(\eta)\\ &= \int_{\mathbb{D}} \frac{1}{|1-\bar{z}\eta|^4}\int_{\mathbb{D}}\frac{(1-|\eta|^2)(1-|\zeta|^2)}{|1-\overline{\eta}\zeta|^2}\mathrm{d}\mu(\zeta)\mathrm{d}A(\eta)\\ &\le \frac{C_2V_\mu(z)}{(1-|z|^2)^2}.\end{aligned} \tag{2.3}$$

By (2.2) and (2.3), we obtain

$$\|K_z\|_{A^2_\mu}^2 \ge \frac{1}{C_2}\frac{1}{(1-|z|^2)^2V_\mu(z)}. \tag{2.4}$$

By (2.1) and (2.4), we get

$$\frac{1}{C_2}\frac{1}{(1-|z|^2)^2V_\mu(z)} \le \|K_z\|_{A^2_\mu}^2 \le \frac{C_1}{(1-|z|^2)^2V_\mu(z)},\ z\in\mathbb{D}.$$

The proof is completed. □

Theorem 2. *Suppose μ is a positive Borel measure on $\mathbb{D}$. Then, there exists an $\alpha \in (0,\frac{1}{8})$ such that for any $z,\eta\in\mathbb{D}$ with $|z-\eta| \le \alpha(1-|z|^2)$, we have*

$$|K_z(\eta)|^2 \approx \frac{1}{(1-|z|^2)^2(1-|\eta|^2)^2V_\mu(z)V_\mu(\eta)}. \tag{2.5}$$

Moreover, for $0<t<1$, there exists a positive constant $C = C(t)$ such that

$$|K_z(\eta)|^2 \le C\|K_z\|_{A^2_\mu}^2\|K_\eta\|_{A^2_\mu}^2\left(\frac{(1-|z|^2)(1-|\eta|^2)}{|z-\eta|^2}\right)^t. \tag{2.6}$$

To show Theorem 2, it is necessary to utilize the following lemmas.

Lemma 2.1. *If f belongs to A^2_μ, then for $z,\eta\in\mathbb{D}$ with $\eta\in\mathbb{D}(\frac{\rho(z)}{4})$, we have the inequality*

$$|f(z)-f(\eta)| \lesssim \frac{|z-\eta|}{1-|z|}\|K_z\|_{A^2_\mu}\|f\|_{A^2_\mu}. \tag{2.7}$$

Proof. By a simple computation,

$$\begin{aligned}|f(z)-f(\eta)| &= \left|\int_\eta^z f'(\xi)\mathrm{d}\xi\right| \\ &= \left|\int_\eta^z \frac{1}{2\pi i}\int_{|w-\xi|=\frac{1-|z|^2}{8}} \frac{f(w)}{(w-\xi)^2}\mathrm{d}\sigma\mathrm{d}\xi\right| \\ &\lesssim \frac{|z-\eta|}{1-|z|}\sup\left\{|f(w)|,\ w\in\mathbb{D}(\frac{\rho(z)}{2})\right\} \\ &= \frac{|z-\eta|}{1-|z|}\sup\left\{|\langle f(\eta), K_w(\eta)\rangle_{A^2_\mu}|,\ w\in\mathbb{D}(\frac{\rho(z)}{2})\right\} \\ &\lesssim \frac{|z-\eta|}{1-|z|}\|f\|_{A^2_\mu}\|K_w\|_{A^2_\mu}.\end{aligned}$$

Note that $\|K_z\|_{A^2_\mu} \approx \|K_w\|_{A^2_\mu}$ for $w \in \mathbb{D}$ with $\eta \in \mathbb{D}(\frac{\rho(z)}{4})$. Thus (2.7) holds. □

Lemma 2.2. *For any $z, \eta \in \mathbb{D}$ with $|z-\eta| \le \alpha(1-|z|^2)$, where $\alpha \in (0, \frac{1}{8})$, there exists a constant $C > 0$ such that*

$$|K_z(\eta)| \approx C\|K_z\|_{A^2_\mu}\|K_\eta\|_{A^2_\mu}.$$

Proof. Applying Lemma 2.1 to $f = K_z$, we find that

$$|K_z(z)-K_z(\eta)| \lesssim \frac{|z-\eta|}{1-|z|}\|K_z\|^2_{A^2_\mu} \lesssim K_z(z).$$

Thus,

$$|K_z(\eta)| \approx K_z(z).$$

Clearly, $\|K_z\|_{A^2_\mu} \approx \|K_\eta\|_{A^2_\mu}$ for $|z-\eta| \le \alpha(1-|z|^2)$. Therefore,

$$|K_z(\eta)| \approx \|K_z\|_{A^2_\mu}\|K_\eta\|_{A^2_\mu}.$$

□

Theorem B ([13, Lemma 4.4.1])**.** *Let $\tau \in C^2(\mathbb{D})$ and Δ be a Laplace operator. Assume that $\Delta\tau > 0$ on $\mathbb{D}$. Then there exists a solution u of the equation $\overline{\partial}u = f$ such that*

$$\int_{\mathbb{D}} |u(w)|^2 e^{-\tau(w)}\mathrm{d}A(w) \le \int_{\mathbb{D}} \frac{|f(w)|^2}{\Delta\tau(w)} e^{-\tau(w)}\mathrm{d}A(w),$$

provided the right-hand side is finite.

Theorem C ([8, Lemma 2.2])**.** *Let $\psi, \phi \in C^2(\mathbb{D})$ and Δ be a Laplace operator. Assume that $\Delta\phi > 0$ on $\mathbb{D}$. If*

$$\left|\frac{\partial\phi(w)}{\partial w}\right|^2 \le r\Delta\phi(w),\ 0 < r < 1,$$

and for any g we can find a solution v of $\overline{\partial}v = g$ such that

$$\int_{\mathbb{D}} |v(z)|^2 e^{-\psi(z)-\phi(z)} \mathrm{d}A(z) \leq C \int_{\mathbb{D}} \frac{|g(z)|^2}{\Delta\phi(z)} e^{-\psi(z)-\phi(z)} \mathrm{d}A(z),$$

then v_0, the solution to $\overline{\partial}v = g$ of minimal norm in $L^2(\mathbb{D}, e^{-\psi(z)}\mathrm{d}A)$, satisfies

$$\int_{\mathbb{D}} |v_0(z)|^2 e^{\phi(z)-\psi(z)} \mathrm{d}A(z) \leq C \int_{\mathbb{D}} \frac{|g(z)|^2}{\Delta\phi(z)} e^{\phi(z)-\psi(z)} \mathrm{d}A(z),$$

where $C = \frac{6}{(1-r)^2}$.

Proof of Theorem 2. By utilizing Lemma 2.2 in conjunction with Theorem 1, we can conclude that (2.5) holds. Fix $\alpha \in (0, \frac{1}{8})$ and let $z, \eta \in \mathbb{D}$. Set $\mathbb{D}(\alpha\rho(z)) := \{\xi \in \mathbb{D}, |\xi - z| \leq \alpha\rho(z)\}$, $\rho(z) = \frac{1}{2}(1 - |z|^2)$. $\mathbb{D}(\alpha\rho(z))$ is an Euclidean disk with center z and radius $\alpha\rho(z)$. If $\mathbb{D}(\alpha\rho(z)) \cap \mathbb{D}(\alpha\rho(\eta)) \neq \varnothing$, it follows that $|z - \eta|^2 \lesssim \rho(z)\rho(\eta)$, which implies the validity of (2.6). Now, let us consider the case where $\mathbb{D}(\alpha\rho(z)) \cap \mathbb{D}(\alpha\rho(\eta)) = \varnothing$. Suppose χ is a smooth real function in $\mathbb{D}$ that satisfies the following conditions:

$$0 \leq \chi \leq 1, \ \mathrm{supp}(\chi) \subset \mathbb{D}(\alpha\rho(\eta)), \ \chi|_{\mathbb{D}(\frac{\alpha\rho(\eta)}{2})} \equiv 1, \ |\bar{\partial}\chi|^2 \leq \frac{\chi}{\rho^2}.$$

For $\xi \in \mathbb{D}(\frac{\alpha\rho(\eta)}{2})$, we find that $V_\mu(\eta) \approx V_\mu(\xi)$. Then

$$\begin{aligned} |K_z(\eta)|^2 V_\mu(\eta) &\lesssim \frac{1}{\rho^2(\eta)} \int_{\mathbb{D}(\frac{\alpha\rho(\eta)}{2})} |K_z(\xi)|^2 V_\mu(\xi) \mathrm{d}A(\xi) \\ &\lesssim \frac{1}{\rho^2(\eta)} \int_{\mathbb{D}} \chi(\xi) K_z(\xi) \overline{K_z(\xi)} V_\mu(\xi) \mathrm{d}A(\xi) \\ &= \frac{1}{\rho^2(\eta)} \|K_z\|^2_{L^2(\mathbb{D}, \chi V_\mu \mathrm{d}A)}. \end{aligned} \tag{2.8}$$

From the duality,

$$\|K_z\|^2_{L^2(\mathbb{D}, \chi V_\mu \mathrm{d}A)} = \sup_f |\langle f, K_z \rangle_{L^2(\mathbb{D}, \chi V_\mu \mathrm{d}A)}|^2,$$

where $f \in H(\mathbb{D})$ satisfying

$$\int_{\mathbb{D}} |f(w)|^2 \chi(w) V_\mu(w) \mathrm{d}A(w) = 1.$$

Thus,

$$P(f\chi)(z) = \langle f, K_z \rangle_{L^2(\mathbb{D}, \chi V_\mu \mathrm{d}A)},$$

where P is the Bergman projection from $L^2(\mathbb{D}, V_\mu \mathrm{d}A)$ into A^2_μ. Therefore, $u_f = f\chi - P(f\chi)$ is the solution with minimal norm in $L^2(\mathbb{D}, V_\mu \mathrm{d}A)$ of the equation

$$\bar{\partial} u_f = \bar{\partial}(f\chi) = f\bar{\partial}\chi. \tag{2.9}$$

Since $(f\chi)(z) = 0$, it can be readily verified that

$$|u_f(z)| = |P(f\chi)(z)| = |\langle f, K_z \rangle_{L^2(\mathbb{D}, \chi V_\mu \mathrm{d}A)}|.$$

From (2.8), we get

$$|K_z(\eta)|^2V_\mu(\eta) \lesssim \frac{1}{\rho^2(\eta)} \sup_f |u_f(z)|^2. \tag{2.10}$$

Since $\mathrm{supp}(\chi) \subset \mathbb{D}(\alpha\rho(\eta))$ and $\mathbb{D}(\alpha\rho(z)) \cap \mathbb{D}(\alpha\rho(\eta)) = \varnothing$, it follows that $z \notin \mathbb{D}(\alpha\rho(\eta))$ and $\chi(z) = 0$. From (2.9), we see that $\bar\partial u_f = 0$. Hence, u_f is analytic in $\mathbb{D}(\alpha\rho(z))$. By the subharmonicity of $|u_f(z)|^2$, we obtain

$$|u_f(z)|^2V_\mu(z) \lesssim \frac{1}{\rho^2(z)} \int_{\mathbb{D}(\frac{\alpha\rho(z)}{2})} |u_f(\xi)|^2V_\mu(\xi)\mathrm{d}A(\xi). \tag{2.11}$$

Since

$$\frac{|1-\bar\eta z|^2}{1-|z|^2} \approx \frac{|1-\bar\eta\xi|^2}{1-|\xi|^2},\ \xi \in \mathbb{D}(\frac{\alpha\rho(z)}{2}),$$

it follows from (2.11) that for $0 < t < 1$,

$$\begin{aligned} &|u_f(z)|^2V_\mu(z)\left(\frac{|1-\bar\eta z|^2}{1-|z|^2}\right)^t \\ &\lesssim \frac{1}{\rho^2(z)} \int_{\mathbb{D}(\frac{\alpha\rho(z)}{2})} |u_f(\xi)|^2V_\mu(\xi)\left(\frac{|1-\bar\eta\xi|^2}{1-|\xi|^2}\right)^t \mathrm{d}A(\xi). \end{aligned} \tag{2.12}$$

Let $\psi(\xi) = -\log(V_\mu(\xi))$ and $\phi(\xi) = t\log\frac{|1-\bar\eta\xi|^2}{1-|\xi|^2}$. Then

$$|u_f(z)|^2e^{\phi(z)-\psi(z)} \lesssim \frac{1}{\rho^2(z)} \int_{\mathbb{D}(\frac{\alpha\rho(z)}{2})} |u_f(\xi)|^2e^{\phi(\xi)-\psi(\xi)}\mathrm{d}A(\xi). \tag{2.13}$$

By a simple computation, we get

$$\frac{\partial}{\partial\xi}\phi(\xi) = t\left(\frac{-\bar\eta}{1-\bar\eta\xi} + \frac{\bar\xi}{1-|\xi|^2}\right) \textit{ and } \Delta\phi(\xi) = \frac{\partial^2}{\bar\partial\partial\xi}\phi(\xi) = \frac{t}{(1-|\xi|^2)^2}.$$

Clearly, $\Delta(\phi+\psi) \geq \Delta\phi > 0$. By Theorem B, there exists a v such that $\bar\partial v = f\bar\partial\chi$ with

$$\begin{aligned} \int_{\mathbb{D}} |v(\xi)|^2e^{-\phi(\xi)-\psi(\xi)}\mathrm{d}A(\xi) &\leq \int_{\mathbb{D}} \frac{|\bar\partial v(\xi)|^2}{\Delta(\phi+\psi)(\xi)}e^{-\phi(\xi)-\psi(\xi)}\mathrm{d}A(\xi) \\ &\leq \int_{\mathbb{D}} \frac{|\bar\partial v(\xi)|^2}{\Delta\phi(\xi)}e^{-\phi(\xi)-\psi(\xi)}\mathrm{d}A(\xi). \end{aligned}$$

Meanwhile, we see that

$$\frac{|\frac{\partial\phi(\xi)}{\partial\xi}|^2}{\Delta\phi(\xi)} = t\left|\frac{-\bar\eta(1-|\xi|^2)}{1-\bar\eta\xi} + \bar\xi\right|^2 = t\left|\frac{\bar\xi-\bar\eta}{1-\bar\eta\xi}\right|^2 \leq t.$$

Note that u_f is the solution of $\overline{\partial} u_f = f\overline{\partial}\chi$ with minimal norm in $L^2(\mathbb{D}, e^{-\psi}\mathrm{d}A)$. Using Theorem C and (2.9), we obtain

$$\begin{aligned}
&\int_{\mathbb{D}} |u_f(\xi)|^2 e^{\phi(\xi)-\psi(\xi)}\mathrm{d}A(\xi) \\
&\lesssim \frac{1}{(1-t)^2}\int_{\mathbb{D}} \frac{|f\overline{\partial}\chi(\xi)|^2}{\Delta\phi(\xi)} e^{\phi(\xi)-\psi(\xi)}\mathrm{d}A(\xi) \\
&\le \frac{1}{t(1-t)^2}\int_{\mathbb{D}} |f(\xi)|^2|\bar{\partial}\chi(\xi)|^2(1-|\xi|^2)^2\frac{|1-\bar{\eta}\xi|^{2t}}{(1-|\xi|^2)^t}V_\mu(\xi)\mathrm{d}A(\xi) \\
&\le \frac{1}{t(1-t)^2}\int_{\mathbb{D}(\alpha\rho(\eta))} |f(\xi)|^2\frac{\chi(\xi)}{\rho^2(\xi)}(1-|\xi|^2)^2\frac{|1-\bar{\eta}\xi|^{2t}}{(1-|\xi|^2)^t}V_\mu(\xi)\mathrm{d}A(\xi) \\
&\lesssim \frac{(1-|\eta|^2)^t}{t(1-t)^2}\int_{\mathbb{D}(\alpha\rho(\eta))} |f(\xi)|^2\chi(\xi)V_\mu(\xi)\mathrm{d}A(\xi) \\
&\lesssim \frac{(1-|\eta|^2)^t}{t(1-t)^2}.
\end{aligned} \tag{2.14}$$

Combining (2.14) and (2.13), we obtain

$$|u_f(z)|^2 \lesssim \frac{(1-|z|^2)^t}{\rho^2(z)V_\mu(z)}\frac{(1-|\eta|^2)^t}{t(1-t)^2|1-\bar{\eta}z|^{2t}}, \tag{2.15}$$

which, together with (2.8) and (2.10), gives

$$\begin{aligned}
|K_z(\eta)|^2V_\mu(\eta) &\lesssim \frac{1}{\rho^2(\eta)}\sup_f |u_f(z)|^2 \\
&\lesssim \frac{1}{\rho^2(\eta)\rho^2(z)V_\mu(z)}\frac{(1-|\eta|^2)^t}{t(1-t)^2}\frac{(1-|z|^2)^t}{|1-\bar{\eta}z|^{2t}}.
\end{aligned}$$

By Theorem 1, there exists a positive constant $C = C(t)$ such that

$$\begin{aligned}
|K_z(\eta)|^2 &\le C\frac{1}{\rho^2(\eta)V_\mu(\eta)\rho^2(z)V_\mu(z)}\frac{(1-|\eta|^2)^t}{t(1-t)^2}\frac{(1-|z|^2)^t}{|1-\bar{\eta}z|^{2t}} \\
&\le C\|K_z\|^2_{A^2_\mu}\|K_\eta\|^2_{A^2_\mu}\left(\frac{(1-|z|^2)(1-|\eta|^2)}{|z-\eta|^2}\right)^t.
\end{aligned}$$

In the last inequality, we use the estimate $|1-\bar{\eta}z| \approx |z-\eta|$ for $\mathbb{D}(\alpha\rho(\eta)) \cap \mathbb{D}(\alpha\rho(z)) = \varnothing$. The proof of Theorem 2 is complete. □

3. Schatten classes of Toeplitz operators on A^2_μ

Let ν be a positive Borel measure on $\mathbb{D}$. The space $L^2_\omega(\nu)$ is defined as the set of all measurable functions f on $\mathbb{D}$ for which the following norm is finite:

$$\|f\|^2_{L^2_\omega(\nu)} = \int_{\mathbb{D}} |f(w)|^2\omega(w)\mathrm{d}\nu(w) < \infty.$$

The Toeplitz operator T_ν acting on A^2_ω is defined by

$$T_\nu f(z) = \int_{\mathbb{D}} K(z,\eta) f(\eta)\omega(\eta)\mathrm{d}\nu(\eta),\ z \in \mathbb{D}.$$

For $f, g \in A^2_\omega$, we have

$$\begin{aligned}\langle Jf, Jg\rangle_{L^2_\omega(\nu)} &= \int_{\mathbb{D}} f(z)\overline{g(z)}\omega(z)\mathrm{d}\nu(z)\\ &= \int_{\mathbb{D}} f(z)\int_{\mathbb{D}} K_z(\zeta)\overline{g(\zeta)}\omega(\zeta)\mathrm{d}A(\zeta)\omega(z)\mathrm{d}\nu(z)\\ &= \int_{\mathbb{D}}\int_{\mathbb{D}} K(\zeta,z)f(z)\omega(z)\mathrm{d}\nu(z)\overline{g(\zeta)}\omega(\zeta)\mathrm{d}A(\zeta)\\ &= \int_{\mathbb{D}} T_\nu f(\zeta)\overline{g(\zeta)}\omega(\zeta)\mathrm{d}A(\zeta)\\ &= \langle T_\nu f(\zeta), g(\zeta)\rangle_{A^2_\omega},\end{aligned}$$

where J is the embedding operator from A^2_ω to $L^2_\omega(\nu)$. Therefore, the operator T_ν can be expressed as the product of the adjoint operator J^* and J, which is denoted as $T_\nu = J^*J$.

Let H be a separable Hilbert space. If T is a compact operator on H, then there exist orthonormal sets $\{e_k\}$ and $\{\sigma_k\}$ in H, as well as a sequence $\{s_k\}$ that decreases to 0, such that

$$Tx = \sum_k s_k \langle x, e_k\rangle_H \sigma_k, \qquad x \in H.$$

For $0 < p < \infty$, we say that an operator T belongs to the Schatten class $S_p(H)$ if

$$\|T\|_{S_p(H)} = \Big(\sum_k s_k^p\Big)^{\frac{1}{p}} < \infty.$$

The Schatten class $S_p(H)$ is a Banach space for $p \geq 1$. As $0 < p < 1$, from [17, Theorem 2.8], we have

$$\|S+T\|^p_{S_p(H)} \leq \|S\|^p_{S_p(H)} + \|T\|^p_{S_p(H)}.$$

The Schatten class S_1 is commonly referred to as the trace class, while S_2 is known as the Hilbert-Schmidt class. Additionally, a compact operator T is said to belong to $S_p(H)$ if and only if its absolute value $|T| = (T^*T)^{\frac{1}{2}}$ also belongs to $S_p(H)$.

Definition. Suppose T is a compact operator on a complex Hilbert space H, and let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous increasing function with $h(0) = 0$. We define the set $S_h(H)$ to be the set of all compact operators T for which there is a constant $c > 0$ such that

$$\sum_{n=1}^{\infty} h(cs_n(T)) < \infty.$$

Here, $S_h(H)$ is called an extend Schatten classes and $S_h(H) = S_p(H)$ for $h(x) = x^p, p \geq 1$; see [11].

Suppose μ is a positive Borel measure on $\mathbb{D}$ and let K denote the reproducing kernel of A^2_μ. An observation shows that

$$\lim_{z\to\mathbb{T}} \|K_z\|_{A^2_\mu} = \infty, \tag{3.1}$$

and for each $\xi \in \mathbb{D}$

$$|K_z(\xi)| = o(\|K_z\|_{A^2_\mu}),\ z \to \mathbb{T}. \tag{3.2}$$

It is easy to know that for $z, \xi \in \mathbb{D}$, there are constants $C_1, C > 0$ such that

$$\|K_z\|_{A^2_\mu}\|K_\xi\|_{A^2_\mu} \leq C|K_z(\xi)| \tag{3.3}$$

for $|z - w| \leq C_1\rho_{V_\mu}(z)$, where $\rho_{V_\mu}(z) = \frac{1}{\sqrt{V_\mu(z)K(z,z)}}, z \in \mathbb{D}$.

The Bergman metric, also known as the hyperbolic metric, is defined on $\mathbb{D}$ for $z, \zeta \in \mathbb{D}$ as follows:

$$\beta(z,\zeta) = \frac{1}{2}\log\frac{1 + d(z,\zeta)}{1 - d(z,\zeta)},$$

where $d(z,\zeta) = |\frac{z-\zeta}{1-\bar{z}\zeta}|$ is the pseudo-hyperbolic metric on $\mathbb{D}$.

A sequence $\{z_n\}_{n\geq 1}$ in $\mathbb{D}$ is called separated if there exists a $\delta > 0$ such that $\beta(z_n, z_m) \geq \delta, n \neq m$.

Theorem D ([11, Corollary 6.3])**.** *If $h : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing convex function and ν is a positive Borel measure on $\mathbb{D}$, then $T_\nu \in S_h(A^2_\mu)$ if and only if there exists positive constant c such that*

$$\sum_n h\Big(c\frac{\nu(\mathbb{D}(\alpha\rho(z_n)))}{|\mathbb{D}(\alpha\rho(z_n))|}\Big) < \infty,$$

where $|\mathbb{D}(\alpha\rho(z_n))|$ denotes the Lebesgue area measure of $\mathbb{D}(\alpha\rho(z_n))$.

Theorem 3. *Let ν be a positive Borel measure on the open unit disk $\mathbb{D}$ and $p > 0$. Then $T_\nu \in S_p(A^2_\mu)$ if and only if*

$$\sum_n \left(\frac{\nu(\mathbb{D}(\alpha\rho(z_n)))}{|\mathbb{D}(\alpha\rho(z_n))|}\right)^p < \infty. \tag{3.4}$$

To demonstrate the necessity of Theorem 3, we will employ the subsequent lemma and its proof, which are derived from the conventional arguments in [14, Lemma 3.7].

Lemma 3.1. *Let $\{e_n\}$ be an orthonormal basis of A^2_μ and $\{b_n\}$ be a separated sequence in $\mathbb{D}$. Then operator L defined by*

$$L\big(\sum_n c_n e_n(z)\big) = \sum_n c_n \frac{\rho^N(b_n)}{(1 - \overline{b_n}z)^N} k_{b_n}(z),\ c_n \in \mathbb{C},\ N > 0,$$

where $k_{b_n} = \frac{K_{b_n}}{\|K_{b_n}\|_{A^2_\mu}}$ *is the normalized reproducing kernels of* A^2_μ*, is bounded on* A^2_μ*.*

Proof. Note that

$$\sup_n \sup_{z\in\mathbb{D}} \frac{\rho^N(b_n)}{|1-\overline{b_n}z|^N} \approx \sup_n \sup_{z\in\mathbb{D}} \frac{(1-|b_n|^2)^N}{|1-\overline{b_n}z|^N} \le 2^N < \infty. \tag{3.5}$$

We need only to show that

$$\|\sum_n c_n k_{b_n}\|_{A^2_\mu} \le \Big(\sum_n |c_n|^2\Big)^{\frac{1}{2}}.$$

Let $f \in A^2_\mu$. Using reproducing formula and Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}\Big|\langle \sum_n c_n k_{b_n}, f\rangle_{A^2_\mu}\Big| &= \Big|\sum_n c_n \|K_{b_n}\|^{-1}_{A^2_\mu}\langle K_{b_n}, f\rangle_{A^2_\mu}\Big| \\ &= \Big|\sum_n c_n \|K_{b_n}\|^{-1}_{A^2_\mu}\overline{f(b_n)}\Big| \\ &\le \Big(\sum_n |c_n|^2\Big)^{\frac{1}{2}}\Big(\sum_n \|K_{b_n}\|^{-2}_{A^2_\mu}|f(b_n)|^2\Big)^{\frac{1}{2}}.\end{aligned}$$

Note that $\{b_n\} \subset \mathbb{D}$ is separate. Applying the subharmonicity of $|f|^2$ and Theorem 1, we get

$$\begin{aligned}\sum_n \|K_{b_n}\|^{-2}_{A^2_\mu}|f(b_n)|^2 &\lesssim \sum_n \frac{1}{\|K_{b_n}\|^2_{A^2_\mu}\rho^2(b_n)}\int_{\mathbb{D}(\rho(b_n))}|f(z)|^2 \mathrm{d}A(z) \\ &\lesssim \sum_n \int_{\mathbb{D}(\rho(b_n))}|f(z)|^2 V_\mu(z)\mathrm{d}A(z) \\ &\lesssim \|f\|^2_{A^2_\mu},\end{aligned}$$

since $V_\mu(b_n) \approx V_\mu(z)$ for $z \in \mathbb{D}(\rho(b_n))$. Thus,

$$\Big|\langle \sum_n c_n k_{b_n}, f\rangle_{A^2_\mu}\Big| \lesssim \Big(\sum_n |c_n|^2\Big)^{\frac{1}{2}}\|f\|_{A^2_\mu},$$

which completes the proof. □

Lemma 3.2 ([22, Lemma 4.9])**.** *Let* $R > 0$*. Suppose* $\{z_n\}$ *is a regular r-lattice in the hyperbolic metric. Then there exists a finite decomposition of* $\{z_n\}$ *into sequences* $\{z_{k1}, z_{k2}, \dots\}$*,* $1 \le k \le N$*, such that for all* k *and all* $i \ne j$*, we have* $\beta(z_{ki}, z_{kj}) \ge R$*.*

Proof of Theorem 3. By Theorem D, it is enough to focus on the scenario where $0 < p < 1$. We adopt the method in [14] to deal with the weighted Bergman spaces. By interpolation, we can establish the outcome for $\frac{1}{2} < p < 1$. Thus, our task is to demonstrate the result for $0 < p \le \frac{1}{2}$.

Note that $T_\nu = J^*J$, where J is embedding operator from A^2_μ to $L^2_\mu(\nu)$, which gives that $\|T_\nu\|^p_{S_p} = \|J\|^{2p}_{S_{2p}}$. Let $\{\mathbb{D}(\alpha\rho(a_j))\}$ be a covering of $\mathbb{D}$ satisfying all the conditions in [11, Proposition 3.1], where α is given in Lemma 2.2. For a partition $\{\sigma_j\}$ of unity subordinate to $\{\mathbb{D}(\alpha\rho(a_j))\}$ and $f \in A^2_\mu$, we have

$$f = \sum_j \sigma_j f.$$

Define an operator from A^2_μ to $L^2_\nu(\mathbb{D}(\alpha\rho(a_j)))$ by

$$J_j f = \sigma_j f$$

and

$$I_j g = g$$

from $L^2_\nu(\mathbb{D}(\alpha\rho(a_j)))$ to $L^2_\nu(\mathbb{D})$. Thus

$$J = \sum_j I_j J_j.$$

Since $0 < 2p \leq 1$, we obtain

$$\|J\|^{2p}_{S_{2p}} \leq \sum_j \|I_j J_j\|^{2p}_{S_{2p}} \leq \sum_j \|J_j\|^{2p}_{S_{2p}}.$$

We first estimate $\|J_j\|^{2p}_{S_{2p}}$. Set

$$A^2_\mu(\mathbb{D}, a_j) = \{f \in A^2_\mu(\mathbb{D}), f(a_j) = 0,\ a_j \in \mathbb{D}\}.$$

Denote by $\mathcal{L}_{a_j}$ the one-dimensional subspace which are spanned by function $k_{a_j}(z) = \frac{K(z,a_j)}{\sqrt{K(a_j,a_j)}}$. Then we have the decomposition

$$A^2_\mu(\mathbb{D}) = A^2_\mu(\mathbb{D}, a_j) \bigoplus \mathcal{L}_{a_j}.$$

Let

$$J^{(1)}_j = J_j|_{A^2_\mu(\mathbb{D},a_j)} : A^2_\mu(\mathbb{D}, a_j) \to L^2_\nu(\mathbb{D}(\alpha\rho(a_j))),$$
$$J^{(2)}_j = J_j|_{\mathcal{L}_{a_j}} : \mathcal{L}_{a_j} \to L^2_\nu(\mathbb{D}(\alpha\rho(a_j))).$$

It is obvious that $J_j = J^{(1)}_j + J^{(2)}_j$. For $0 < 2p < 1$, we have

$$\|J_j\|^{2p}_{S_{2p}} \leq \|J^{(1)}_j\|^{2p}_{S_{2p}} + \|J^{(2)}_j\|^{2p}_{S_{2p}}. \tag{3.6}$$

The fact that $J^{(2)}_j$ is a rank one operator implies

$$\begin{aligned}
\|J^{(2)}_j\|_{S_{2p}} &= \|J^{(2)}_j\|_{S_2} \\
&= \left(\sum_m |\langle J^{(2)}_j e_m(z), e_m(z)\rangle_{L^2_\nu(\mathbb{D}(\alpha\rho(a_j)))}|^2 \right)^{\frac{1}{2}} \\
&\leq \left(\int_{\mathbb{D}(\alpha\rho(a_j))} \sum_m |e_m(z)|^2 V_\mu(z) \mathrm{d}\nu(z) \right)^{\frac{1}{2}}
\end{aligned}$$

$$= \left(\int_{\mathbb{D}(\alpha\rho(a_j))} K(z,z) V_\mu(z) \mathrm{d}\nu(z) \right)^{\frac{1}{2}}. \tag{3.7}$$

Next, we estimate $\|J_j^{(1)}\|_{S_{2p}}^{2p}$. Set

$$S_j : A_\mu^2(\mathbb{D}, a_j) \to A_\mu^2(\mathbb{D}),\ (S_j f)(z) = \frac{f(z)}{z - a_j},$$

and

$$T_j : L_\nu^2(\mathbb{D}(\alpha\rho(a_j))) \to L_\nu^2(\mathbb{D}(\alpha\rho(a_j))),\ (T_j f)(z) = f(z)(z - a_j).$$

It is easy to show that the operator $J_j^{(1)}$ has a decomposition $J_j^{(1)} = T_j J_j S_j$ and

$$\|J_j^{(1)}\|_{S_{2p}} \leq \|T_j\| \|S_j\| \|J_j\|_{S_{2p}}. \tag{3.8}$$

Now we estimate the norms of S_j and T_j.

Since S_j maps $A_\mu^2(\mathbb{D}, a_j)$ into $A_\mu^2(\mathbb{D})$, we have

$$\begin{aligned} \|S_j f\|_{A_\mu^2}^2 = & \int_{\mathbb{D}(\alpha\rho(a_j))} |S_j f(z)|^2 V_\mu(z) \mathrm{d}A(z) \\ & + \int_{\mathbb{D} \setminus \mathbb{D}(\alpha\rho(a_j))} |S_j f(z)|^2 V_\mu(z) \mathrm{d}A(z). \end{aligned} \tag{3.9}$$

Using the reproducing formula,

$$S_j f(z) = \langle S_j f(w), K_z(w) \rangle_{A_\mu^2} = \int_{\mathbb{D}} S_j f(w) \overline{K_z(w)} V_\mu(w) \mathrm{d}A(w).$$

This implies that

$$\begin{aligned} & \int_{\mathbb{D}(\alpha\rho(a_j))} |S_j f(z)|^2 V_\mu(z) \mathrm{d}A(z) \\ \leq & \int_{\mathbb{D}(\alpha\rho(a_j))} \|K_z\|_{A_\mu^2}^2 \|S_j f\|_{A_\mu^2}^2 V_\mu(z) \mathrm{d}A(z) \\ = & \|S_j f\|_{A_\mu^2}^2 \int_{\mathbb{D}(\alpha\rho(a_j))} \|K_z\|_{A_\mu^2}^2 V_\mu(z) \mathrm{d}A(z). \end{aligned} \tag{3.10}$$

For $z \in \mathbb{D}(\alpha\rho(a_j))$, we know $\rho(z) \approx \rho(a_j)$. By Theorem 1, we deduce

$$\begin{aligned} \int_{\mathbb{D}(\alpha\rho(a_j))} \|K_z\|_{A_\mu^2}^2 V_\mu(z) \mathrm{d}A(z) & \leq C \int_{\mathbb{D}(\alpha\rho(a_j))} \frac{1}{\rho^2(z)} \mathrm{d}A(z) \\ & \leq \frac{C}{\rho^2(a_j)} \int_{\mathbb{D}(\alpha\rho(a_j))} \mathrm{d}A(z) \\ & \leq C\alpha^2, \end{aligned} \tag{3.11}$$

where C is independent of a_j. Choose $\alpha \in (0, \frac{1}{8})$ small enough such that $C\alpha^2 < 1$ in (3.11). Combining (3.9), (3.10) and (3.11) we conclude that

$$\begin{aligned}\|S_j f\|^2_{A^2_\mu} &\le C \int_{\mathbb{D}\setminus\mathbb{D}(\alpha\rho(a_j))} |S_j f(z)|^2 V_\mu(z) \mathrm{d}A(z) \\ &= C \int_{\mathbb{D}\setminus\mathbb{D}(\alpha\rho(a_j))} \left|\frac{f(z)}{z-a_j}\right|^2 V_\mu(z)\mathrm{d}A(z) \\ &\le \frac{C}{\rho^2(a_j)} \|f\|^2_{A^2_\mu}. \end{aligned} \tag{3.12}$$

Therefore, there exists a constant C (independent of a_j),

$$\|S_j\| \le \frac{C}{\rho(a_j)}. \tag{3.13}$$

It is obvious that

$$\|T_j\| \le \alpha\rho(a_j). \tag{3.14}$$

Thus, there exist a constant $0 < \varrho < 1$ such that

$$\|T_j\|\|S_j\| \le \varrho, \tag{3.15}$$

where $\varrho = C\alpha$. By (3.6), (3.7) and (3.8) we get

$$\|J_j\|^{2p}_{S_{2p}} \le \frac{1}{1-\varrho^{2p}} \left(\int_{\mathbb{D}(\alpha\rho(a_j))} \|K_z\|^2_{A^2_\mu} V_\mu(z) \mathrm{d}\nu(z) \right)^p. \tag{3.16}$$

Using Theorem 1, we conclude

$$\|J_j\|^{2p}_{S_{2p}} \le C \left(\frac{\nu(\mathbb{D}(\alpha\rho(a_j)))}{|\mathbb{D}(\alpha\rho(a_j))|} \right)^p.$$

Therefore,

$$\|T_\nu\|^p_{S_p} = \|J\|^{2p}_{S_{2p}} = \sum_j \|J_j\|^{2p}_{S_{2p}} \le C \sum_j \left(\frac{\nu(\mathbb{D}(\alpha\rho(a_j)))}{|\mathbb{D}(\alpha\rho(a_j))|} \right)^p < \infty.$$

The proof of the necessity of Theorem 3 follows closely those of Theorem 1 in [12]. Suppose $T_\nu \in S_p(A^2_\mu)$. Lemma 3.2 tells us that it only needs to find an $R > 0$ such that

$$\sum_n \Big(\frac{\nu(\mathbb{D}(\alpha\rho(b_n)))}{|\mathbb{D}(\alpha\rho(b_n))|} \Big)^p < \infty$$

holds for $\{b_n\}$ satisfying $\beta(b_n, b_m) > R$ ($n \ne m$), where α is appeared in Lemma 2.2.

Let

$$\lambda = \sum_n \nu\chi_n,$$

where χ_n denotes the characteristic function of $\mathbb{D}(\alpha\rho(b_n))$. Obviously,

$$0 \le T^p_\lambda \le T^p_\nu, \ \|T_\lambda\|_{S_p} \le \|T_\nu\|_{S_p}.$$

Let $\{e_n\}$ be an orthonormal basis of A^2_μ and L be the operator on A^2_ω defined by

$$Le_n(z) = f_n(z) = \frac{\rho^N(b_n)}{(1-\overline{b_n}z)^N} k_{b_n}(z),\ z \in \mathbb{D}.$$

By Lemma 3.1, L is bounded on A^2_ω. Since $T_\lambda \in S_p(A^2_\mu)$, the operator $T = L^* T_\lambda L \in S_p(A^2_\mu)$ with $\|T\|_{S_p} \leq \|L\|^2 \|T_\sigma\|_{S_p(A^2_\mu)}$. We decompose the operator T into the sum of D and F, i.e., $T = D + F$, where

$$Df = \sum_n \langle Te_n, e_n\rangle_{A^2_\omega} \langle f, e_n\rangle_{A^2_\omega} e_n,\ Ff = \sum_{n,k;n\neq k} \langle Te_k, e_n\rangle_{A^2_\omega} \langle f, e_k\rangle_{A^2_\omega} e_n,$$

on A^2_μ. Using triangle inequality, we get

$$\|T\|^p_{S_p(A^2_\mu)} \geq \|D\|^p_{S_p(A^2_\mu)} - \|F\|^p_{S_p(A^2_\mu)}. \tag{3.17}$$

Since D is the diagonal operator and $\lambda = \nu$ on $\mathbb{D}(\alpha\rho(b_n))$, applying Theorem 1 and Lemma 2.2, we get

$$\begin{aligned}
\|D\|^p_{S_p(A^2_\mu)} = \sum_n \langle Te_n, e_n\rangle^p_{A^2_\mu} &= \sum_n \langle L^*T_\lambda Le_n, e_n\rangle^p_{A^2_\mu} = \sum_n \langle T_\lambda f_n, f_n\rangle^p_{A^2_\mu} \\
&= \sum_n \Big(\int_{\mathbb{D}} |f_n(z)|^2 V_\mu(z) d\lambda(z)\Big)^p \\
&\geq \sum_n \Big(\int_{\mathbb{D}(\alpha\rho(b_n))} \frac{\rho^{2N}(b_n)|k_{b_n}(z)|^2 V_\mu(z)}{|1-\overline{b_n}z|^{2N}} d\lambda(z)\Big)^p \\
&\approx \sum_n \Big(\int_{\mathbb{D}(\alpha\rho(b_n))} \frac{|K_{b_n}(z)|^2 V_\mu(z)}{\|K_{b_n}\|^2_{A^2_\mu}} d\nu(z)\Big)^p \\
&\approx \sum_n \Big(\int_{\mathbb{D}(\alpha\rho(b_n))} \|K_z\|^2_{A^2_\mu} V_\mu(z) \mathrm{d}\nu(z)\Big)^p \\
&\approx \sum_n \Big(\int_{\mathbb{D}(\alpha\rho(b_n))} \frac{V_\mu(z)}{(1-|z|^2)^2 V_\mu(z)} \mathrm{d}\nu(z)\Big)^p \\
&\approx \sum_n \Big(\int_{\mathbb{D}(\alpha\rho(b_n))} \frac{1}{(1-|b_n|^2)^2} \mathrm{d}\nu(z)\Big)^p.
\end{aligned}$$

Thus, there is a $C > 0$ such that

$$\|D\|^p_{S_p(A^2_\mu)} \geq C \sum_n \Big(\frac{\nu(\mathbb{D}(\alpha\rho(b_n)))}{|\mathbb{D}(\alpha\rho(b_n))|}\Big)^p. \tag{3.18}$$

Now, we estimate the upper bound of $\|F\|^p_{S_p(A^2_\mu)}$. From [22, Proposition 1.29], we know that

$$\begin{aligned}
&\|F\|^p_{S_p(A^2_\mu)} \\
&\leq \sum_k \sum_n |\langle Fe_n, e_k\rangle_{A^2_\mu}|^p = \sum_{n,k;n\neq k} |\langle T_\lambda f_n, f_k\rangle_{A^2_\mu}|^p
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n,k;n\neq k}\left|\int_{\mathbb{D}} f_n(z)\overline{f_k(z)}V_\mu(z)d\lambda(z)\right|^p\\
&\leq \sum_{n,k;n\neq k}\sum_j\left(\int_{\mathbb{D}(\alpha\rho(b_n))}|k_{b_n}(z)||k_{b_k}(z)|\frac{\rho^N(b_n)\rho^N(b_k)}{|1-\overline{b_j}z|^N|1-\overline{b_k}z|^N}V_\mu(z)\mathrm{d}\nu(z)\right)^p\\
&= Q_{n,k}(\nu).
\end{aligned}
$$

Note that $\beta(b_n,b_k)\geq R$ for $n\neq k$. Since $\max\{\beta(z,b_n),\beta(z,b_k)\}\geq\frac{R}{2}$ for $z\in\mathbb{D}(\alpha\rho(b_n))$, we are able to suppose $\beta(z,b_n)\geq\frac{R}{2}$. By [12, p.482], the function $\sinh(z)=\frac{e^z-e^{-z}}{2}$ satisfies

$$
\sinh^2(\beta(\zeta,\eta))=\frac{|\zeta-\eta|^2}{(1-|\zeta|^2)(1-|\eta|^2)},\ \zeta,\eta\in\mathbb{D}. \tag{3.19}
$$

Combing (3.19) and (2.6), for $0<\gamma<\frac{1}{2}$, there is a constant $C_\gamma>0$ such that

$$
\begin{aligned}
|k_{b_n}(z)| &= \frac{|K_{b_n}(z)|^{\frac{1}{2}}}{\|K_{b_n}\|^{\frac{1}{2}}_{A^2_\mu}}|k_{b_n}(z)|^{\frac{1}{2}}\\
&\leq \frac{C_\gamma\|K_{b_n}\|^{\frac{1}{2}}_{A^2_\mu}\|K_z\|^{\frac{1}{2}}_{A^2_\mu}}{\|K_{b_n}\|^{\frac{1}{2}}\sinh^{\frac{\gamma}{2}}(\beta(b_n,z))}|k_{b_n}(z)|^{\frac{1}{2}}\\
&\leq \frac{C_\gamma\|K_z\|^{\frac{1}{2}}}{\sinh^{\frac{\gamma}{2}}(\frac{R}{2})}|k_{b_n}(z)|^{\frac{1}{2}}=r_\gamma\|K_z\|^{\frac{1}{2}}_{A^2_\mu}|k_{b_n}(z)|^{\frac{1}{2}}\\
&\leq r_\gamma\|K_z\|_{A^2_\mu}\Big(\frac{\rho(b_n)\rho(z)}{|1-\overline{b_n}z|^2}\Big)^{\frac{\gamma}{2}},
\end{aligned}
$$

where $r_\gamma=\frac{C_\gamma}{\sinh^{\frac{\gamma}{2}}(\frac{R}{2})}$. By Theorem 2,

$$
\begin{aligned}
|k_{b_k}(z)| &= \frac{|K_{b_n}(z)|^{\frac{1}{2}}}{\|K_{b_n}\|^{\frac{1}{2}}_{A^2_\mu}}|k_{b_n}(z)|^{\frac{1}{2}}=\frac{|\langle K_{b_n},K_z\rangle|^{\frac{1}{2}}}{\|K_{b_n}\|^{\frac{1}{2}}_{A^2_\mu}}|k_{b_n}(z)|^{\frac{1}{2}}\\
&\leq |k_{b_k}(z)|^{\frac{1}{2}}\|K_z\|^{\frac{1}{2}}_{A^2_\mu}\\
&\lesssim \|K_z\|_{A^2_\mu}\Big(\frac{\rho(b_k)\rho(z)}{|1-\overline{b_k}z|^2}\Big)^{\frac{\gamma}{2}}.
\end{aligned}
$$

For $z\in\mathbb{D}(\alpha\rho(b_j))$, it follows that

$$
\begin{aligned}
|k_{b_n}(z)||k_{b_k}(z)| &\lesssim r_\gamma\Big(\frac{\rho(b_k)\rho(b_n)\rho^2(z)}{|1-\overline{b_n}z|^2|1-\overline{b_k}z|^2}\Big)^{\frac{\gamma}{2}}\|K_z\|^2_{A^2_\mu}\\
&\lesssim r_\gamma\Big(\frac{\rho(b_k)\rho(b_n)\rho^2(b_j)}{|1-\overline{b_n}b_j|^2|1-\overline{b_k}b_j|^2}\Big)^{\frac{\gamma}{2}}\|K_{b_j}\|^2_{A^2_\mu}.
\end{aligned}
$$

Therefore,

$$Q_{n,k}(\nu) \lesssim \sum_{n,k;n\neq k} \Big(r_\gamma \sum_j \Big(\frac{\rho(b_k)\rho(b_n)\rho^2(b_j)}{|1-\overline{b_n}b_j|^2|1-\overline{b_k}b_j|^2} \Big)^{\frac{\gamma}{2}} \times \frac{\rho^N(b_n)\rho^N(b_k)}{|1-\overline{b_n}b_j|^N|1-\overline{b_k}b_j|^N} \frac{\nu(\mathbb{D}(\alpha\rho(b_j)))}{|\mathbb{D}(\alpha\rho(b_j))|} \Big)^p.$$

For all N with $p(N+\frac{\gamma}{2}) \geq 1$, we have

$$\begin{aligned}
\|F\|^p_{S_p(A^2_\mu)} &\leq Q_{n,k}(\nu) \\
&\lesssim \sum_{n,k;n\neq k} \Big(r_\gamma \sum_j \Big(\frac{\rho(b_k)\rho(b_n)\rho^2(b_j)}{|1-\overline{b_n}b_j|^2|1-\overline{b_k}b_j|^2} \Big)^{\frac{\gamma}{2}} \\
&\quad \times \frac{\rho^N(b_n)\rho^N(b_k)}{|1-\overline{b_n}b_j|^N|1-\overline{b_k}b_j|^N} \frac{\nu(\mathbb{D}(\alpha\rho(b_j)))}{|\mathbb{D}(\alpha\rho(b_j))|} \Big)^p \\
&\lesssim r_\gamma^p \sum_j (\rho(b_j))^{p\gamma} \Big(\frac{\nu(\mathbb{D}(\alpha\rho(b_j)))}{|\mathbb{D}(\alpha\rho(b_j))|} \Big)^p \\
&\quad \times \sum_{n,k;n\neq k} \frac{\rho^{p(N+\frac{\gamma}{2})}(b_n)\rho^{p(N+\frac{\gamma}{2})}(b_k)}{|1-\overline{b_n}b_j|^{p(N+\gamma)}|1-\overline{b_k}b_j|^{p(N+\gamma)}} \\
&\lesssim r_\gamma^p \sum_j (\rho(b_j))^{p\gamma} \Big(\frac{\nu(\mathbb{D}(\alpha\rho(b_j)))}{|\mathbb{D}(\alpha\rho(b_j))|} \Big)^p \Big(\sum_n \frac{\rho^{p(N+\frac{\gamma}{2})}(b_n)}{|1-\overline{b_n}b_j|^{p(N+\gamma)}} \Big)^2 \\
&\lesssim r_\gamma^p \sum_j \Big(\frac{\nu(\mathbb{D}(\alpha\rho(b_j)))}{|\mathbb{D}(\alpha\rho(b_j))|} \Big)^p.
\end{aligned}$$

The last inequality can be found in [15, Lemma 4]. Since $r_\gamma \lesssim \frac{1}{R}$, we get

$$\|F\|^p_{S_p(A^2_\mu)} \lesssim \frac{1}{R^p} \sum_j \Big(\frac{\nu(\mathbb{D}(\alpha\rho(b_j)))}{|\mathbb{D}(\alpha\rho(b_j))|} \Big)^p. \tag{3.20}$$

Setting R large enough and combining (3.17), (3.18) and (3.20), we have

$$\begin{aligned}
\infty > \|T\|^p_{S_p(A^2_\mu)} &\geq \|D\|^p_{S_p(A^2_\mu)} - \|F\|^p_{S_p(A^2_\mu)} \\
&\geq C \sum_j \Big(\frac{\nu(\mathbb{D}(\alpha\rho(b_j)))}{|\mathbb{D}(\alpha\rho(b_j))|} \Big)^p - \frac{1}{R^p} \sum_j \Big(\frac{\nu(\mathbb{D}(\alpha\rho(b_j)))}{|\mathbb{D}(\alpha\rho(b_j))|} \Big)^p \\
&= (C - \frac{1}{R^p}) \sum_j \Big(\frac{\nu(\mathbb{D}(\alpha\rho(b_j)))}{|\mathbb{D}(\alpha\rho(b_j))|} \Big)^p.
\end{aligned}$$

Thus

$$\sum_j \Big(\frac{\nu(\mathbb{D}(\alpha\rho(b_j)))}{|\mathbb{D}(\alpha\rho(b_j))|} \Big)^p \leq \|T\|^p_{S_p(A^2_\mu)} < \infty.$$

The proof of Theorem 3 is completed. $\square$

4. Schatten class composition operators on $\mathcal{D}_\mu$

The Nevanlinna counting function associated with a positive Borel measure μ on the unit disk $\mathbb{D}$ and an analytic self-map φ of $\mathbb{D}$ is defined as follows:

$$N_{\varphi,\mu}(z) = \sum_{z=\varphi(w)} V_\mu(w),\ z \in \mathbb{D}.$$

The measure μ_φ on $\mathbb{D}$ is defined as follows:

$$\mu_\varphi(B) = \int_B \frac{N_{\varphi,\mu}(z)}{V_\mu(z)} \mathrm{d}A(z), \tag{4.1}$$

where B is a Borel set in $\mathbb{D}$. In [16] the authors investigated composition operators on standard Bergman spaces A^2_α which belong to the Schatten class. Constantin [9] described the membership of composition operators belonging to the Schatten class on weighted Bergman spaces with Bekolle weights. Pau and Pérez [19] studied the Schatten class of composition operators on standard weighted Dirichlet type spaces. The main result presented in this section is the following theorem. The proof provided here is based on previous work by [19] and [12].

Theorem 4. *Assume that μ is a positive Borel measure on $\mathbb{D}$ and $p > 0$. Suppose φ is an analytic self-map of $\mathbb{D}$. Then $C_\varphi \in S_p(\mathcal{D}_\mu)$ if and only if*

$$\sum_n \Big(\frac{\mu_\varphi(\mathbb{D}(\alpha\rho(z_n)))}{|\mathbb{D}(\alpha\rho(z_n))|}\Big)^{\frac{p}{2}} < \infty.$$

To prove Theorem 4, we require the following lemma.

Lemma 4.1. *Suppose μ is a positive Borel measure on $\mathbb{D}$, and φ is an analytic self-map of $\mathbb{D}$. Assume that $h : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function and $h(0) = 0$. Then $C_\varphi \in S_h(\mathcal{D}_\mu)$ if and only if $T_{\mu_\varphi} \in S_{h\circ\sqrt{\cdot}}(A^2_\mu)$.*

Proof. Define a linear mapping $I_\mu : \mathcal{D}_\mu \to A^2_\mu$ by $I_\mu f = f'$. It is easy to see that I_μ is bounded. By [12, Lemma 2.1], we know that $C_\varphi \in S_h(\mathcal{D}_\mu)$ if and only if $W_\varphi := I_\mu C_\varphi I^*_\mu \in S_h(A^2_\mu)$ since $S_h(\mathcal{D}_\mu)$ is a two-sided idea. For $h \in \mathcal{D}_\mu$ and $g \in A^2_\mu$, we find that

$$\langle I_\mu h(w), g(w)\rangle_{A^2_\mu} = \int_{\mathbb{D}} h'(w)\overline{g(w)}V_\mu(w)\mathrm{d}A(w) = \langle h(w), I^*_\mu g(w)\rangle_{\mathcal{D}_\mu}$$

and

$$I^*_\mu g(w) = \int_0^w g(\xi)\mathrm{d}\xi.$$

For $f \in A^2_\mu$, we get that

$$W_\varphi f(w) = I_\mu C_\varphi I^*_\mu f(w) = \varphi'(w)f(\varphi(w))$$

and

$$W^*_\varphi W_\varphi f(z) = \langle W_\varphi f, W_\varphi K_z\rangle = \int_{\mathbb{D}} f(\varphi(\eta))\overline{K_z(\varphi(\eta))}|\varphi'(\eta)|^2 V_\mu(\eta)\mathrm{d}A(\eta).$$

Using (4.1), by a change of variables, we have

$$W_\varphi^* W_\varphi f(z) = \int_{\mathbb{D}} \frac{f(\zeta)\overline{K_z(\zeta)} V_\mu(\zeta) N_{\varphi,\mu}(\zeta)}{V_\mu(\zeta)} \mathrm{d}A(\zeta) = T_{\mu_\varphi} f(z),$$

which means that the singular values of T_{μ_φ} are the squares of those of W_φ. Thus, C_φ belongs to $S_h(\mathcal{D}_\mu)$ if and only if $T_{\mu_\varphi} \in S_{h\circ\sqrt{\cdot}}(A^2_\mu)$. $\square$

Proof of Theorem 4. Let $h(x) = x^p$. By Lemma 4.1, $C_\varphi \in S_p(\mathcal{D}_\mu)$ if and only if $T_{\mu_\varphi} \in S_{\frac{p}{2}}(A^2_\mu)$. Theorem 3 shows that $T_{\mu_\varphi} \in S_{\frac{p}{2}}(A^2_\mu)$ if and only if

$$\sum_n \Big(\frac{\mu_\varphi(\mathbb{D}(\alpha\rho(z_n)))}{|\mathbb{D}(\alpha\rho(z_n))|} \Big)^{\frac{p}{2}} < \infty.$$

Thus, we have established the theorem. $\square$

For the Hilbert-Schmidt class, we have

Theorem 5. *Assume that φ be an analytic self-map of $\mathbb{D}$ and μ be a positive Borel measure on $\mathbb{D}$. Then $C_\varphi \in S_2(\mathcal{D}_\mu)$ if and only if*

$$\int_{\mathbb{D}} \frac{V_\mu(\zeta)}{(1-|\varphi(\zeta)|^2)^2 V_\mu(\varphi(\zeta))} |\varphi'(\zeta)|^2 \mathrm{d}A(\zeta) < \infty.$$

Proof. Assume that $\{e_n\}$ forms an orthonormal basis for A^2_μ. By Lemma 4.1, we know that C_φ belongs to $S_2(\mathcal{D}_\mu)$ if and only if T_{μ_φ} is in $S_1(A^2_\mu)$. By Theorem 1,

$$\begin{aligned}
\sum_{n=0}^{\infty} \langle T_{\mu_\varphi} e_n, e_n \rangle_{A^2_\mu} &= \sum_{n=0}^{\infty} \int_{\mathbb{D}} |e_n(z)|^2 V_\mu(z) \mathrm{d}\mu_\varphi(z) \\
&= \int_{\mathbb{D}} \big(\sum_{n=0}^{\infty} |e_n(z)|^2 \big) V_\mu(z) \mathrm{d}\mu_\varphi(z) \\
&= \int_{\mathbb{D}} \|K_z\|^2_{A^2_\mu} V_\mu(z) \mathrm{d}\mu_\varphi(z) \approx \int_{\mathbb{D}} \frac{\mathrm{d}\mu_\varphi(z)}{(1-|z|^2)^2}.
\end{aligned}$$

Note that $T_{\mu_\varphi} \in S_1(A^2_\mu)$ if and only if $\sum_{n=0}^{\infty} \langle T_{\mu_\varphi} e_n, e_n \rangle_{A^2_\mu} < \infty$. Hence, $C_\varphi \in S_2(\mathcal{D}_\mu)$ if and only if $\int_{\mathbb{D}} \frac{\mathrm{d}\mu_\varphi(z)}{(1-|z|^2)^2} < \infty$. A change of variables shows

$$\begin{aligned}
\int_{\mathbb{D}} \frac{\mathrm{d}\mu_\varphi(z)}{(1-|z|^2)^2} &= \int_{\mathbb{D}} \frac{N_{\varphi,\mu}(z)}{(1-|z|^2)^2 V_\mu(z)} \mathrm{d}A(z) \\
&= \int_{\mathbb{D}} \frac{\sum_{z=\varphi(\zeta)} V_\mu(\zeta)}{(1-|z|^2)^2 V_\mu(z)} \mathrm{d}A(z) \\
&= \int_{\mathbb{D}} \frac{V_\mu(\zeta)}{(1-|\varphi(\zeta)|^2)^2 V_\mu(\varphi(\zeta))} |\varphi'(\zeta)|^2 \mathrm{d}A(\zeta) \\
&< \infty.
\end{aligned}$$

The proof has been completed. $\square$

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