Commun. Korean Math. Soc. **39** (2024), No. 3, pp. 775–784
https://doi.org/10.4134/CKMS.c230199
pISSN: 1225-1763 / eISSN: 2234-3024

SUPER AND STRONG $\gamma\mathcal{H}$-COMPACTNESS IN HEREDITARY m-SPACES

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Abstract. Let $(X, m, \mathcal{H})$ be a hereditary m-space and $\gamma : m \to P(X)$ be an operation on m. A subset A of X is said to be $\gamma\mathcal{H}$-compact relative to X [3] if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of A by m-open sets of X, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$. In this paper, we define and investigate two kinds of strong forms of $\gamma\mathcal{H}$-compact relative to X.

1. Introduction

In 1967, Newcomb [10] introduced the notion of compactness modulo an ideal. Rančin [13] and Hamlett and Janković [6] further investigated this notion and obtained some more properties of compactness modulo an ideal. Császár [5] introduced the notion of hereditary classes as a generalization of ideals. In [12], a minimal structure and a minimal space (X, m) are introduced and investigated. Let $(X, m, \mathcal{H})$ be a hereditary m-space and $\gamma : m \to P(X)$ be an operation on m. A subset A of X is said to be $\gamma\mathcal{H}$-compact relative to X [3] if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of A by m-open sets of X, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$. Recently, [4] introduced and studied the notions of θ-$\mathcal{H}$-compact in hereditary m-space. Several characterizations of minimal structures with notion of hereditary class were provided in [1, 2].

In this paper, we define a subset A of a hereditary m-space $(X, m, \mathcal{H})$ to be *super* $\gamma\mathcal{H}$-compact relative to X if for every family $\{U_\alpha : \alpha \in \Delta\}$ of m-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$, there exists a finite subset Δ_0 of Δ such that $A \subset \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}$. Similarly, we define a subset called *strongly* $\gamma\mathcal{H}$-compact relative to X and investigate their properties.

Received August 3, 2023; Revised February 17, 2024; Accepted April 16, 2024.
2020 *Mathematics Subject Classification.* Primary 54D20, 54D30.
Key words and phrases. Hereditary m-space, $\gamma\mathcal{H}$-compactness, strong $\gamma\mathcal{H}$-compactness, super $\gamma\mathcal{H}$-compactness.

2. Preliminaries

Definition 2.1. Let $\mathcal{P}(X)$ be the power set of a nonempty set X. A subfamily m of $\mathcal{P}(X)$ is called a *minimal structure* (briefly *m-structure*) [12] on X if m satisfies the following conditions:
(1) $\emptyset \in m$ and $X \in m$,
(2) The union of any family of subsets belonging to m belongs to m.

A set X with an m-structure m on X is denoted by (X, m) and is called an *m-space*. Each member of m is said to be *m-open* and the complement of an m-open set is said to be *m-closed*.

Definition 2.2. Let (X, m) be an m-space and A a subset of X. The *m-closure* $\mathrm{mCl}(A)$ and the *m-interior* $\mathrm{mInt}(A)$ of A [9] are defined as follows:
(1) $\mathrm{mCl}(A) = \cap\{F \subset X : A \subset F, X \setminus F \in m\}$,
(2) $\mathrm{mInt}(A) = \cup\{U \subset X : U \subset A, U \in m\}$.

Lemma 2.3 ([12])**.** *Let (X, m) be an m-space and A a subset of X.*
(1) *$x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m(x)$, where $m(x)$ denotes the family $\{U : x \in U \in m\}$.*
(2) *A is m-closed if and only if $\mathrm{mCl}(A) = A$.*

Definition 2.4. A nonempty subfamily $\mathcal{H}$ of $\mathcal{P}(X)$ is called a *hereditary class* on X [5] if it satisfies the following properties: $A \in \mathcal{H}$ and $B \subset A$ implies $B \in \mathcal{H}$. A hereditary class $\mathcal{H}$ is called an *ideal* ([8], [14]) if it satisfies the additional condition: $A \in \mathcal{H}$ and $B \in \mathcal{H}$ implies $A \cup B \in \mathcal{H}$.

Let $X = \{a, b, c\}$. If $\mathcal{H} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}\}$, then $\mathcal{H}$ is a hereditary class but is not an ideal. Since $\mathcal{H}$ does not contain $\{a, b\}$ so, $\mathcal{H}$ is not an ideal.

A minimal space (X, m) with a hereditary class $\mathcal{H}$ on X is called a *hereditary minimal space* (briefly *hereditary m-space*) and is denoted by $(X, m, \mathcal{H})$. The notion of ideals has been introduced in [8] and [14] and further investigated in [7].

Definition 2.5. Let (X, m) be an m-space. Let $m\gamma : m \to P(X)$ be a function from m into $P(X)$ such that $U \subset m\gamma(U)$ for each $U \in m$. The function $m\gamma$ is called an *$m\gamma$-operation* on m [11] and the image $m\gamma(U)$ is simply denoted by $\gamma(U)$. In this paper, an $m\gamma$-operation is simply called a γ-operation.

Let $\gamma = Cl$ (closure). Then $\gamma(A \cup B) = \gamma(A) \cup \gamma(B)$ for any subsets A and B of X.

Definition 2.6. Let (X, m) be an m-space and $\gamma : m \to P(X)$ be a γ-operation. A subset A of X is said to be *γ-open* [11] if for each $x \in A$ there exists $U \in m$ such that $x \in U \subset \gamma(U) \subset A$. The complement of a γ-open set is said to be *γ-closed*. The family of all γ-open sets of (X, m) is denoted by $\gamma(X)$. The *γ-closure* of A, $\gamma\mathrm{Cl}(A)$, is defined as follows: $\gamma\mathrm{Cl}(A) = \cap\{F \subset X : A \subset F, X \setminus F \in \gamma(X)\}$.

Example 2.7. Let $X = \{a, b, c\}$ with $m = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\gamma(A) = Cl(A)$ for any subset A of X. Then, $\{a, b\}$ is an open set but not γ-open. Because when $a \in \{a, b\}$. If $a \in U \in \tau$, then $U = \{a\}, \{a, b\}$ and X. If $U = \{a\}$, then $a \in U \subset \gamma(U) = Cl(U) = \{a, c\}$ and $\gamma(U)$ does not contain in $\{a, b\}$. If $U = \{a, b\}$, then $a \in U \subset \gamma(U) = Cl(U) = X$ and hence $\gamma(U)$ does not contain in $\{a, b\}$. If $U = X$, then $a \in U \subset \gamma(U) = Cl(U) = X$ and $\gamma(U)$ does not contain in $\{a, b\}$. Therefore, $\{a, b\}$ is not γ-open.

Definition 2.8. Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. A subset A of X is said to be *$\gamma\mathcal{H}$-compact relative to X* [3] (resp. *γ-compact relative to X*) if for each cover $\{U_\alpha : \alpha \in \Delta\}$ of A by m-open sets of X, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$ (resp. $A \subset \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}$).

Definition 2.9. Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. The space $(X, m, \mathcal{H})$ is said to be *$\gamma\mathcal{H}$-compact* [3] (resp. *γ-compact* [11]) if X is *$\gamma\mathcal{H}$-compact relative to X* (resp. *γ-compact relative to X*).

3. Super $\gamma\mathcal{H}$-compact spaces

Definition 3.1. Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m.

(1) A subset A of X is said to be *super $\gamma\mathcal{H}$-compact relative to X* if for every family $\{U_\alpha : \alpha \in \Delta\}$ of m-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$, there exists a finite subset Δ_0 of Δ such that $A \subset \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}$.

(2) $(X, m\ \mathcal{H})$ is called a *super $\gamma\mathcal{H}$-compact space* if X is super $\gamma\mathcal{H}$-compact relative to X.

Remark 3.2. Let $(X, m, \mathcal{H})$ be a hereditary m-space. If $\mathcal{H} = \{\emptyset\}$, then "super $\gamma\mathcal{H}$-compact relative to X" coincides with "γ-compact relative to X".

Theorem 3.3. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. For a subset A of X, the following properties are equivalent*:

(1) *A is super $\gamma\mathcal{H}$-compact relative to X*;

(2) *for every family $\{F_\alpha : \alpha \in \Delta\}$ of m-closed sets of X such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$, there exists a finite subset Δ_0 of Δ such that $A \cap (\cap\{[X \setminus \gamma(X \setminus F_\alpha)] : \alpha \in \Delta_0\}) = \emptyset$.*

Proof. (1) ⇒ (2): Let $\{F_\alpha : \alpha \in \Delta\}$ be any family of m-closed sets of X such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$. Then, we have

$$\begin{aligned} A \setminus (\cup\{X \setminus F_\alpha : \alpha \in \Delta\}) &= A \setminus (X \setminus \cap\{F_\alpha : \alpha \in \Delta\}) \\ &= A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in \mathcal{H}. \end{aligned}$$

Since $X \setminus F_\alpha$ is m-open for each $\alpha \in \Delta$, by (1) there exists a finite subset Δ_0 of Δ such that $A \subset \cup\{X \setminus F_\alpha : \alpha \in \Delta_0\} \subset \cup\{\gamma(X \setminus F_\alpha) : \alpha \in \Delta_0\}$. Therefore, we have

$$A \cap [X \setminus (\cup\{\gamma(X \setminus F_\alpha) : \alpha \in \Delta_0\})]$$

$$= A \cap (\cap\{[X \setminus \gamma(X \setminus F_\alpha)] : \alpha \in \Delta_0\})$$
$$= \emptyset.$$

(2) ⇒ (1): Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of m-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Then, $\{X \setminus U_\alpha : \alpha \in \Delta\}$ is a family of m-closed sets such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} = A \cap (X \setminus \cup\{U_\alpha : \alpha \in \Delta\}) = A \cap (\cap\{X \setminus U_\alpha : \alpha \in \Delta\})$ and hence $A \cap (\cap\{X \setminus U_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$. By (2), there exists a finite subset Δ_0 of Δ such that $A \cap (\cap[X \setminus \gamma(X \setminus (X \setminus U_\alpha)) : \alpha \in \Delta_0]) = A \cap (\cap[X \setminus \gamma(U_\alpha) : \alpha \in \Delta_0]) = \emptyset$. Therefore, $A \cap (X \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}) = \emptyset$ and hence, $A \subset \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}$. This shows that A is super $\gamma\mathcal{H}$-compact relative to X. □

Corollary 3.4. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. Then, the following properties are equivalent*:

(1) *$(X, m, \mathcal{H})$ is super $\gamma\mathcal{H}$-compact*;

(2) *for every family $\{F_\alpha : \alpha \in \Delta\}$ of m-closed sets of X such that $\cap\{F_\alpha : \alpha \in \Delta\} \in \mathcal{H}$, there exists a finite subset Δ_0 of Δ such that $\cap\{[X \setminus \gamma(X \setminus F_\alpha)] : \alpha \in \Delta_0\} = \emptyset$.*

Definition 3.5. Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. A subset A of X is said to be *$\mathcal{H}\gamma g$-closed* if $\gamma\mathrm{Cl}(A) \subset U$ whenever, $A \setminus U \in \mathcal{H}$ and U is m-open.

Example 3.6. Let $X = \{a, b, c\}$, $m = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $A = \{a\}$ and $\mathcal{H} = \{\emptyset, \{c\}\}$. Then, $(X, m, \mathcal{H})$ is a hereditary m-space and let $\gamma = Cl$. Let $U = \{a\}$. Then $A \subseteq U$ and $Cl(A) \setminus U = \{a, c\} \setminus \{a\} = \{c\} \in \mathcal{H}$. Let $U = \{a, b\}$. Then $A \subseteq U$ and $Cl(A) \setminus U = \{a, c\} \setminus \{a, b\} = \{c\} \in \mathcal{H}$. Let $U = X$. Then $A \subseteq U$ and $Cl(A) \setminus U = \{a, c\} \setminus X = \emptyset \in \mathcal{H}$. Therefore, A is an $\mathcal{H}_\gamma g$-closed set.

Theorem 3.7. *Let $(X, m, \mathcal{H})$ be a hereditary m-space, γ be a γ-operation on m and A, B be subsets of X such that $A \subset B \subset \gamma\mathrm{Cl}(A)$ and A is $\mathcal{H}\gamma g$-closed, then the following properties hold*:

(1) *if $\gamma\mathrm{Cl}(A)$ is γ-compact relative to X, then B is super $\gamma\mathcal{H}$-compact relative to X,*

(2) *if B is γ-compact relative to X, then A is super $\gamma\mathcal{H}$-compact relative to X.*

Proof. (1): Suppose that $\gamma\mathrm{Cl}(A)$ is $\gamma\mathcal{H}$-compact relative to X. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of m-open sets of X such that $B \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Then, $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Since A is $\mathcal{H}\gamma g$-closed, $\gamma\mathrm{Cl}(A) \subset \cup\{U_\alpha : \alpha \in \Delta\}$. Since $\gamma\mathrm{Cl}(A)$ is γ-compact relative to X, there exists a finite subset Δ_0 of Δ such that $\gamma\mathrm{Cl}(A) \subset \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}$. Since $B \subset \gamma\mathrm{Cl}(A)$, we have $B \subset \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}$. Therefore, B is super $\gamma\mathcal{H}$-compact relative to X.

(2): Suppose that B is γ-compact relative to X. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of m-open sets in X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Since A is $\mathcal{H}\gamma g$-closed, $\gamma\mathrm{Cl}(A) \subset \cup\{U_\alpha : \alpha \in \Delta\}$. Hence, we have $B \subset \gamma\mathrm{Cl}(A) \subset \cup\{U_\alpha :$

$\alpha \in \Delta\}$. Since B is γ-compact relative to X, there exists a finite subset Δ_0 of Δ such that $B \subset \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}$. Since $A \subset B, A \subset \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}$. Therefore, A is super $\gamma\mathcal{H}$-compact relative to X. □

Theorem 3.8. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. If subsets A and B of X are super $\gamma\mathcal{H}$-compact relative to X, then $A \cup B$ is super $\gamma\mathcal{H}$-compact relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of m-open sets of X such that $(A \cup B) \setminus \cup\{U_\alpha \in \Delta\} \in \mathcal{H}$. Then, we have $A \setminus \cup\{U_\alpha \in \Delta\} \in \mathcal{H}$ and $B \setminus \cup\{U_\alpha \in \Delta\} \in \mathcal{H}$. Since A and B are super $\gamma\mathcal{H}$-compact relative to X, there exist finite subsets Δ_A and Δ_B of Δ such that $A \subset \cup\{\gamma\mathrm{Cl}(U_\alpha) : \alpha \in \Delta_A\}$ and $B \subset \cup\{\gamma\mathrm{Cl}(U_\alpha) : \alpha \in \Delta_B\}$. Hence, we have $A \cup B \subset \cup\{\gamma\mathrm{Cl}(U_\alpha) : \alpha \in \Delta_A \cup \Delta_B\}$. $\Delta_A \cup \Delta_B$ is a finite subset of Δ. Therefore, $A \cup B$ is super $\gamma\mathcal{H}$-compact relative to X. □

Theorem 3.9. *Let $(X, m, \mathcal{H})$ be a hereditary m-space, γ be a γ-operation on m and A, B be subsets of X. If A is super $\gamma\mathcal{H}$-compact relative to X and B is γ-closed, then $A \cap B$ is super $\gamma\mathcal{H}$-compact relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of m-open sets of X such that $(A \cap B) \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Since B is γ-closed, $X \setminus B$ is γ-open and for each $x \in X \setminus B$, there exists $V_x \in m$ such that $x \in V_x \subset \gamma(V_x) \subset X \setminus B$. Hence $\{U_\alpha : \alpha \in \Delta\} \cup [\cup\{V_x : x \in X \setminus B\}]$ is a family of m-open sets of X. $(A \cap B) \setminus \cup\{U_\alpha : \alpha \in \Delta\} = A \setminus [(X \setminus B) \cup (\cup\{U_\alpha : \alpha \in \Delta\})] = A \setminus [(\cup\{V_x : x \in X \setminus B\}) \cup (\cup\{U_\alpha : \alpha \in \Delta\})] \in \mathcal{H}$. Since A is super $\gamma\mathcal{H}$-compact relative to X, there exist finite subset Δ_0 of Δ and finite points $x_1, x_2, \ldots, x_n$ in $X \setminus B$ such that $A \subset [(\cup\{\gamma(V_{x_i}) : i = 1, 2, \ldots, n\}) \cup (\cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\})]$. Since $B \cap \gamma(V_{x_i}) = \emptyset$ for each x_i $(i = 1, 2, \ldots, n)$, $A \cap B \subset [\cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}] \cap B \subset \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}$. Therefore, $A \cap B$ is super $\gamma\mathcal{H}$-compact relative to X. □

Corollary 3.10. *If a hereditary m-space $(X, m, \mathcal{H})$ is super $\gamma\mathcal{H}$-compact and B is γ-closed, then B is super $\gamma\mathcal{H}$-compact relative to X.*

Definition 3.11. A function $f : (X, m) \to (Y, n)$ is said to be (γ, δ)-*closed* if for each $y \in Y$ and $U \in m$ containing $f^{-1}(y)$, there exists $V \in n$ containing y such that $f^{-1}(\delta(V)) \subseteq \gamma(U)$.

Definition 3.12. Let $(X, m, \mathcal{H})$ be a hereditary m-space.

(1) A subset A of X is said to be super $\mathcal{H}$-*compact relative to* X if for every family $\{U_\alpha : \alpha \in \Delta\}$ of m-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$, there exists a finite subset Δ_0 of Δ such that $A \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$.

(2) $(X, m\ \mathcal{H})$ is called a super $\mathcal{H}$-*compact space* if X is super $\mathcal{H}$-compact relative to X.

Theorem 3.13. *Let $f : (X, m) \to (Y, n, \mathcal{H})$ be a (γ, δ)-closed surjective function such that $\gamma(U \cup V) = \gamma(U) \cup \gamma(V)$ for each $U, V \in m$. If $f^{-1}(y)$ is super $\mathcal{H}$-compact relative to X for each $y \in Y$ and B is δ-compact relative to Y, then $f^{-1}(B)$ is super $\gamma f^{-1}(\mathcal{H})$-compact relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of m-open sets of X such that $f^{-1}(B) \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in f^{-1}(\mathcal{H})$. Then, for each $y \in B$, since $f^{-1}(y)$ is super $\mathcal{H}$-compact relative to X, there exists a finite subset $\Delta(y)$ of Δ such that $f^{-1}(y) \subseteq \cup\{U_\alpha : \alpha \in \Delta(y)\} = U_y$. Since U_y is an m-open set of X containing $f^{-1}(y)$ and f is (γ, δ)-closed there exists an n-open set V_y containing y such that $f^{-1}(\delta(V_y)) \subseteq \gamma(U_y)$. Since $\{V_y : y \in B\}$ is an n-open cover of B and B is δ-compact relative to Y, there exists a finite subset B_0 of B such that $B \subseteq \cup\{\delta(V_y) : y \in B_0\}$. Hence, we have

$$\begin{aligned} f^{-1}(B) &\subseteq \cup\,\{f^{-1}(\delta(V_y)) : y \in B_0\} \\ &\subseteq \cup\,\{\gamma(U_y) : y \in B_0\} \\ &\subseteq \cup\,\{\gamma(U_\alpha) : \alpha \in \Delta(y), y \in B_0\}. \end{aligned}$$

We obtain $f^{-1}(B) \subseteq \cup\{\gamma(U_\alpha) : \alpha \in \Delta(y), y \in B_0\}$. This shows that $f^{-1}(B)$ is super $\gamma f^{-1}(\mathcal{H})$-compact relative to Y. □

Corollary 3.14. *Let $f : (X, m) \to (Y, n, \mathcal{H})$ be a (γ, δ)-closed surjective function such that $\gamma(U \cup V) = \gamma(U) \cup \gamma(V)$ for each $U, V \in m$. If $f^{-1}(y)$ is super $\mathcal{H}$-compact relative to X for each $y \in Y$ and B is super $\delta\mathcal{H}$-compact relative to Y, then $f^{-1}(B)$ is super $\gamma f^{-1}(\mathcal{H})$-compact relative to X.*

Corollary 3.15. *Let $f : (X, m) \to (Y, n, \mathcal{H})$ be a (γ, δ)-closed surjective function such that $\gamma(U \cup V) = \gamma(U) \cup \gamma(V)$ for each $U, V \in m$. If $f^{-1}(y)$ is super $\mathcal{H}$-compact relative to X for each $y \in Y$ and Y is δ-compact, then X is super $\gamma f^{-1}(\mathcal{H})$-compact.*

4. Strongly $\gamma\mathcal{H}$-compact spaces

Definition 4.1. Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m.

(1) A subset A of X is said to be *strongly $\gamma\mathcal{H}$-compact relative to X* if for every family $\{U_\alpha : \alpha \in \Delta\}$ of m-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$.

(2) $(X, m, \mathcal{H})$ is said to be *strongly $\gamma\mathcal{H}$-compact* if X is *strongly $\gamma\mathcal{H}$-compact relative to X*.

Theorem 4.2. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ be a γ-operation on m. For a subset A of X, the following properties are equivalent*:

(1) *A is strongly $\gamma\mathcal{H}$-compact relative to X*;

(2) *for every family $\{F_\alpha : \alpha \in \Delta\}$ of m-closed sets of X such that*

$$A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in \mathcal{H},$$

there exists a finite subset Δ_0 of Δ such that

$$A \cap (\cap\{[X \setminus \gamma(X \setminus F_\alpha)] : \alpha \in \Delta_0\}) \in \mathcal{H}.$$

Proof. (1) $\Rightarrow$ (2): Let $\{F_\alpha : \alpha \in \Delta\}$ be any family of m-closed sets of X such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$. Then, $A \setminus \cup\{X \setminus F_\alpha : \alpha \in \Delta\}) = A \setminus (X \setminus \cap\{F_\alpha : \alpha \in \Delta\}) = A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$. Since $X \setminus F_\alpha$ is m-open for each $\alpha \in \Delta$ and A is strongly $\gamma\mathcal{H}$-compact relative to X by (1), there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{\gamma(X \setminus F_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$. This implies that $A \cap (\cap\{[X \setminus \gamma(X \setminus F_\alpha)] : \alpha \in \Delta_0\}) = A \setminus (X \setminus (\cap\{[X \setminus \gamma(X \setminus F_\alpha)] : \alpha \in \Delta_0\})) = A \setminus \cup\{\gamma(X \setminus F_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$.

(2) $\Rightarrow$ (1): Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of m-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Then, $\{X \setminus U_\alpha : \alpha \in \Delta\}$ is a family of m-closed sets of X and also $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} = A \cap (X \setminus \cup\{U_\alpha : \alpha \in \Delta\}) = A \cap (\cap\{X \setminus U_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$. Thus, by (2) there exists a finite subset Δ_0 of Δ such that $A \cap (\cap\{X \setminus \gamma(U_\alpha) : \alpha \in \Delta_0\}) \in \mathcal{H}$. Therefore, we have $A \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} = A \cap (X \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}) = A \cap (\cap\{X \setminus \gamma(U_\alpha) : \alpha \in \Delta_0\}) \in \mathcal{H}$. This shows that A is strongly $\gamma\mathcal{H}$-compact relative to X. □

Corollary 4.3. *For a hereditary m-space $(X, m, \mathcal{H})$, the following properties are equivalent, where γ is a γ-operation on m:*

(1) *$(X, m, \mathcal{H})$ is strongly $\gamma\mathcal{H}$-compact*;

(2) *for every family $\{F_\alpha : \alpha \in \Delta\}$ of m-closed sets of X such that $\cap\{F_\alpha : \alpha \in \Delta\} \in \mathcal{H}$, there exists a finite subset Δ_0 of Δ such that $\cap\{[X \setminus \gamma(X \setminus F_\alpha)] : \alpha \in \Delta_0\} \in \mathcal{H}$.*

Theorem 4.4. *Let $(X, m, \mathcal{H})$ be a hereditary m-space, γ be a γ-operation on m and A, B be subsets of X such that A is $\mathcal{H}\gamma g$-closed and $A \subset B \subset \gamma\mathrm{Cl}(A)$, then the following properties hold*:

(1) *if $\gamma\mathrm{Cl}(A)$ is $\gamma\mathcal{H}$-compact relative to X, then B is strongly $\gamma\mathcal{H}$-compact relative to X,*

(2) *if B is $\gamma\mathcal{H}$-compact relative to X, then A is strongly $\gamma\mathcal{H}$-compact relative to X.*

Proof. (1): Suppose that $\gamma\mathrm{Cl}(A)$ is $\gamma\mathcal{H}$-compact relative to X. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of m-open sets of X such that $B \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Then, $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$ and $\cup\{U_\alpha : \alpha \in \Delta\} \in m$. Since A is $\mathcal{H}mg$-closed, $\gamma\mathrm{Cl}(A) \subset \cup\{U_\alpha : \alpha \in \Delta\}$. Since $\gamma\mathrm{Cl}(A)$ is $\gamma\mathcal{H}$-compact relative to X, there exists a finite subset Δ_0 of Δ such that $\gamma\mathrm{Cl}(A) \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$ and hence $B \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$. Therefore, B is strongly $\gamma\mathcal{H}$-compact relative to X.

(2): Suppose that B is $\gamma\mathcal{H}$-compact relative to X. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of m-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Since A is $\mathcal{H}mg$-closed, we have $B \subset \gamma\mathrm{Cl}(A) \subset \cup\{U_\alpha : \alpha \in \Delta\}$. Since B is $\gamma\mathcal{H}$-compact relative to X, there exists a finite subset Δ_0 of Δ such that $B \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$. Since $A \subset B$, $A \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$. Hence, A is strongly $\gamma\mathcal{H}$-compact relative to X. □

Theorem 4.5. *Let $(X, m, \mathcal{H})$ be an ideal m-space and γ be a γ-operation on m. If subsets A and B of X are strongly $\gamma\mathcal{H}$-compact relative to X, then $A \cup B$ is strongly $\gamma\mathcal{H}$-compact relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of m-open sets of X such that $(A \cup B) \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Then, $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$ and $B \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Since A and B are strongly $\gamma\mathcal{H}$-compact relative to X, there exist finite subsets Δ_A and Δ_B of Δ and subsets H_A and H_B of $\mathcal{H}$ such that $A \subset \cup\{U_\alpha : \alpha \in \Delta_A\} \cup H_A$ and $B \subset \cup\{U_\alpha : \alpha \in \Delta_B\} \cup H_B$. Hence, we have $(A \cup B) \subset \cup\{U_\alpha : \alpha \in \Delta_A \cup \Delta_B\} \cup (H_A \cup H_B)$. Since $\mathcal{H}$ is an ideal, we have $(A \cup B) \setminus \cup\{U_\alpha : \alpha \in \Delta_A \cup \Delta_B\} \in \mathcal{H}$. This shows that $A \cup B$ is strongly $\gamma\mathcal{H}$-compact relative to X. □

Theorem 4.6. *Let $(X, m, \mathcal{H})$ be a hereditary m-space, γ a γ-operation on m and A, B be subsets of X. If A is strongly $\gamma\mathcal{H}$-compact relative to X and B is γ-closed, then $A \cap B$ is strongly $\gamma\mathcal{H}$-compact relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of m-open sets of X such that $(A \cap B) \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Since B is γ-closed, $X \setminus B$ is γ-open and for each $x \in X \setminus B$, there exists $V_x \in m$ such that $x \in V_x \subset \gamma(V_x) \subset X \setminus B$. Hence, $\{U_\alpha : \alpha \in \Delta\} \cup [\cup\{V_x : x \in X \setminus B\}]$ is a family of m-open sets of X. $(A \cap B) \setminus \cup\{U_\alpha : \alpha \in \Delta\} = A \setminus [(X \setminus B) \cup (\cup\{U_\alpha : \alpha \in \Delta\})] = A \setminus [\cup\{V_x : x \in X \setminus B\} \cup (\cup\{U_\alpha : \alpha \in \Delta\})] \in \mathcal{H}$. Since A is strongly $\gamma\mathcal{H}$-compact relative to X, there exist finite subset Δ_0 of Δ and finite points $x_1, x_2, ..., x_n$ in $X \setminus B$ such that $A \setminus [\cup\{\gamma(V_{x_i}) : i = 1, 2, ..., n\} \cup (\cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\})] \in \mathcal{H}$. Since $B \cap \gamma(V_{x_i}) = \emptyset$ for each x_i $(i = 1, 2, ..., n)$, $A \cap B \setminus [\cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}] \in \mathcal{H}$. Therefore, $A \cap B$ is strongly $\gamma\mathcal{H}$-compact relative to X. □

Corollary 4.7. *If a hereditary m-space $(X, m, \mathcal{H})$ is strongly $\gamma\mathcal{H}$-compact and B is γ-closed, then B is strongly $\gamma\mathcal{H}$-compact relative to X.*

Theorem 4.8. *Let $f : (X, m) \to (Y, n, \mathcal{H})$ be a (γ, δ)-closed surjective function such that $\gamma(U \cup V) = \gamma(U) \cup \gamma(V)$ for each $U, V \in m$. If $f^{-1}(y)$ is super $\mathcal{H}$-compact relative to X for each $y \in Y$ and B is $\delta\mathcal{H}$-compact relative to Y, then $f^{-1}(B)$ is strongly $\gamma f^{-1}(\mathcal{H})$-compact relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of m-open sets of X such that $f^{-1}(B) \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in f^{-1}(\mathcal{H})$. Then, for each $y \in B$, since $f^{-1}(y)$ is super $\mathcal{H}$-compact relative to X, there exists a finite subset $\Delta(y)$ of Δ such that $f^{-1}(y) \subseteq \cup\{U_\alpha : \alpha \in \Delta(y)\} = U_y$. Since U_y is an m-open set of X containing $f^{-1}(y)$ and f is (γ, δ)-closed, there exists an n-open set V_y containing y such that $f^{-1}(\delta(V_y)) \subseteq \gamma(U_y)$. Since $\{V_y : y \in B\}$ is an n-open cover of B and B is $\delta\mathcal{H}$-compact relative to Y, there exists a finite subset B_0 of B such that $B \setminus \cup\{\delta(V_y) : y \in B_0\} \in \mathcal{H}$. Therefore, $B \subseteq \cup\{\delta(V_y) : y \in B_0\} \cup H_0$, where $H_0 \in \mathcal{H}$. Hence, we have

$$f^{-1}(B) \subseteq \cup\{f^{-1}(\delta(V_y)) : y \in B_0\} \cup f^{-1}(H_0)$$

$$\subseteq \cup \{\gamma(U_y) : y \in B_0\} \cup f^{-1}(H_0)$$
$$\subseteq \cup \{\gamma(U_\alpha) : \alpha \in \Delta(y), y \in B_0\} \cup f^{-1}(H_0).$$

We obtain $f^{-1}(B) \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta(y), y \in B_0\} \in f^{-1}(\mathcal{H})$. This shows that $f^{-1}(B)$ is strongly $\gamma f^{-1}(\mathcal{H})$-compact relative to Y. □

Corollary 4.9. *Let $f : (X, m) \to (Y, n, \mathcal{H})$ be a (γ, δ)-closed surjective function such that $\gamma(U \cup V) = \gamma(U) \cup \gamma(V)$ for each $U, V \in m$. If $f^{-1}(y)$ is super $\mathcal{H}$-compact relative to X for each $y \in Y$ and B is strongly $\delta\mathcal{H}$-compact relative to Y, then $f^{-1}(B)$ is strongly $\gamma f^{-1}(\mathcal{H})$-compact relative to X.*

Corollary 4.10. *Let $f : (X, m) \to (Y, n, \mathcal{H})$ be a (γ, δ)-closed surjective function such that $\gamma(U \cup V) = \gamma(U) \cup \gamma(V)$ for each $U, V \in m$. If $f^{-1}(y)$ is super $\mathcal{H}$-compact relative to X for each $y \in Y$ and Y is $\delta\mathcal{H}$-compact, then X is strongly $\gamma f^{-1}(\mathcal{H})$-compact.*

Remark 4.11. We have the following relationships:

super $\gamma\mathcal{H}$-compact relative to X $\Rightarrow$ strongly $\gamma\mathcal{H}$-compact relative to X
$\Downarrow$ $\Downarrow$
γ-compact relative to X $\Rightarrow$ $\gamma\mathcal{H}$-compact relative to X

Remark 4.12. The following examples show that "γ-compact relative to X" and "strongly $\gamma\mathcal{H}$-compact relative to X" are independent of each other. Therefore, the converse of the above four implications are not necessarily true.

Example 4.13. Let $\mathcal{R}$ be the set of real numbers with the usual topology, $X = [1, 2]$ and $m = \{X \cap (a, b) : a < b, a, b \in \mathcal{R}\}$. Then, it is clear that (X, m) is a topological space and an m-space. Let $\mathcal{H} = \{\emptyset, \{1\}, \{2\}\}$. Let γ be a γ-operation on m such that $\gamma(U) = \mathrm{Cl}(U)$ for each $U \in m$. Observe that (X, m) is γ-compact relative to X but $(X, m, \mathcal{H})$ is not strongly $\gamma\mathcal{H}$-compact relative to X. In fact if $U_n = (1 + \frac{1}{n}, 2]$ for all integer numbers $n > 1$, then $X \setminus \cup_{n>1} U_n = \{1\} \in \mathcal{H}$. If we take $N = \max\{n_1, n_2, \ldots, n_k\}$, $k \in \mathbf{Z}$ and $n_1, n_2, \ldots, n_k$ are integer numbers, then $X \setminus \cup_{i=1}^{k} \gamma(U_{n_i}) = X \setminus [1 + \frac{1}{N}, 2] = [1, 1 + \frac{1}{N}) \notin \mathcal{H}$.

Example 4.14. Let $\mathcal{R}$ be the set of real numbers with the usual topology τ. Let $X = (0, 1)$, m the relative topology of τ on X, $\mathcal{H} = \{A : A \subseteq (0, 1)\}$ and $\gamma(U) = \mathrm{Cl}(U)$ for each $U \in m$. Then $(X, m, \mathcal{H})$ is strongly $\gamma\mathcal{H}$-compact relative to X but (X, m) is not γ-compact relative to X. Because an m-open cover $\{(0 + \frac{1}{n}, 1 - \frac{1}{n}) : n \in \mathbf{Z}^+\}$ of X has no finite γ-closure subcover.

Acknowledgments. The authors are highly grateful to editors and referees for their valuable comments and suggestions for improving the paper.

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