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A VARIANT OF D'ALEMBERT'S AND WILSON'S FUNCTIONAL EQUATIONS FOR MATRIX VALUED FUNCTIONS

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Abstract. Given M a monoid with a neutral element e. We show that the solutions of d'Alembert's functional equation for $n \times n$ matrices

$$\Phi(pr, qs) + \Phi(sp, rq) = 2\Phi(r, s)\Phi(p, q), \quad p, q, r, s \in M$$

are abelian. Furthermore, we prove under additional assumption that the solutions of the n-dimensional mixed vector-matrix Wilson's functional equation

$$\begin{cases} f(pr, qs) + f(sp, rq) = 2\Phi(r, s)f(p, q), \\ \Phi(p, q) = \Phi(q, p), \quad p, q, r, s \in M \end{cases}$$

are abelian. As an application we solve the first functional equation on groups for the particular case of $n = 3$.

1. Introduction

During their investigations of distance measures, Chung, Kannappan, Ng, and Sahoo [6, Lemma 2.2] found the solutions $f :]0, 1[\times]0, 1[\longrightarrow \mathbb{R}$ of the functional equation

$$f(pr, qs) + f(sp, rq) = f(p, q)f(r, s), \quad p, q, r, s \in]0, 1[. \tag{1.1}$$

In [16] Stetkær obtained the general solution $f : S \longrightarrow \mathbb{C}$ of the variant of d'Alembert's functional equation

$$f(xy) + f(\sigma(y)x) = 2f(x)f(y), \quad x, y \in S \tag{1.2}$$

on a possibly non-commutative semigroup S, where $\sigma : S \longrightarrow S$ is an involutive automorphism. That is $\sigma(xy) = \sigma(x)\sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in S$. The solutions of (1.2) are the functions $f = \frac{\chi+\chi\circ\sigma}{2}$, where $\chi : S \longrightarrow \mathbb{C}$ is a multiplicative function.

If S is a semigroup, then the switch map $\sigma(x, y) := (y, x)$ is an involutive automorphism of the product semigroup $S \times S$. By help of σ and the

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component-wise multiplication $(p,q)(r,s) = (pr, qs)$ we reformulate (1.1) as

$$f((p,q)(r,s)) + f(\sigma(r,s)(p,q)) = f(p,q)f(r,s), \quad (p,q),(r,s) \in S \times S.$$

Then (1.1) is a special instance of (1.2), if we work with $g = f/2$ instead of f.

In this paper we study the n-dimensional version of the variant of d'Alembert's functional equation

$$\Phi(pr, qs) + \Phi(sp, rq) = 2\Phi(r,s)\Phi(p,q) \quad p,q,r,s \in M, \tag{1.3}$$

and the vector-matrix variant of Wilson's functional equation

$$\begin{cases} f(pr,qs) + f(sp,rq) = 2\Phi(r,s)f(p,q), \\ \Phi(p,q) = \Phi(q,p), \quad p,q,r,s \in M, \end{cases} \tag{1.4}$$

where M is a monoid, $f : M \times M \longrightarrow \mathbb{C}^n$, $\Phi : M \times M \to \mathcal{M}_n(\mathbb{C})$ are the unknown functions.

Our first purpose is to prove that the solutions Φ of the functional equation (1.3) are abelian as well as showing that the solutions (f, Φ) of the functional equation (1.4) are abelian since the components of f are linearly independent. Moreover, we find that f remains an abelian function even if we avoid the last condition. Secondly, as an application we solve the functional equation (1.3) on groups for the particular case $n = 3$.

The matrix or even operator version of d'Alembert's functional equation

$$\Phi(xy) + \Phi(\sigma(y)x) = 2\Phi(y)\Phi(x), \quad x, y \in M, \tag{1.5}$$

on abelian groups $M = G$ with $\sigma = -id$ and $\Phi(e) = I$ has been treated by Fattorini [8], Kurepa [11], Baker and Davidson [1], Kisyński [9,10], Székelyhidi [17], Chojnacki [4,5], Sinopoulos [12,13] and Stetkær [15], Bouikhalene, Elqorachi and Manar [2] for general involutions σ. In non-abelian groups and non abelian monoids generated by their squares, the solutions of (1.5) taking their values in $\mathcal{M}_2(\mathbb{C})$ were recently obtained by Chahbi and Elqorachi [3]. The solutions described in [3] are not necessarily abelian.

Wilson's functional equation has been studied in the mixed vector-matrix form

$$f(xy) + f(\sigma(y)x) = 2\Phi(y)f(x), \quad x, y \in G, \tag{1.6}$$

by P. Sinopoulos [12,13], with $\sigma(x) = x^{-1}, x \in G$, by Stetkær [15] as well as Bouikhalene, Elqorachi and Manar [2] with a general involutive automorphism σ on abelian groups.

The solutions of (1.6) taking their values in $\mathbb{C}^2$ are obtained in [3] under the condition that Φ is a solution of d'Alembert's matrix functional equation (1.5).

2. Notation, terminology and some preliminary results

In this section we present a general set-up and auxiliary results which will be used in the next sections.

Notation and terminology

Throughout this paper S denotes an arbitrary semigroup, while M and G are respectively a monoid and a group with neutral element e.

$\sigma : S \longrightarrow S$ will be any involutive automorphism. For the sake of convenience, we will denote $G \times G$ by $\overline{G}$, $M \times M$ by $\overline{M}$ and (e, e) by $\mathbf{e}$. Then $\overline{G}$ (or $\overline{M}$) is a group (or a monoid) with a neutral element $\mathbf{e}$ under component-wise multiplication. That is, $(p, q)(r, s) = (pr, qs)$. We denote by $\mathscr{M}(G)$ the set of all homomorphisms $\mu : G \longrightarrow \mathbb{C}$ on G valued in $(\mathbb{C}, \cdot)$: $\mu(xy) = \mu(x)\mu(y)$ for all $x, y \in G$, and $\mathscr{M}^+(G) := \{\mu \in \mathscr{M}(G) : \mu \circ \sigma = \mu\}$. Let $\mathscr{A}(G)$ be the set of all additive maps $a : G \longrightarrow \mathbb{C}$ of G into $(\mathbb{C}, +)$: $a(xy) = a(x) + a(y)$ for all $x, y \in G$, and $\mathcal{A}^{\pm}(G) := \{a \in \mathscr{A}(G) : a \circ \sigma = \pm a\}$. $\mathscr{S}(G)$ denotes the set of maps $Q : G \to \mathbb{C}$ defined by $Q(x) = q(x, x), x \in G$, with $q : G \times G \to \mathbb{C}$ being a symmetric bi-additive map and $\mathscr{S}^-(G)$ is the subset of $\mathscr{S}(G)$ for which q satisfies $q(\sigma(x), y) = -q(x, y)$ for any $x, y \in G$. For a function f, the new functions $f^e := \frac{f + f\circ\sigma}{2}$ and $f^o := \frac{f - f\circ\sigma}{2}$ denote respectively the even and the odd part of f.

$\mathcal{F}_n$ denotes the set of all $\mathbb{C}^n$-valued functions on M with linearly independent components. We should note for $f : M \longrightarrow \mathbb{C}^n$ that $f \in \mathcal{F}_n \Longleftrightarrow$ *span* $\{f(x)|x \in M\} = \mathbb{C}^n$. We define that a function f on S is abelian if f is central: $f(xy) = f(yx)$ for all $x, y \in S$, and f satisfies the Kannappan condition: $f(xyz) = f(xzy)$ for all $x, y, z \in S$. Finally, $\mathcal{M}_n(\mathbb{C})$ is the set of all $n \times n$ matrices over $\mathbb{C}$, $GL(n, \mathbb{C})$ is the group of $n \times n$ invertible matrices, I_n is the unit matrix of $\mathcal{M}_n(\mathbb{C})$ and the transpose of a matrix A is denoted by A^T.

The next lemma was obtained in [3].

Lemma 2.1. *Let σ be an involutive automorphism of M. If $\Phi : M \longrightarrow \mathcal{M}_n(\mathbb{C})$ is a solution of the functional equation*

$$\begin{cases} \Phi(xy) + \Phi(\sigma(y)x) = 2\Phi(x)\Phi(y), \quad x, y \in M, \\ \Phi(e) = I_n. \end{cases} \tag{2.1}$$

Then

(i) $\Phi \circ \sigma = \Phi$
(ii) $\Phi(x)\Phi(y) = \Phi(y)\Phi(x)$ *for all* $x, y \in M$.

Remark 1. The Lemma 2.1 remains true for the following variant of d'Alembert's matrix functional equation:

$$\begin{cases} \Phi(xy) + \Phi(\sigma(y)x) = 2\Phi(y)\Phi(x) \quad x, y \in M \\ \Phi(e) = I_n. \end{cases} \tag{2.2}$$

Lemma 2.2. *Let $\Phi : M \longrightarrow \mathcal{M}_n(\mathbb{C})$ be a central solution of* (2.1) *or of* (2.2), *then Φ is abelian.*

Proof. Replacing x by xy and y by z in (2.1) we get

$$\Phi(xyz) = 2\Phi(xy)\Phi(z) - \Phi(\sigma(z)xy).$$

Applying (2.1) to the term $\Phi(\sigma(z)xy)$ gives

$$\Phi(yz\sigma(x)) + \Phi(\sigma(z)xy) = 2\Phi(y)\Phi(\sigma(z)x).$$

So we get

$$\Phi(xyz) = 2\Phi(xy)\Phi(z) + \Phi(yz\sigma(x)) - 2\Phi(y)\Phi(z\sigma(x)).$$

Doing the same for the terms $\Phi(yz\sigma(x))$ and $\Phi(z\sigma(x))$ leads to

$$\begin{aligned}\Phi(xyz) =& 2\Phi(xy)\Phi(z) + 2\Phi(yz)\Phi(\sigma(x)) - \Phi(xyz)\\ &- 4\Phi(y)\Phi(z)\Phi(\sigma(x)) + 2\Phi(y)\Phi(xz).\end{aligned}$$

Taking into account ((i), Lemma 2.1) that $\Phi \circ \sigma = \Phi$ we obtain the identity

$$\Phi(xyz) = \Phi(x)\Phi(yz) + \Phi(y)\Phi(xz) + \Phi(z)\Phi(xy) - 2\Phi(y)\Phi(z)\Phi(x),$$

for all $x, y, z \in M$. Since $\Phi(x), \Phi(y)$ and $\Phi(z)$ commute with each other and Φ is central, we deduce that $\Phi(xyz) = \Phi(xzy)$ for all $x, y, z \in M$, which implies that Φ is abelian. □

Proposition 2.1. *Let the pair $f : M \longrightarrow \mathbb{C}^n, \Phi : M \longrightarrow \mathcal{M}_n(\mathbb{C})$ be a solution of the matrix variant of Wilson's functional equation*

$$f(xy) + f(\sigma(y)x) = 2\Phi(y)f(x) \quad x, y \in M \tag{2.3}$$

such that

$$\begin{cases} \Phi(x)\Phi(y)f(e) = \Phi(y)\Phi(x)f(e), \\ \Phi(xy)f(e) = \Phi(yx)f(e) \quad \text{for all } x, y \in M. \end{cases} \tag{2.4}$$

Then

(1) *For all $y \in M$*

$$\Phi(y)(span\{f(x) \in \mathbb{C}^n | x \in M\}) \subseteq span\{f(x) \in \mathbb{C}^n | x \in M\}. \tag{2.5}$$

(2) *f is central.*

(3) *The restriction Ψ of Φ to $U := span\{f(x) \in \mathbb{C}^n | x \in M\}$ is a solution of the matrix variant of d'Alembert's functional equation*

$$\Psi(xy) + \Psi(x\sigma(y)) = 2\Psi(y)\Psi(x), \quad x, y \in M \tag{2.6}$$

satisfying $\Psi(e) = I_n|_U$.

(4) *If $f \in \mathcal{F}_n$ then Φ is a solution of the functional equation*

$$\begin{cases} \Phi(xy) + \Phi(x\sigma(y)) = 2\Phi(y)\Phi(x), \quad x, y \in M, \\ \Phi(e) = I_n. \end{cases} \tag{2.7}$$

Proof. It follows directly from (2.3) that $\Phi(y)$ leaves the space $span\{f(x) \in \mathbb{C}^n | x \in M\}$ invariant.

To prove the second statement we will need the following:

Lemma 2.3. *Let the pair* $f : M \longrightarrow \mathbb{C}^n, \Phi : M \longrightarrow \mathcal{M}_n(\mathbb{C})$ *be a solution of the functional equation* (2.3). *Then the identity*

$$f(xyz) = \Phi(z)f(xy) + \Phi(y)f(xz) + \Phi(yz)f(x) - 2\Phi(y)\Phi(z)f(x), \tag{2.8}$$

holds for all $x, y, z \in M$.

Proof. By replacing x by xy and y by z, equation (2.3) becomes

$$f((xy)z) + f(\sigma(z)xy) = 2\Phi(z)f(xy) \quad x, y, z \in M.$$

If we replace y by yz in (2.3) we get

$$f(x(yz)) + f(\sigma(y)\sigma(z)x) = 2\Phi(yz)f(x), \quad x, y, z \in M.$$

By replacing x by $\sigma(z)x$ in (2.3) we obtain

$$\begin{aligned} f(\sigma(z)xy) + f(\sigma(y)\sigma(z)x) &= 2\Phi(y)f(\sigma(z)x) \\ &= 2\Phi(y)[2\Phi(z)f(x) - f(xz)], \quad x, y, z \in M. \end{aligned}$$

Subtracting the last identity from the sum of the two firsts gives the desired identity. □

Rest of proof of Proposition 2.1. By replacing x by e in (2.8) we find that

$$f(yz) = \Phi(z)f(y) + \Phi(y)f(z) + \Phi(yz)f(e) - 2\Phi(y)\Phi(z)f(e), \quad x, y, z \in M.$$

Since (2.4) holds, the centrality of f is immediate. Adding the two identities that we obtain from (2.3) by replacing y by yz and $y\sigma(z)$ respectively we find that

$$\begin{aligned} &f(xyz) + f(\sigma(y)\sigma(z)x) + f(xy\sigma(z)) + f(\sigma(y)zx) \\ &= 2[\Phi(yz) + \Phi(y\sigma(z))]f(x). \end{aligned} \tag{2.9}$$

Taking into account that f is central we can rewrite (2.9) as follows

$$\begin{aligned} &f(xyz) + f(\sigma(z)xy) + f(x\sigma(y)z) + f(\sigma(z)x\sigma(y)) \\ &= 2[\Phi(yz) + \Phi(y\sigma(z))]f(x). \end{aligned} \tag{2.10}$$

Using (2.3) again, (2.10) becomes

$$2\Phi(z)[f(xy) + f(x\sigma(y))] = 2[\Phi(yz) + \Phi(y\sigma(z))]f(x),$$

which implies that

$$[\Phi(yz) + \Phi(y\sigma(z))]f(x) = 2\Phi(z)\Phi(y)f(x) \text{ for all } x, y, z \in M.$$

This shows that Ψ is a solution of the functional equation (2.6). Putting $y = e$ in the original functional equation (2.3) we see that $\Psi(e) = I_n$ on $span\{f(x) \in \mathbb{C}^n | x \in M\}$. This proves (3), and consequently (4) holds since $f \in \mathcal{F}_n$. □

3. A variant of d'Alembert's functional equation for matrices

At first, it is interesting to recall that the solutions $\Phi : G \longrightarrow \mathcal{M}_2(\mathbb{C})$ of (1.5) with $\Phi(e) = I_2$ for a general involutive automorphism are not necessarily abelian (see [3] p. 13 for more details). By contrast, the main result of the present section is the fact that any solution of equation (1.3) (which is an instance of (1.5)) is abelian. This allows us to give in this case an exhaustive list of solutions of the functional equation (1.3) for the particular case $n = 3$.

Proposition 3.1. *Let* $\Phi : \overline{M} \longrightarrow \mathcal{M}_n(\mathbb{C})$ *be a solution of* (1.3) *satisfying* $\Phi(e, e) = I_n$. *Then* Φ *is an abelian function.*

Proof. Let $\Phi : \overline{M} \longrightarrow \mathcal{M}_n(\mathbb{C})$ be a solution of (1.3). Letting $p = q = e$ in (1.3) shows that Φ is symmetric: That is $\Phi(s, r) = \Phi(r, s)$ for all $r, s \in M$.
Now, setting $q = s = e$ in (1.3) and taking into account Remark 1 we get

$$\Phi(pr, e) + \Phi(p, r) = 2\Phi(p, e)\Phi(r, e) \text{ for all } p, r \in M. \tag{3.1}$$

Defining a function $g : M \longrightarrow \mathcal{M}_n(\mathbb{C})$ by $g := \Phi(\cdot, e)$, the equation (3.1) can be written as the following

$$\Phi(p, r) = 2g(p)g(r) - g(pr) \text{ for all } p, r \in M. \tag{3.2}$$

Since Φ is symmetric, we have

$$\Phi(p, r) = 2g(r)g(p) - g(rp) \text{ for all } p, r \in M. \tag{3.3}$$

Subtracting (3.3) from (3.2) and using Remark 1 yield

$$g(pr) = g(rp) \text{ for all } p, r \in M.$$

Hence g is central. Now, switching p and q in (1.3) and using the fact that Φ is symmetric we get

$$\Phi(qr, ps) + \Phi(sq, rp) = 2\Phi(p, q)\Phi(r, s) \text{ for all } p, q, r, s \in M.$$

Then

$$\Phi(pr, qs) + \Phi(sp, rq) = \Phi(qr, ps) + \Phi(sq, rp) \text{ for all } p, q, r, s \in M.$$

Using (3.2) we get

$$\begin{aligned} &2g(pr)g(qs) - g(prqs) + 2g(sp)g(rq) - g(sprq) \\ &= 2g(qr)g(ps) - g(qrps) + 2g(sq)g(rp) - g(sqrp) \end{aligned}$$

for all $p, r, q, s \in M$. Since g is central and satisfies $g(a)g(b) = g(b)g(a)$ for all $a, b \in M$, it simplifies to

$$g(prqs) = g(qrps) = g(rpsq) \text{ for all } p, q, r, s \in M.$$

Using (3.2) to compute $\Phi(pr, qs)$ and $\Phi(rp, sq)$ we get

$$\Phi(pr, qs) = 2g(pr)g(qs) - g(prqs) \text{ for all } p, q, r, s \in M,$$

and

$$\Phi(rp, sq) = 2g(rp)g(sq) - g(rpsq) \text{ for all } p, q, r, s \in M.$$

Consequently, it follows

$$\Phi(pr, qs) = \Phi(rp, sq) \text{ for all } p, q, r, s \in M,$$

or equivalently

$$\Phi((p, q)(r, s)) = \Phi((r, s)(p, q)) \text{ for all } (p, q), (r, s) \in \overline{M}.$$

This shows that Φ is central. Finally, with the condition $\Phi(e, e) = I_n$ equation (1.3) is an instance of (2.2), so we can use Lemma 2.2 to obtain the desired result. □

Let $\Phi : \overline{M} \longrightarrow \mathcal{M}_n(\mathbb{C})$ be a solution of (1.3). Putting $x = y = \mathbf{e}$ in (1.3) shows that $\Phi(\mathbf{e})\Phi(\mathbf{e}) = \Phi(\mathbf{e})$, from which we see that $\Phi(\mathbf{e})$ is a projection. So there are $n+1$ possibilities: $\Phi(\mathbf{e}) = I_n$, $\Phi(\mathbf{e})$ is a k-dimensional projection for $k \in \{1; 2; \ldots; n-1\}$, or $\Phi(\mathbf{e}) = 0$. However, the last possibility is uninteresting because it implies that $\Phi = 0$. The case $\Phi(\mathbf{e}) = I_n$ was covered in Theorem 3.2 above, while the other cases are treated in Proposition 3.2 below.

Proposition 3.2. *Let $\Phi : \overline{M} \longrightarrow \mathcal{M}_n(\mathbb{C})$ be a solution of* (1.3) *such that $\Phi(e, e)$ is an k-dimensional projection for $k \in \{1; 2; \ldots; n-1\}$. Then Φ is an abelian function.*

Proof. Recalling that (1.3) is an instance of (1.5), then (1.3) can be reformulated as follows:

$$\Phi(xy) + \Phi(\sigma(y)x) = 2\Phi(y)\Phi(x) \quad x, y \in \overline{M}. \tag{3.4}$$

Up to a similarity the k-projection $\Phi(\mathbf{e})$ has the form

$$\Phi(\mathbf{e}) = (\theta_{ij})_{i;j\in\{1,2,\ldots,n\}} \text{ such that } \theta_{ij} = \begin{cases} \delta_i^j \text{ if } i; j \in \{1, 2, \ldots, k\}, \\ 0 \text{ otherwise,} \end{cases} \tag{3.5}$$

for $k \in \{1; 2; \ldots; n-1\}$, where δ_i^j is the delta Kronecker. Discarding for simplicity of writing the similarity matrix we assume that $\Phi(\mathbf{e})$ is one of these $n-1$ matrices. We use the notation

$$\Phi = (\phi_{ij})_{i;j\in\{1,2,\ldots,n\}}. \tag{3.6}$$

If $\Phi(\mathbf{e})$ has the form (3.5) then $\phi_{ij}(\mathbf{e}) = \delta_i^j$ for $i, j \in \{1, 2, \ldots, k\}$ and by putting $y = \mathbf{e}$ in (3.4) we get that $\phi_{ij} = 0$ for $i \in \{k+1, k+2, \ldots, n\}$, $j \in \{1, 2, \ldots, n\}$. Then identity (3.4) means that the block matrix $\Phi_k := (\phi_{ij})_{i;j\in\{1,2,\ldots,k\}}$ is a solution of k-dimensional variant of d'Alembert's functional equations:

$$\begin{cases} \Phi_k(xy) + \Phi_k(\sigma(y)x) = 2\Phi_k(y)\Phi_k(x) \quad x, y \in \overline{M}, \\ \Phi_k(\mathbf{e}) = I_k. \end{cases} \tag{3.7}$$

And for $l \in \{k+1, k+2, \ldots, n\}$ the vectors $\varphi_l := [\phi_{1l}, \phi_{2l}, \ldots, \phi_{kl}]^T$ are solutions of the $n-k$ k-dimensional Wilson functional equations

$$\varphi_l(xy) + \varphi_l(\sigma(y)x) = 2\Phi_k(y)\varphi_l(x) \quad x, y \in \overline{M}. \tag{3.8}$$

According to Proposition 3.1, Φ_k is abelian. Then by using the identity (2.8) of Lemma 2.3, the functional equations (3.8) shows that the $n-k$ vectors φ_l are also abelian. Consequently Φ is abelian. This completes the proof. □

Theorem 3.1. *Let* $\Phi : \overline{M} \longrightarrow \mathcal{M}_n(\mathbb{C})$ *be a solution of* (1.3)*. Then* Φ *is an abelian function.*

Proof. The theorem is an immediate consequence of Proposition 3.1 and Proposition 3.2. □

Theorem 3.2. *Let* $\Phi : \overline{G} \longrightarrow \mathcal{M}_3(\mathbb{C})$ *be a solution of the matrix functional equation* (1.3) *satisfying* $\Phi(e,e) = I_3$*. Then there exists* $C \in GL(3,\mathbb{C})$ *such that* Φ *has one of the following 9 forms*:

(i)

$$\Phi = C \begin{pmatrix} (\mu+\mu\circ\sigma)/2 & 0 & 0 \\ 0 & (\gamma+\gamma\circ\sigma)/2 & 0 \\ 0 & 0 & (\eta+\eta\circ\sigma)/2 \end{pmatrix} C^{-1}, \tag{3.9}$$

where $\left(\frac{\mu+\mu\circ\sigma}{2}\right)(p,q) = \frac{\mu_1(p)\mu_2(q)+\mu_1(q)\mu_2(p)}{2}$, $p,q \in G$, $\left(\frac{\gamma+\gamma\circ\sigma}{2}\right)(p,q) = \frac{\gamma_1(p)\gamma_2(q)+\gamma_1(q)\gamma_2(p)}{2}$, $p,q \in G$ *and* $\left(\frac{\eta+\eta\circ\sigma}{2}\right)(p,q) = \frac{\eta_1(p)\eta_2(q)+\eta_1(q)\eta_2(p)}{2}$, $p,q \in G$ *such that* $\mu,\gamma,\eta \in \mathscr{M}(\overline{G}) \setminus \{0\}$ *and* $\mu_1,\mu_2,\gamma_1,\gamma_2,\eta_1,\eta_2 \in \mathscr{M}(G) \setminus \{0\}$.

(ii)

$$\Phi = C \begin{pmatrix} \mu^+ & \mu^+(a^+ + Q^-) & 0 \\ 0 & \mu^+ & 0 \\ 0 & 0 & (\eta+\eta\circ\sigma)/2 \end{pmatrix} C^{-1}, \tag{3.10}$$

where $\mu^+(p,q) = \mu_0(pq)$, $p,q \in G$, $a^+(p,q) = a_0(pq)$, $p,q \in G$, $Q^-(p,q) = \psi_0(pq^{-1})$, $p,q \in G$ *and* $\left(\frac{\eta+\eta\circ\sigma}{2}\right)(p,q) = \frac{\eta_1(p)\eta_2(q)+\eta_1(q)\eta_2(p)}{2}$, $p,q \in G$ *such that* $\mu^+ \in \mathscr{M}^+(\overline{G}) \setminus \{0\}$, $\eta \in \mathscr{M}(\overline{G}) \setminus \{0\}$, $\mu_0,\eta_1,\eta_2 \in \mathscr{M}(G) \setminus \{0\}$, $a^+ \in \mathscr{A}^+(\overline{G})$, $a_0 \in \mathscr{A}(G)$, $Q^- \in \mathscr{S}^-(\overline{G})$ *and* $\psi_0 \in \mathscr{S}(G)$*. Furthermore* $a^+ + Q^- \neq 0$.

(iii)

$$\Phi = C \begin{pmatrix} \frac{\mu+\mu\circ\sigma}{2} & \frac{\mu+\mu\circ\sigma}{2}a^+ + \frac{\mu-\mu\circ\sigma}{2}a^- & 0 \\ 0 & \frac{\mu+\mu\circ\sigma}{2} & 0 \\ 0 & 0 & \frac{\eta+\eta\circ\sigma}{2} \end{pmatrix} C^{-1}, \tag{3.11}$$

where $\left(\frac{\mu\pm\mu\circ\sigma}{2}\right)(p,q) = \frac{\mu_1(p)\mu_2(q)\pm\mu_1(q)\mu_2(p)}{2}$, $p,q \in G$, $\left(\frac{\eta+\eta\circ\sigma)}{2}\right)(p,q) = \frac{\eta_1(p)\eta_2(q)+\eta_1(q)\eta_2(p)}{2}$, $p,q \in G$, $a^+(p,q) = a_0(pq)$, $p,q \in G$, $a^-(p,q) = a_1(pq^{-1})$ $p,q \in G$ *such that* $\mu,\eta \in \mathscr{M}(\overline{G}) \setminus \{0\}$ *with* $\mu \neq \mu\circ\sigma$, $\mu_1,\mu_2,\eta_1,\eta_1 \in \mathscr{M}(G) \setminus \{0\}$ *with* $\mu_1 \neq \mu_2$ *and* $a_0,a_1 \in \mathscr{A}(G)$.

(iv)

$$\Phi = C \begin{pmatrix} \mu^+ & 0 & \mu^+(a_1^+ + Q_1^-) \\ 0 & \mu^+ & \mu^+(a_2^+ + Q_2^-) \\ 0 & 0 & \mu^+ \end{pmatrix} C^{-1}, \tag{3.12}$$

where $\mu^+(p,q)=\mu_0(pq)$, $p,q\in G$, $a_i^+(p,q)=b_i(pq)$, $p,q\in G$ *and* $Q_i^-(p,q)=\psi_i(pq^{-1})$ $p,q\in G$ *such that* $\mu^+\in\mathscr{M}^+(\overline{G})\backslash\{0\}$, $\mu_0\in\mathscr{M}(G)\backslash\{0\}$, $a_i^+\in\mathscr{A}^+(\overline{G})$, $a_0\in\mathscr{A}(G)$, $b_i\in\mathscr{A}(G)$, $Q_i^-\in\mathscr{S}^-(\overline{G})$ *and* $\psi_i\in\mathscr{S}(G)$ *for i=1, 2.*

(v)

$$\Phi=C\begin{pmatrix}\mu^+ & \mu^+(a_2^++Q_2^-) & \mu^+(a_1^++Q_1^-)\\ 0 & \mu^+ & 0\\ 0 & 0 & \mu^+\end{pmatrix}C^{-1}, \tag{3.13}$$

where $\mu^+(p,q)=\mu_0(pq)$, $p,q\in G$, $a_i^+(p,q)=b_i(pq)$, $p,q\in G$ *and* $Q_i^-(p,q)=\psi_i(pq^{-1})$ $p,q\in G$ *such that* $\mu^+\in\mathscr{M}^+(\overline{G})\backslash\{0\}$, $\mu_0\in\mathscr{M}(G)\backslash\{0\}$, $a_i^+\in\mathscr{A}^+(\overline{G})$, $a_0\in\mathscr{A}(G)$, $b_i\in\mathscr{A}(G)$, $Q_i^-\in\mathscr{S}^-(\overline{G})$ *and* $\psi_i\in\mathscr{S}(G)$ *for i=1, 2.*

(vi)

(3.14)

$$\Phi=C\begin{pmatrix}\mu^+ & d^{-1}\mu^+(a^++(a^-)^2) & \mu^+(\frac{(a^+)^2}{2}+a^+(a^-)^2+\frac{(a^-)^4}{6}+a_1^++Q^-)\\ 0 & \mu^+ & d\mu^+(a^++(a^-)^2)\\ 0 & 0 & \mu^+\end{pmatrix}C^{-1},$$

where $\mu^+(p,q)=\mu_0(pq)$, $p,q\in G$, $a^+(p,q)=b(pq)$ $p,q\in G$, $a_1^+(p,q)=b_1(pq)$, $p,q\in G$, $a^-(p,q)=b_0(pq^{-1})$ *and* $Q^-(p,q)=\psi_0(pq^{-1})$, $p,q\in G$ *such that* $\mu^+\in\mathscr{M}^+(\overline{G})\setminus\{0\}$, $\mu_0\in\mathscr{M}(G)\setminus\{0\}$, $a^+,a_1^+\in\mathscr{A}^+(\overline{G})$, $a^-\in\mathscr{A}^-(\overline{G})$, $b,b_0,b_1\in\mathscr{A}(G)$, $Q^-\in\mathscr{S}^-(\overline{G})$, $\psi_0\in\mathscr{S}(G)$ *and* $d\in\mathbb{C}\backslash\{0\}$.

(vii)

(3.15)

$$\Phi=C\begin{pmatrix}\frac{\mu+\mu\circ\sigma}{2} & \frac{\lambda_1}{\lambda}(\frac{\mu+\mu\circ\sigma}{2}a^++\frac{\mu-\mu\circ\sigma}{2}a^-) & *\\ 0 & \frac{\mu+\mu\circ\sigma}{2} & \frac{\lambda_2}{\lambda}(\frac{\mu+\mu\circ\sigma}{2}a^++\frac{\mu-\mu\circ\sigma}{2}a^-)\\ 0 & 0 & \frac{\mu+\mu\circ\sigma}{2}\end{pmatrix}C^{-1},$$

with $*=\frac{\mu+\mu\circ\sigma}{2}a_1^++\frac{\mu-\mu\circ\sigma}{2}a_1^-+\frac{1}{4}(\mu(a^++a^-)^2+\mu\circ\sigma(a^+-a^-)^2)$, *and where* $\big(\frac{\mu\pm\mu\circ\sigma}{2}\big)(p,q)=\frac{\mu_1(p)\mu_2(q)\pm\mu_1(q)\mu_2(p)}{2}$, $p,q\in G$, $a^+(p,q)=a_0(pq)$, $a_1^+(p,q)=b_1(pq)$, $a^-(p,q)=a_2(pq^{-1})$, $a_1^-(p,q)=a_3(pq^{-1})$, $p,q\in G$ *and* $\lambda^2=\lambda_1\lambda_2$ *such that* $\mu\in\mathscr{M}(\overline{G})\setminus\{0\}$ *with* $\mu\neq\mu\circ\sigma$, $\mu_1,\mu_2\in\mathscr{M}(G)\setminus\{0\}$ *verifying* $\mu_1\neq\mu_2$, $a^+\in\mathscr{A}^+(\overline{G})$, $a^-\in\mathscr{A}^-(\overline{G})$, $a_0,b_1,a_2,a_3\in\mathscr{A}(G)$ *and* $\lambda_1,\lambda_2\in\mathbb{C}\backslash\{0\}$.

(viii)

$$\Phi=C\begin{pmatrix}\frac{\mu+\mu\circ\sigma}{2} & 0 & \frac{\mu+\mu\circ\sigma}{2}a_1^++\frac{\mu-\mu\circ\sigma}{2}a_1^-\\ 0 & \frac{\mu+\mu\circ\sigma}{2} & \frac{\mu+\mu\circ\sigma}{2}a_2^++\frac{\mu-\mu\circ\sigma}{2}a_2^-\\ 0 & 0 & \frac{\mu+\mu\circ\sigma}{2}\end{pmatrix}C^{-1}, \tag{3.16}$$

where $\big(\frac{\mu\pm\mu\circ\sigma}{2}\big)(p,q)=\frac{\mu_1(p)\mu_2(q)\pm\mu_1(q)\mu_2(p)}{2}$, $p,q\in G$ *such that* $\mu\in\mathscr{M}(\overline{G})\setminus\{0\}$ *with* $\mu\neq\mu\circ\sigma$, $\mu_1,\mu_2\in\mathscr{M}(G)\setminus\{0\}$ *verifying* $\mu_1\neq\mu_2$ *and where* $a_1^+(p,q)=b_1(pq)$, $a_2^+(p,q)=b_2(pq)$, $a_1^-(p,q)=a_3(pq^{-1})$, $a_2^-(p,q)=a_4(pq^{-1})$, $p,q\in G$ *such that* $a_1^+,a_2^+\in\mathscr{A}^+(\overline{G})$, $a_1^-,a_2^-\in\mathscr{A}^-(\overline{G})$, $b_1,b_2,a_3,a_4\in\mathscr{A}(G)$.

(ix)

(3.17)
$$\Phi = C\begin{pmatrix} \frac{\mu+\mu\circ\sigma}{2} & \frac{\mu+\mu\circ\sigma}{2}a_2^+ + \frac{\mu-\mu\circ\sigma}{2}a_2^- & \frac{\mu+\mu\circ\sigma}{2}a_1^+ + \frac{\mu-\mu\circ\sigma}{2}a_1^- \\ 0 & \frac{\mu+\mu\circ\sigma}{2} & 0 \\ 0 & 0 & \frac{\mu+\mu\circ\sigma}{2} \end{pmatrix} C^{-1},$$

where $\left(\frac{\mu\pm\mu\circ\sigma}{2}\right)(p,q) = \frac{\mu_1(p)\mu_2(q)\pm\mu_1(q)\mu_2(p)}{2}$, $p,q \in G$ *such that* $\mu \in \mathscr{M}(\overline{G})\setminus\{0\}$ *with* $\mu \neq \mu\circ\sigma$, $\mu_1,\mu_2 \in \mathscr{M}(G)\setminus\{0\}$ *verifying* $\mu_1 \neq \mu_2$ *and where* $a_1^+(p,q) = b_1(pq)$, $a_2^+(p,q) = b_2(pq)$, $a_1^-(p,q) = a_3(pq^{-1})$, $a_2^-(p,q) = a_4(pq^{-1})$, $p,q \in G$ *such that* $a_1^+, a_2^+ \in \mathscr{A}^+(\overline{G})$, $a_1^-, a_2^- \in \mathscr{A}^-(\overline{G})$, $b_1, b_2, a_3, a_4 \in \mathscr{A}(G)$. *Conversely, the formulas of* (i),(ii),...,(viii) *and* (ix) *define solutions of* (1.3) *satisfying* $\Phi(e,e) = I_3$.

Proof. It is laborious, but elementary to check that all of the possibilities listed in Theorem 3.2 define solutions of (1.3) satisfying $\Phi(e,e) = I_3$, so it is left to show that each solution has one of the listed forms.

Since the matrices $\Phi(x)$, $x \in \overline{G}$ commute with one other (Lemma 2.1), Lemma 1 of [13] shows that there exists $C \in GL(3,\mathbb{C})$ such that

$$\Phi(x) = C\begin{pmatrix} \phi_1(x) & \lambda_1\phi(x) & \phi_0(x) \\ 0 & \phi_2(x) & \lambda_2\phi(x) \\ 0 & 0 & \phi_3(x) \end{pmatrix} C^{-1},\ x \in \overline{G} \tag{3.18}$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$. Since Φ is a solution of (2.1) with $\Phi(e,e) = I_3$, Proposition 3.1 shows that ϕ, ϕ_1, ϕ_2, ϕ_3 and ϕ_0 are abelian scalar functions on $\overline{G}$. Furthermore, they satisfy the following functional equations

$$\phi_i(xy) + \phi_i(\sigma(y)x) = 2\phi_i(y)\phi_i(x),\ \text{for } i = 1,2,3, \tag{3.19}$$

$$\lambda_1\phi(xy) + \lambda_1\phi(\sigma(y)x) = 2\lambda_1\phi_1(y)\phi(x) + 2\lambda_1\phi(y)\phi_2(x), \tag{3.20}$$

$$\phi_0(xy) + \phi_0(\sigma(y)x) = 2\phi_1(y)\phi_0(x) + 2\lambda_1\lambda_2\phi(y)\phi(x) + 2\phi_0(y)\phi_3(x), \tag{3.21}$$

$$\lambda_2\phi(xy) + \lambda_2\phi(\sigma(y)x) = 2\lambda_2\phi_2(y)\phi(x) + 2\lambda_2\phi(y)\phi_3(x), \tag{3.22}$$

for all $x, y \in \overline{G}$. To show that the solutions are expressed in terms of multiplicative, additive and quadratic scalar functions on G we can refer to [16] and [7]. The rest of the proof can be found in [2]. □

Proposition 3.3. *Let* $\Phi : \overline{G} \longrightarrow \mathcal{M}_3(\mathbb{C})$ *be a solution of the matrix functional equation* (1.3).

(1) If $\Phi(e,e)$ is a 1-dimensional projection then there exist $\mu,\mu_1,\mu_2 \in \mathscr{M}(G)\setminus\{0\}$ with $\mu_1 \neq \mu_2$, $a_1,a_2 \in \mathscr{A}(G)$, $c,c' \in \mathbb{C}$ and $C \in GL(3,\mathbb{C})$ such that

(3.23)
$$\Phi(p,q)=C\begin{bmatrix} \frac{\mu_1(p)\mu_2(q)+\mu_2(p)\mu_1(q)}{2} & c\frac{\mu_1(p)\mu_2(q)-\mu_2(p)\mu_1(q)}{2} & c'\frac{\mu_1(p)\mu_2(q)-\mu_2(p)\mu_1(q)}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}C^{-1},$$

for all $p,q \in G$, or

$$\Phi(p,q) = C\mu(pq)\begin{pmatrix} 1 & a_1(pq^{-1}) & a_2(pq^{-1}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}C^{-1} \quad p,q \in G. \tag{3.24}$$

(2) If $\Phi(e,e)$ is a 2-dimensional projection then there exists $C_1 \in GL(3,\mathbb{C})$ such that

$$\Phi = C_1\begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ 0 & 0 & 0 \end{pmatrix}C_1^{-1}, \tag{3.25}$$

in which the block matrices

$$\begin{pmatrix} \phi_{13} \\ \phi_{23} \end{pmatrix} \text{ and } \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

are given by

$$\begin{pmatrix} \phi_{13} \\ \phi_{23} \end{pmatrix} = C(\mathcal{U}\alpha + \mathcal{U}\circ\sigma\beta) \text{ and } \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} = C\frac{\mathcal{U}+\mathcal{U}\circ\sigma}{2}C^{-1}, \tag{3.26}$$

where $\alpha,\beta \in \mathbb{C}^2$ and $\mathcal{U} : \overline{G} \longrightarrow \mathcal{M}_2(\mathbb{C})$ has one of the following 6 forms:

$$\mathcal{U}_1(p,q) = \begin{pmatrix} \mu_1(p)\mu_2(q) & 0 \\ 0 & \gamma_1(p)\gamma_2(q) \end{pmatrix} \quad p,q \in G,$$

$$\mathcal{U}_2(p,q) = \begin{pmatrix} \mu_1(p)\mu_2(q) & 0 \\ 0 & \gamma(pq)(1+a(pq^{-1})) \end{pmatrix} \quad p,q \in G,$$

$$\mathcal{U}_3(p,q) = \begin{pmatrix} \mu(pq)(1+a_1(pq^{-1})) & 0 \\ 0 & \gamma(pq)(1+a_2(pq^{-1})) \end{pmatrix} \quad p,q \in G,$$

$$\mathcal{U}_4(p,q) = \mu_1(p)\mu_2(q)\begin{pmatrix} 1 & a_1(p)+a_2(q) \\ 0 & 1 \end{pmatrix} \quad p,q \in G,$$

$$\mathcal{U}_5(p,q) = \mu(pq)\begin{pmatrix} 1 & a_1(p)+a_2(q)+\psi(pq^{-1}) \\ 0 & 1 \end{pmatrix} \quad p,q \in G,$$

$$\mathcal{U}_6(p,q) = \mu(pq)\begin{pmatrix} 1+a(pq^{-1}) & * \\ 0 & 1+a(pq^{-1}) \end{pmatrix} \quad p,q \in G,$$

with $* = c(a(pq^{-1}))^3 + 3c(a(pq^{-1}))^2 + a(pq) + a(pq)a(pq^{-1}) + a_1(pq^{-1})$, in which $C \in GL(2,\mathbb{C})$, $\mu,\gamma,\mu_1,\mu_2,\gamma_1,\gamma_2, \in \mathscr{M}(G)\setminus\{0\}$, $a,a_1,a_2 \in \mathscr{A}(G)$, $\psi \in \mathscr{S}(G)$ and $c \in \mathbb{C}$.

Proof. We use similar computations to those used in the proof of Proposition 3.2. The equation (1.3) can be reformulated as follows:

$$\Phi(xy) + \Phi(\sigma(y)x) = 2\Phi(y)\Phi(x) \quad x, y \in \overline{G}. \tag{3.27}$$

Writing

$$\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{pmatrix}. \tag{3.28}$$

Up to a similarity if $\Phi(\mathbf{e})$ is a projection then it can be taken as the orthogonal projection on the first canonical basis vector of $\mathbb{C}^3$, so that

$$\Phi(\mathbf{e}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.29}$$

in the case of 1-dimensional projection and

$$\Phi(\mathbf{e}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.30}$$

in the case of 2-dimensional projection. Taking $y = \mathbf{e}$ in (3.27) we find that $\Phi(x) = \Phi(\mathbf{e})\Phi(x)$ for all $x \in \overline{G}$ then, if $\Phi(\mathbf{e})$ has the form (3.29) we get $\phi_{21} = \phi_{22} = \phi_{23} = \phi_{31} = \phi_{32} = \phi_{33} = 0$ and ϕ_{11}, ϕ_{12}, ϕ_{13} are solutions of the scalar d'Alembert's and Wilson's functional equations respectively:

$$\phi_{11}(xy) + \phi_{11}(\sigma(y)x) = 2\phi_{11}(y)\phi_{11}(x) \quad x, y \in \overline{G}, \tag{3.31}$$

$$\phi_{12}(xy) + \phi_{12}(\sigma(y)x) = 2\phi_{11}(y)\phi_{12}(x) \quad x, y \in \overline{G}, \tag{3.32}$$

$$\phi_{13}(xy) + \phi_{13}(\sigma(y)x) = 2\phi_{11}(y)\phi_{13}(x) \quad x, y \in \overline{G}, \tag{3.33}$$

such that $\phi_{11}(\mathbf{e}) = 1$ and $\phi_{12}(\mathbf{e}) = \phi_{13}(\mathbf{e}) = 0$. Finally the formulas of [14] imply the first statement.

If $\Phi(\mathbf{e})$ has the form (3.30) then $\phi_{31} = \phi_{32} = \phi_{33} = 0$ and

$$\varphi_2 := \begin{pmatrix} \phi_{13} \\ \phi_{23} \end{pmatrix} \text{ and } \Phi_2 := \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

verify the 2-dimensional variants of d'Alembert's and Wilson's functional equations respectively:

$$\begin{cases} \Phi_2(xy) + \Phi_2(\sigma(y)x) = 2\Phi_2(y)\Phi_2(x) \quad x, y \in \overline{G}, \\ \Phi_2(\mathbf{e}) = I_2. \end{cases} \tag{3.34}$$

and

$$\varphi_2(xy) + \varphi_2(\sigma(y)x) = 2\Phi_2(y)\varphi_2(x) \quad x, y \in \overline{G}. \tag{3.35}$$

It is obvious that Φ_2 is abelian (In fact its matrix elements are some of the matrix elements of Φ), then using [14, Theorem 3.3] allows us to conclude that

$$\varphi_2 = C(\mathcal{U}\alpha + \mathcal{U}\circ\sigma\beta) \text{ and } \Phi_2 = C\frac{\mathcal{U}+\mathcal{U}\circ\sigma}{2}C^{-1}, \tag{3.36}$$

such that $\alpha, \beta \in \mathbb{C}^2$ and $\mathcal{U} : \overline{G} \longrightarrow \mathcal{M}_2(\mathbb{C})$ has one of the 6 forms cited in the second statement of the proposition. □

4. Vector-matrix variant of Wilson's functional equation

The present section is dedicated to show that the solutions of the functional equation (1.4) are abelian if the unknown function f is belonging to $\mathcal{F}_n$, and furthermore that Φ is a solution of the n-dimensional version of the variant of d'Alembert's functional equation (1.3). A set of main results are established for that goal, which is essentially Theorem 4.1.

All results of this section (Lemmata 4.1, 4.2 and 4.3 and Theorem 4.1) contain the hypothesis that Φ is symmetric, that is $\Phi = \Phi \circ \sigma$.

Lemma 4.1. *Let the pair $f : \overline{M} \longrightarrow \mathbb{C}^n$, $\Phi : \overline{M} \longrightarrow \mathcal{M}_n(\mathbb{C})$ be a solution of* (1.4)*. Then* (2.4) *holds and f is central.*

Proof. By replacing (p, q) by (e, e) in equation (1.4) we get

$$\Phi(r, s)f(e, e) = \frac{f(r, s) + f(s, r)}{2} \text{ for all } r, s \in M.$$

By using that Φ is symmetric and by a simple computation we get

$$[\Phi(pr, qs) + \Phi(sp, rq)]f(e, e) = 2\Phi(r, s)\Phi(p, q)f(e, e) \quad p, q, r, s \in M. \tag{4.1}$$

By similar computations to those of proofs of Lemma 2.1 and Proposition 3.1, it follows

$$\Phi(p, q)\Phi(r, s)f(e, e) = \Phi(r, s)\Phi(p, q)f(e, e) \text{ for all } (p, q), (r, s) \in \overline{M},$$

and

$$\Phi((p, q)(r, s))f(e, e) = \Phi((r, s)(p, q))f(e, e) \text{ for all } (p, q), (r, s) \in \overline{M}.$$

This can be written as follows

$$\begin{cases} \Phi(x)\Phi(y)f(\mathbf{e}) = \Phi(y)\Phi(x)f(\mathbf{e}), \\ \Phi(xy)f(\mathbf{e}) = \Phi(yx)f(\mathbf{e}) \text{ for all } x, y \in \overline{M}. \end{cases}$$

Since (2.4) holds, Proposition 2.1 shows that f is central. □

Lemma 4.2. *Let the pair $f : \overline{M} \longrightarrow \mathbb{C}^n$, $\Phi : \overline{M} \longrightarrow \mathcal{M}_n(\mathbb{C})$ be a solution of* (1.4) *such that $f \in \mathcal{F}_n$. Then*

$$\Phi(w, e)\Phi(q, e) = \Phi(q, e)\Phi(w, e) \textit{ for all } q, w \in M. \tag{4.2}$$

Proof. First, we can easily show that f^e is also a solution of (1.4) since Φ is symmetric. Then we have a right to use the identity (2.8), so

$$f^e(xyz) = \Phi(z)f^e(xy) + \Phi(y)f^e(xz) + \Phi(yz)f^e(x) - 2\Phi(y)\Phi(z)f^e(x)$$

for all $x, y, z \in \overline{M}$. Using this for $x = (p, u)$; $y = (q, e)$; $z = (e, w)$ yields

$$\begin{aligned} f^e(pq, uw) = &\Phi(e, w)f^e(pq, u) + \Phi(q, e)f^e(p, uw) + \Phi(q, w)f^e(p, u) \\ &- 2\Phi(q, e)\Phi(e, w)f^e(p, u). \end{aligned}$$

Switching p with u and q with w and taking into consideration that Φ and f^e are both symmetric lead to

$$\Phi(w, e)\Phi(q, e)f^e(u, p) = \Phi(q, e)\Phi(w, e)f^e(u, p),$$

that is

$$\Phi(w, e)\Phi(q, e)f^e = \Phi(q, e)\Phi(w, e)f^e \text{ for all } q, w \in M. \tag{4.3}$$

On the other hand f^o is also a solution of (1.4), so by using (2.8), we can write

$$f^o(xyz) = \Phi(z)f^o(xy) + \Phi(y)f^o(xz) + \Phi(yz)f^o(x) - 2\Phi(y)\Phi(z)f^o(x) \tag{4.4}$$

for all $x, y, z \in \overline{M}$. Taking into account that $f^o(\mathbf{e}) = 0$ the last identity with $x = \mathbf{e}$ implies

$$f^o(yz) = \Phi(z)f^o(y) + \Phi(y)f^o(z). \tag{4.5}$$

So, we get

$$f^o(xyz) = \Phi(yz)f^o(x) + \Phi(x)f^o(yz) \text{ for all } x, y, z \in \overline{M}. \tag{4.6}$$

Then (4.4) and (4.6) yield

$$2\Phi(y)\Phi(z)f^o(x) = \Phi(z)f^o(xy) + \Phi(y)f^o(xz) - \Phi(x)f^o(yz).$$

By switching y with z and taking heed of the fact that f^o is central (identity (4.5)) we deduce

$$\Phi(y)\Phi(z)f^o(x) = \Phi(z)\Phi(y)f^o(x) \text{ for all } x, y, z \in \overline{M}.$$

Particularly, for $y = (w, e)$; $z = (q, e)$ we have

$$\Phi(w, e)\Phi(q, e)f^o = \Phi(q, e)\Phi(w, e)f^o \text{ for all } q, w \in M. \tag{4.7}$$

Since $f = f^o + f^e$, adding (4.3) to (4.7) leads to the desired result. □

Lemma 4.3. *Let the pair* $f : \overline{M} \longrightarrow \mathbb{C}^n$, $\Phi : \overline{M} \longrightarrow \mathcal{M}_n(\mathbb{C})$ *be a solution of* (1.4). *If* $f \in \mathcal{F}_n$ *then*

(i) *The map* $g := \Phi(\cdot, e)$ *is central.*
(ii) *The maps* $f_1 := f(\cdot, e)$ *and* $f_2 := f(e, \cdot)$ *satisfy the Kannappan condition* : $f_1(pqr) = f_1(prq)$ *and* $f_2(pqr) = f_2(prq)$ *for all* $p, q, r \in M$.

Proof. Since the pair $f:\overline{M}\longrightarrow\mathbb{C}^n$, $\Phi:\overline{M}\longrightarrow\mathcal{M}_n(\mathbb{C})$ is a solution of (1.4), Lemma 4.1 ensures that (2.4) holds, then according to Proposition 2.1, Φ is a solution of the functional equation

$$\Phi(pr,qs)+\Phi(ps,qr)=2\Phi(r,s)\Phi(p,q) \text{ for all } p,q,r,s\in M.$$

Then for $q=s=e$ we have

$$\Phi(pr,e)=2\Phi(r,e)\Phi(p,e)-\Phi(p,r) \text{ for all } p,r\in M.$$

So

$$g(pr)=2g(r)g(p)-\Phi(p,r) \text{ for all } p,r\in M.$$

Since $g(r)$ and $g(p)$ commute (Lemma 4.2) and $\Phi(p,r)=\Phi(r,p)$ for all $p,r\in M$, g is central. This proves (i).

By setting $x=(p,e)$; $y=(r,e)$ and $z=(s,e)$ in the identity (2.8) we deduce

$$\begin{aligned} f(prs,e)&=\Phi(s,e)f(pr,e)+\Phi(r,e)f(ps,e)\\ &\qquad+\Phi(rs,e)f(p,e)-2\Phi(r,e)\Phi(s,e)f(p,e). \end{aligned}$$

That is

$$f_1(prs)=g(s)f_1(pr)+g(r)f_1(ps)+g(rs)f_1(p)-2g(r)g(s)f_1(p).$$

Since $g(r)$ and $g(s)$ commute and g is central, the map f_1 satisfies the Kannappan condition. Also we prove by similar computations that f_2 satisfies the same condition. This completes the proof. □

Theorem 4.1. *Let the pair* $f:\overline{M}\longrightarrow\mathbb{C}^n$, $\Phi:\overline{M}\longrightarrow\mathcal{M}_n(\mathbb{C})$ *be a solution of* (1.4) *such that* $f\in\mathcal{F}_n$. *Then*

(1) Φ *is an abelian solution of the functional equation* (1.3) *such that* $\Phi(e,e)=I_n$.
(2) f *is abelian.*

Proof. Let the pair $f:\overline{M}\longrightarrow\mathbb{C}^n$, $\Phi:\overline{M}\longrightarrow\mathcal{M}_n(\mathbb{C})$ be a solution of (1.4). Using the same arguments as in the proof of Lemma 4.3 (i) we have

$$\Phi(pr,qs)+\Phi(ps,qr)=2\Phi(r,s)\Phi(p,q) \text{ for all } p,q,r,s\in M. \tag{4.8}$$

To prove the first statement we just have to check that Φ is a central map. Let us first show that $g:=\Phi(\cdot,e)=\Phi(e,\cdot)$ is an abelian function from M into $\mathcal{M}_n(\mathbb{C})$. Setting $s=e$ and $r=abc$ for some $a,b,c\in M$ and taking into account that f is central (Lemma 4.1), the equation (1.4) shows that

$$f(pabc,q)+f(p,qabc)=2\Phi(abc,e)f(p,q) \text{ for all } p,q,a,b,c\in M, \tag{4.9}$$

which we write

$$f((pabc,e)(e,q))+f((e,qabc)(p,e))=2\Phi(abc,e)f(p,q). \tag{4.10}$$

Using (2.3) to expand the left-hand side of (4.10) with $x=(pabc,e),y=(e,q)$ for the first term and with $x=(e,qabc),y=(p,e)$ for the second, we get

$$2\Phi(abc,e)f(p,q)=2\Phi(e,q)f(pabc,e)-f(qpabc,e)$$

$$+ 2\Phi(p,e)f(e,qabc) - f(e,pqabc).$$

Switching b and c then using Lemma 4.3 allow us to obtain the following

$$2\Phi(acb,e)f(p,q) = 2\Phi(abc,e)f(p,q) \text{ for all } p,q,a,b,c \in M.$$

Since $f \in \mathcal{F}_n$, we conclude that $g := \Phi(\cdot, e)$ satisfies the Kannappan condition. Then it is an abelian function. As a result of (4.8) we have

$$\Phi((p,r)(q,s)) = \Phi(pq,rs) = \Phi((pq,e)(e,rs)) = 2\Phi(e,rs)\Phi(pq,e) - \Phi(pqrs,e)$$

for all $p,q,r,s \in M$. Then

$$\Phi((p,r)(q,s)) = 2g(rs)g(pq) - g(pqrs) \text{ for all } p,q,r,s \in M, \tag{4.11}$$

and

$$\Phi((q,s)(p,r)) = 2g(sr)g(qp) - g(qpsr) \text{ for all } p,q,r,s \in M. \tag{4.12}$$

Since g is abelian, we conclude from (4.11) and (4.12) that Φ is central. Moreover Proposition 3.1 shows that Φ is abelian. This proves (1).

Taking into consideration the centrality of Φ and f and the fact that the matrices $\Phi(y)$ and $\Phi(z)$ commute (This follows from Lemma 4.1 in combination with Proposition 2.1 (4)), the identity (2.8) shows that f is abelian. This proves (2) and completes the proof. □

Note 1. Let (f,Φ) satisfies (1.4) such that $f \notin \mathcal{F}_n$ then f remains abelian. To show this we first need to recall that equation (1.4) can be reformulated as

$$\begin{cases} f(xy) + f(\sigma(y)x) = 2\Phi(y)f(x) & x,y \in \overline{M}, \\ \Phi(x) = \Phi \circ \sigma(x) & x \in \overline{M}. \end{cases} \tag{4.13}$$

If $n = 1$ then $f \notin \mathcal{F}_n$ means that $f = 0$, so f is clearly abelian. If $n > 1$ the sub-case $\dim\langle\{f(x) \in \mathbb{C}^n | x \in \overline{M}\}\rangle = 0$ means that $f = 0$, then f is abelian.

From now we may assume that $\dim\langle\{f(x) \in \mathbb{C}^n | x \in \overline{M}\}\rangle = k$ for some $k \in \mathbb{N}^*$ strictly less than n, that is

$$U := span\{f(x) \in \mathbb{C}^n | x \in \overline{M}\} = span\,\{u_i \in \mathbb{C}^n | i = 1,\dots,k\}$$

for some linearly independent vectors $(u_i)_{i\in\{1,\dots,k\}} \in \mathbb{C}^n$. Then there exists a set of scalar functions on $\overline{M}$: $(f_i)_{i\in\{1,\dots,k\}}$ such that

$$f(x) = \sum_{i=1}^{k} f_i(x)u_i \quad x \in \overline{M}. \tag{4.14}$$

Using (2.5) ensures the existence of a set of scalar functions on $\overline{M}$: ϕ_{ij}, $i,j \in \{1,\dots,k\}$ such that

$$\Phi(x)u_j = \sum_{i=1}^{k} \phi_{ij}(x)u_i \quad x \in \overline{M}, \tag{4.15}$$

for $j \in \{1, \ldots, k\}$. Substituting f and Φ in (4.13) shows that $\varphi_k := [f_1, \ldots, f_k]^T$ and $\Phi_k := (\phi_{ij})_{i,j\in\{1,\ldots,k\}}$ satisfy:

$$\varphi_k(xy) + \varphi_k(\sigma(y)x) = 2\Phi_k(y)\varphi_k(x), \quad x, y \in \overline{M}.$$

Since $(u_i)_{i\in\{1,\ldots,k\}}$ are linearly independent, the components of φ_k are linearly independent, that is, $\varphi_k \in \mathcal{F}_k$. Then Theorem 4.1 shows that φ_k is abelian. Consequently, we deduce from (4.14) that f is abelian.

Note 2. If $n > 1$ and $\dim\langle\{f(x) \in \mathbb{C}^n | x \in \overline{M}\}\rangle = k$ for some $k \in \mathbb{N}^*$ strictly less than n then it is immediate to see from the formula (4.15) (because Φ_k is abelian by Theorem 4.1), that the operator valued function $x \mapsto \Phi(x)|_U$ from $\overline{M}$ to $\mathcal{L}(U)$ is abelian.

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