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ON $\mathbf{H}_2$-PROPER TIMELIKE HYPERSURFACES IN LORENTZ 4-SPACE FORMS

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Abstract. The ordinary mean curvature vector field **H** on a submanifold M of a space form is said to be *proper* if it satisfies equality $\Delta\mathbf{H} = a\mathbf{H}$ for a constant real number a. It is proven that every hypersurface of an Riemannian space form with proper mean curvature vector field has constant mean curvature. In this manuscript, we study the Lorentzian hypersurfaces with proper second mean curvature vector field of four dimensional Lorentzian space forms. We show that the scalar curvature of such a hypersurface has to be constant. In addition, as a classification result, we show that each Lorentzian hypersurface of a Lorentzian 4-space form with proper second mean curvature vector field is C-biharmonic, C-1-type or C-null-2-type. Also, we prove that every $\mathbf{H}_2$-proper Lorentzian hypersurface with constant ordinary mean curvature in a Lorentz 4-space form is 1-minimal.

1. Introduction

Among the differential geometric research subjects, the study of constant mean curvature submanifolds is of great importance. Clearly, every such a submanifold of an Euclidean space satisfies the proper condition. On the contrary, this is a question that has remained unanswered in some cases and is closely related to a famous conjecture of Bang-Yen Chen which says that every submanifold of an Euclidean space with harmonic mean curvature vector field has zero mean curvature [6]. It has several improvements (for instance) in [1, 5, 8, 9]. In this field, Defever has proved that the mean curvature of a hypersurface in $\mathbb{E}^4$ is constant if its mean curvature vector is proper ([7]). The subject of hypersurfaces in semi-Riemannian manifolds has been studied in the last two decades (see [2, 4, 17, 20]).

By definition, a hypersurface is **H**-proper if it satisfies the condition $\Delta\mathbf{H} = a\mathbf{H}$ for a constant real number a, where Δ is the Laplace operator. In this paper, we take an extended version of this condition by putting the Cheng-Yau operator C instead of Δ. The operator C denotes the linear operator arisen

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from the first variation of the second mean curvature (see [3, 11, 15, 16, 19]). We study the $\mathbf{H}_2$-proper timelike (i.e. Lorentzian) hypersurfaces of Lorentz 4-space forms. Since there are four possible matrix forms for the shape operator of such a hypersurface, we discuss the subject in four different cases.

2. Preliminaries

We recall some notations and formulae from [10,13–16,21]. We use the semi-Euclidean 5-space $\mathbb{E}^5_\xi$ of index $\xi = 1, 2$, equipped with the product defined by $\langle \mathbf{v}, \mathbf{w}\rangle = -\sum_{i=1}^{\xi} v_i w_i + \sum_{i=\xi+1}^{5} v_i w_i$, for each vectors $\mathbf{v} = (v_1, \dots, v_5)$ and $\mathbf{w} = (w_1, \dots, w_5)$ in $\mathbb{R}^5$. In fact, we deal with the 4-dimensional Lorentzian space forms with the following common notation

$$\mathbb{M}^4_1(c) = \begin{cases} \mathbb{S}^4_1(r) & (\text{if } c = 1/r^2) \\ \mathbb{L}^4 = \mathbb{E}^4_1 & (\text{if } c = 0) \\ \mathbb{H}^4_1(-r) & (\text{if } c = -1/r^2), \end{cases}$$

where, for $r > 0$, $\mathbb{S}^4_1(r) = \{\mathbf{v} \in \mathbb{E}^5_1 | \langle \mathbf{v}, \mathbf{v}\rangle = r^2\}$ denotes the 4-pseudosphere of radius r and curvature $1/r^2$, and $\mathbb{H}^4_1(-r) = \{\mathbf{v} \in \mathbb{E}^5_2 | \langle \mathbf{v}, \mathbf{v}\rangle = -r^2, v_1 > 0\}$ denotes the pseudo-hyperbolic 4-space of radius $-r$ and curvature $-1/r^2$. In the canonical cases $c = \pm 1$, we get the de Sitter 4-space $d\mathbb{S}^4 := \mathbb{S}^4_1(1)$ and anti de Sitter 4-space $Ad\mathbb{S}^4 = \mathbb{H}^4_1(-1)$. Also, for $c = 0$ we get the Lorentz-Minkowski 4-space $\mathbb{L}^4 := \mathbb{E}^4_1$.

We consider a Lorentzian (timelike) hypersurface M^3_1 of a canonical Lorentzian 4-space form (i.e. $\mathbb{M}^4_1(c)$ for $c = 0, \pm 1$) defined by an isometric immersion $\mathbf{x} : M^3_1 \to \mathbb{M}^4_1(c)$. The set of all smooth tangent vector fields on M^3_1 is denoted by $\chi(M^3_1)$. The symbols ∇ and $\bar{\nabla}$ denote the Levi-Civita connections on M^3_1 and $\mathbb{M}^4_1(c)$, respectively. Also, ∇^0 denotes the Levi-Civita connection on $\mathbb{E}^5_\nu$ (for $\nu = 1, 2$). The Weingarten formula on M^3_1 is $\bar{\nabla}_V W = \nabla_V W + \langle \mathrm{S}V, W\rangle \mathbf{n}$, for each $V, W \in \chi(M^3_1)$, where S is the shape operator associated to a unit normal vector field $\mathbf{n}$ on M^3_1. Furthermore, in the case $|c| = 1$, $\mathbb{M}^4(c)$ is a 4-hyperquadric with the unit normal vector field $\mathbf{x}$ and the Gauss formula $\nabla^0_V W = \bar{\nabla}_V W - c\langle V, W\rangle \mathbf{x}$.

According to the Lorentz metric on M^3_1 induced from $\mathbb{M}^4(c)$, we can determine the possible states for a base of the tangent space of M^3_1. For a detailed study, one can refer to the references [12, 13, 18]. In general, a basis $\Omega := \{w_1, w_2, w_3\}$ of a Lorentz linear 3-space is said to be *orthonormal* if it satisfies equalities $\langle w_1, w_1\rangle = -1$, $\langle w_2, w_2\rangle = \langle w_3, w_3\rangle = 1$ and $\langle w_i, w_j\rangle = 0$ for each $i \neq j$. Also, Ω is called *pseudo-orthonormal* if it satisfies $\langle w_1, w_1\rangle = \langle w_2, w_2\rangle = 0$, $\langle w_1, w_2\rangle = -1$ and $\langle w_i, w_3\rangle = \delta^3_i$ for $i = 1, 2, 3$. As usual, δ is the Kronecker Delta.

Associated to a basis chosen on M^3_1, the second fundamental form (shape operator) S has four different matrix forms. When the metric on M^3_1 has diagonal form $\mathcal{G}_1 := \mathrm{diag}[-1,1,1]$, then S is of form $\mathcal{D}_1 = \mathrm{diag}[\lambda_1,\lambda_2,\lambda_3]$ or

$$\mathcal{D}_2 = \mathrm{diag}[\begin{bmatrix} \lambda_1 & \lambda_2 \\ -\lambda_2 & \lambda_1 \end{bmatrix}, \lambda_3],\ (\lambda_2 \neq 0).$$

In the non-diagonal metric case $\mathcal{G}_2 = \mathrm{diag}[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, 1]$ the shape operator is of form

$$\mathcal{D}_3 = \mathrm{diag}[\begin{bmatrix} \lambda_1 + \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \lambda_1 - \frac{1}{2} \end{bmatrix}, \lambda_2] \text{ or } \mathcal{D}_4 = \begin{bmatrix} \lambda & 0 & \frac{\sqrt{2}}{2} \\ 0 & \lambda & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \lambda \end{bmatrix}.$$

When $S = \mathcal{D}_k$, we say that M_1^3 is a $\mathcal{D}_k$-*hypersurface*. To unify symbols, we define the ordered triple $\{\kappa_1;\kappa_2;\kappa_3\}$ of principal curvatures as follows:

$$\{\kappa_1;\kappa_2;\kappa_3\} = \begin{cases} \{\lambda_1;\lambda_2;\lambda_3\} & (\text{if } S = \mathcal{D}_1) \\ \{\lambda_1 + i\lambda_2;\lambda_1 - i\lambda_2;\lambda_3\} & (\text{if } S = \mathcal{D}_2) \\ \{\lambda_1;\lambda_1;\lambda_2\} & (\text{if } S = \mathcal{D}_3) \\ \{\lambda;\lambda;\lambda\} & (\text{if } S = \mathcal{D}_4). \end{cases}$$

We apply the symmetric functions

$$s_0 := 1, s_1 = \sum_{j=1}^{3} \kappa_j, s_2 := \sum_{1 \le i_1 < i_2 \le 3} \kappa_{i_1}\kappa_{i_2} \text{ and } s_3 := \kappa_1\kappa_2\kappa_3,$$

in the definition of *jth mean curvature* of M_1^3 given by $H_j = \frac{1}{\binom{3}{j}} s_j$ (where $j = 0,1,2,3$). When H_{j+1} is identically null, M_1^3 is called *j-minimal*. By definition, a $\mathcal{D}_1$-hypersurface M_1^3 is isoparametric if it has constant principal curvatures. For $k = 2,3,4$, a $\mathcal{D}_k$-hypersurface M_1^3 is isoparametric if the coefficients of minimal polynomial of its shape operator are constant. By a theorem in [12], each timelike hypersurface of $\mathbb{M}_1^4(c)$ with complex principal curvatures is non-isoparametric. The Newton operators on M_1^3 are given by the inductive definitions $\mathrm{N}_0 = I$ and $\mathrm{N}_j = s_j I - \mathrm{S} \circ \mathrm{N}_{j-1}$ for $j = 1,2,3$. As usual, I denotes the identity operator (see [15, 16]).

In special case, H_1 is the ordinary mean curvature H. The second mean curvature H_2 and the normalized scalar curvature R satisfy the equality $H_2 := n(n-1)(1-R)$.

We apply the Newton map on M_1^3 by expression

$$\mathrm{N}_0 = I,\ \mathrm{N}_1 = -s_1 I + \mathrm{S},\ \mathrm{N}_2 = s_2 I - s_1 \mathrm{S} + \mathrm{S}^2. \tag{2.1}$$

We need to certify the matrix form of N_1 and N_2 in four cases $\mathrm{S} = \mathcal{D}_k$ ($k = 1,2,3,4$). When $\mathrm{S} = \mathcal{D}_1$, we have $\mathrm{N}_1 = \mathrm{diag}[\lambda_2 + \lambda_3, \lambda_1 + \lambda_3, \lambda_1 + \lambda_2]$ and $\mathrm{N}_2 = \mathrm{diag}[\lambda_2\lambda_3, \lambda_1\lambda_3, \lambda_1\lambda_2]$.

In the case $\mathrm{S} = \mathcal{D}_2$,

$$\mathrm{N}_1 = \mathrm{diag}[\begin{bmatrix} \lambda_1 + \lambda_3 & -\lambda_2 \\ \lambda_2 & \lambda_1 + \lambda_3 \end{bmatrix}, 2\kappa] \text{ and } \mathrm{N}_2 = \mathrm{diag}[\begin{bmatrix} \lambda_1\lambda_3 & -\lambda_2\lambda_3 \\ \lambda_2\lambda_3 & \lambda_1\lambda_3 \end{bmatrix}, \lambda_1^2 + \lambda_2^2].$$

When $\mathrm{S} = \mathcal{D}_3$,

$$\mathrm{N}_1 = \mathrm{diag}[\begin{bmatrix} \lambda_1+\lambda_2-\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \lambda_1+\lambda_2+\frac{1}{2} \end{bmatrix}, 2\lambda_1] \text{ and}$$

$$\mathrm{N}_2 = \mathrm{diag}[\begin{bmatrix} (\lambda_1-\frac{1}{2})\lambda_2 & -\frac{1}{2}\lambda_2 \\ \frac{1}{2}\lambda_2 & (\lambda_1+\frac{1}{2})\lambda_2 \end{bmatrix}, \lambda_1^2].$$

In the case $\mathrm{S} = \mathcal{D}_4$,

$$\mathrm{N}_1 = \begin{bmatrix} 2\lambda & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 2\lambda & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2\lambda \end{bmatrix} \text{ and } \mathrm{N}_2 = \begin{bmatrix} \lambda^2-\frac{1}{2} & -\frac{1}{2} & -\frac{\sqrt{2}}{2}\lambda \\ \frac{1}{2} & \lambda^2+\frac{1}{2} & \frac{\sqrt{2}}{2}\lambda \\ \frac{\sqrt{2}}{2}\lambda & \frac{\sqrt{2}}{2}\lambda & \lambda^2 \end{bmatrix}.$$

We have the following formulae for the Newton transformations:

$$\begin{aligned} &\mathrm{tr}(\mathrm{N}_j) = c_j H_j,\ \mathrm{tr}(S\circ \mathrm{N}_j) = c_j H_{j+1}, \\ &\mathrm{tr}(S^2\circ \mathrm{N}_1) = 9H_1H_2 - 3H_3,\ \mathrm{tr}(S^2\circ \mathrm{N}_2) = 3H_1H_3, \end{aligned} \tag{2.2}$$

where $j = 0, 1, 2$, $c_0 = c_2 = 3$ and $c_1 = 6$.

Now, we consider the Cheng-Yau operator $\mathrm{C} : \mathcal{C}^\infty(M_1^3) \to \mathcal{C}^\infty(M_1^3)$ given by $\mathrm{C}(f) = \mathrm{tr}(\mathrm{N}_1 \circ \nabla^2 f)$, where, $\nabla^2 f : \chi(M) \to \chi(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f which is defined by $(\nabla^2 f)X = \nabla_X(\nabla f)$ for every smooth vector fields X on M_1^3, where $\nabla f = \sharp df$. In other words, $\mathrm{C}(f)$ is given by $\mathrm{C}(f) = \sum_{i=1}^3 \mu_{i,1}(e_ie_if - \nabla_{e_i}e_if)$.

A hypersurface is said to be $\boldsymbol{H}_2$*-proper* if its second mean curvature vector field satisfies the condition $\mathrm{C}\mathbf{H}_2 = a\mathbf{H}_2$, for a constant number a. Clearly, this condition has a simpler expression by two equations as:

$$\begin{aligned} &\text{(i) } \mathrm{C}H_2 = H_2(a + 9H_1H_2 - 3H_3), \\ &\text{(ii) } \mathrm{N}_2\nabla H_2 = \frac{9}{2}H_2\nabla H_2. \end{aligned} \tag{2.3}$$

Now we recall the definition of an C-finite type hypersurface from [10]. The structure equations of $\mathbb{E}_1^4$ are given by

$$d\omega_i = \sum_{j=1}^4 \omega_{ij}\wedge\omega_j,\ \omega_{ij} + \omega_{ji} = 0$$

and

$$d\omega_{ij} = \sum_{l=1}^4 \omega_{il}\wedge\omega_{lj}.$$

With restriction to M_1^3, we have $\omega_4 = 0$ and then,

$$0 = d\omega_4 = \sum_{i=1}^3 \omega_{4,i}\wedge\omega_i.$$

A lemma due to Cartan gives the decomposition

$$\omega_{4,i} = \sum_{j=1}^3 h_{ij}\omega_j$$

for smooth functions h_{ij} satisfying the equality $B = \sum h_{ij}\omega_i\omega_j e_4$, where B is the second fundamental form of M. The mean curvature H is given by

$$H = \frac{1}{3}\sum_{i=1}^{3} h_{ii}.$$

So, the structure equations of M are

$$d\omega_i = \sum_{j=1}^{3} \omega_{ij} \wedge \omega_j, \ \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l=1}^{3} R_{ijkl}\omega_k \wedge \omega_l$$

for $i, j = 1, 2, 3$, and the Gauss equations $R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk})$, where R_{ijkl} denotes the components of the Riemannian curvature tensor of M. Now, let h_{ijk} denote the covariant derivative of h_{ij}. We have

$$dh_{ij} = \sum_{k=1}^{3} h_{ijk}\omega_k + \sum_{k=1}^{3} h_{kj}\omega_{ik} + \sum_{k=1}^{3} h_{ik}\omega_{jk}.$$

One can choose $e_1, \ldots, e_n$ such that $h_{ij} = \kappa_i\delta_{ij}$. On the other hand, the Levi-Civita connection of M^3 satisfies $\nabla_{e_i} e_j = \sum_k \omega_{jk}(e_i)e_k$, and we have $e_i(k_j) = \omega_{ij}(e_j)(\kappa_i - \kappa_j)$ and

$$\omega_{ij}(e_l)(\kappa_i - \kappa_j) = \omega_{il}(e_j)(\kappa_i - \kappa_l)$$

whenever i, j, l are distinct.

Definition 2.1. An isometrically immersed hypersurface $\psi : M_1^3 \to \mathbb{M}_1^4(c)$ (for $c = \pm 1$) is said to be of C-finite type if ψ has a finite decomposition $\psi = \sum_{i=0}^{m} \psi_i$, for some positive integer m, satisfying the condition $\mathrm{C}\psi_i = \gamma_i\psi_i$, for some real numbers $\gamma_i \in \mathbb{R}$ and vector mappings $\psi_i : M_1^3 \to \mathbb{E}_s^5$ (where $s = 1$ or $s = 2$) for $i = 1, 2, \cdots, m$, and ψ_0 is a constant vector. M^n is called C-m-type if all γ_i's are mutually distinct. An C-m-type hypersurface is said to be null if for at least one i $(1 \leq i \leq m)$ we have $\gamma_i = 0$.

3. $\mathbf{H}_2$-proper Hypersurfaces

In this section, we study the $\mathbf{H}_2$-proper timelike hypersurfaces of $\mathbb{M}_1^4(c)$.

Theorem 3.1. *Every $\boldsymbol{H}_2$-proper $\mathcal{D}_1$-hypersurface $\boldsymbol{x} : M_1^3 \to \mathbb{M}_1^4(c)$ satisfies one of the following conditions*:

(i) *M_1^3 is C-biharmonic,*
(ii) *M_1^3 is of C-1-type,*
(iii) *M_1^3 is of C-null-2-type.*

Proof. By assumption, $\mathbf{H}_2$ is proper, which means that it satisfies condition $\mathrm{C}\mathbf{H}_2 = c\mathbf{H}_2$ for a constant real number c. If $c = 0$, then M_1^3 is a C-biharmonic hypersurface, which gives (i). In the case $c \neq 0$, taking $\mathbf{x}_1 = \frac{1}{c}\mathrm{C}\mathbf{x}$ and $\mathbf{x}_0 = \mathbf{x} - \mathbf{x}_1$, we have

$$\mathrm{C}\mathbf{x}_1 = \frac{1}{c}\mathrm{C}^2\mathbf{x} = \frac{6}{c}\mathrm{C}\mathbf{H}_2 = 6\mathbf{H}_2 = \mathrm{C}\mathbf{x}.$$

Hence, M is either of C-1-type or of C-null-2-type, depending on $\mathbf{x}_0$ is a constant or non-constant. The converse is easy to verify. □

Theorem 3.2. *The C-finite type Lorentz hypersurfaces of the 4-dimensional Lorentz space form can not be C-biharmonic.*

Proof. Let $\mathbf{x} : M_1^3 \to \mathbb{M}_1^4(c)$ be a C-k-type Lorentzian hypersurface. There is a decomposition as

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_1 + \cdots + \mathbf{x}_k, \tag{3.1}$$

with $\mathrm{C}\mathbf{x}_0 = 0$ and $\mathrm{C}\mathbf{x}_i = \lambda_i \mathbf{x}_i$ for nonzero distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ of C. From (3.1) we get

$$\mathrm{C}^s\mathbf{x} = \lambda_1^s\mathbf{x}_1 + \cdots + \lambda_k^s\mathbf{x}_k, \tag{3.2}$$

for $s = 1, 2, 3, \ldots$.

Now, assume that $\mathbf{x}$ is C-biharmonic (i.e $\mathrm{C}^2\mathbf{x} = \mathrm{C}^3\mathbf{x} = 0$). So, from (3.2) we get

$$\begin{cases} \lambda_1^2\mathbf{x}_1 + \cdots + \lambda_k^2\mathbf{x}_k = 0 \\ \lambda_1^3\mathbf{x}_1 + \cdots + \lambda_k^3\mathbf{x}_k = 0, \end{cases}$$

Since $\lambda_1, \ldots, \lambda_k$ are mutually distinct eigenvalues of C, the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are linearly independent. Hence, we have $\lambda_1 = \cdots = \lambda_k = 0$, which is a contradiction. □

Proposition 3.3. *Each* $\boldsymbol{H}_2$*-proper* $\mathcal{D}_1$*-hypersurface of* $\mathbb{M}_1^4(c)$ *with three distinct principal curvatures has constant scalar curvature.*

Proof. Let $\mathbf{x} : M_1^3 \to \mathbb{M}_1^4(c)$ be such a hypersurface. It is enough to prove that H_2 is constant. By the method of reasoning on reductio ad absurdum, we assume that $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$ is non-empty. Since the shape operator S is of type $\mathcal{D}_1$, for mutually distinct λ_i's $(i = 1, 2, 3)$ we have $\mathrm{S}e_i = \lambda_i e_i$ and $\mathrm{N}_2 e_i = \mu_{i,2} e_i$. By decomposition $\nabla H_2 = \sum_{i=1}^{3} \epsilon_i e_i(H_2) e_i$, the condition (2.3)(ii) gives

$$e_i(H_2)(\mu_{i,2} - \frac{9}{2}H_2) = 0, \ (i = 1, 2, 3). \tag{3.3}$$

Around each point in U, there exists an open neighborhood on which we have $e_i(H_2) \neq 0$ for at least one i. We can assume (without loss of generality) that $e_1(H_2) \neq 0$ and then we get $\mu_{1,2} = \frac{9}{2}H_2$ (locally) on U, which gives $\lambda_2\lambda_3 = \frac{9}{2}H_2$.

The continuation of the proof is the confirmation of several equalities, which is done in three steps.

Step 1: $e_2(H_2) = e_3(H_2) = 0$.
If $e_2(H_2) \neq 0$ or $e_3(H_2) \neq 0$, then by (3.3) we get $\mu_{1,2} = \mu_{2,2} = \frac{9}{2}H_2$ or $\mu_{1,2} = \mu_{3,2} = \frac{9}{2}H_2$, which give $(\lambda_1 - \lambda_2)\lambda_3 = 0$ or $(\lambda_1 - \lambda_3)\lambda_2 = 0$. Since λ_i's are distinct, we have $\lambda_3 = 0$ or $\lambda_2 = 0$, and then $H_2 = 0$ on U. The result contradicts with the definition of U.

Step 2: $e_2(\lambda_1) = e_3(\lambda_1) = 0$.
From the assumption that H is constant, it follows that

$$e_2(\lambda_1) = e_2(3H - \lambda_1 - \lambda_2) = -e_2(\lambda_1) - e_2(\lambda_2).$$

On the other hand, by Step 1 we have $e_2(H_2) = 0$ and $\lambda_2\lambda_3 = \frac{9}{2}H_2$ and then we have

$$e_2(\lambda_1\lambda_3) + e_2(\lambda_1\lambda_2) = e_2(3H_2 - \frac{9}{2}H_2) = 0,$$

which gives $\lambda_1 e_2(\lambda_2 + \lambda_3) + (\lambda_2 + \lambda_3)e_2\lambda_1 = 0$, and then we have

$$\begin{aligned}&\lambda_1 e_2(3H - \lambda_1) + (\lambda_2 + \lambda_3)e_2\lambda_1\\ =&\lambda_1 e_2(-\lambda_1) + (\lambda_2 + \lambda_3)e_2\lambda_1\\ =&(-\lambda_1 + \lambda_2 + \lambda_3)e_2\lambda_1 = 0.\end{aligned}$$

Therefore, assuming $e_2(\lambda_1) \neq 0$, we get $\lambda_1 = \lambda_2+\lambda_3$ which gives a contradiction

$$e_2(\lambda_1) = e_2(\lambda_2 + \lambda_3) = e_2(3H - \lambda_1) = -e_2(\lambda_1).$$

Consequently, $e_2(\lambda_1) = 0$.

Similarly, one can show $e_3(\lambda_1) = 0$.

Step 3: $e_2(\lambda_3) = e_3(\lambda_2) = 0$.
From the covariant derivatives

$$\nabla_{e_i} e_j = \sum_{k=1}^{3} \omega_{ij}^k e_k \ (i, j = 1, 2, 3),$$

using the compatibility condition $\nabla_{e_k}\langle e_i, e_j\rangle = 0$, we get

$$\omega_{ki}^i = 0,\ \omega_{ki}^j + \omega_{kj}^i = 0 \ (i, j, k = 1, 2, 3),$$

and by the Codazzi equation

$$(\nabla_V S)W = \nabla_W S)V \ (\forall V, W \in \chi(M)),$$

for distinct $i, j, k \in \{1, 2, 3\}$, we obtain

$$\begin{aligned}&\text{(i) } e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j,\\ &\text{(ii) } (\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j.\end{aligned}$$

A simple computation on the components of the identity $(\nabla_{e_i} S)e_j - (\nabla_{e_j} S)e_i \equiv 0$ for distinct $i, j = 1, 2, 3,$ gives $[e_2,\ e_3](H_2) = 0$, $\omega_{12}^1 = \omega_{13}^1 = \omega_{13}^2 = \omega_{21}^3 =$

$\omega_{32}^1 = 0$ and

$$\omega_{21}^2 = \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2},\ \omega_{31}^3 = \frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3},\ \omega_{23}^2 = \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2},\ \omega_{32}^3 = \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}.$$

Hence, we have $\nabla_{e_1} e_k = 0$ for $k = 1, 2, 3$, and

(3.4)

$$\nabla_{e_2} e_1 = \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2} e_2,\ \nabla_{e_3} e_1 = \frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3} e_3, \nabla_{e_2} e_2 = \frac{e_1(\lambda_2)}{\lambda_2 - \lambda_1} e_1,$$
$$\nabla_{e_3} e_2 = \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3} e_3, \nabla_{e_2} e_3 = \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2} e_2,\ \nabla_{e_3} e_3 = \frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1} e_1 + \frac{e_2(\lambda_3)}{\lambda_3 - \lambda_2} e_2.$$

The Gauss equations $\langle R(e_2, e_3)e_1,\ e_2\rangle$ and $\langle R(e_2, e_3)e_1,\ e_3\rangle$ give

$$e_3\left(\frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}\right) = \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2}\left(\frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3} - \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}\right), \tag{3.5}$$

$$e_2\left(\frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3}\right) = \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3} - \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}\right). \tag{3.6}$$

Also equations $\langle R(e_1, e_2)e_1,\ e_2\rangle$ and $\langle R(e_3, e_1)e_1,\ e_3\rangle$ give

(3.7)

$$e_1\left(\frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}\right) + \left(\frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}\right)^2 = \lambda_1\lambda_2,\ e_1\left(\frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3}\right) + \left(\frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1}\right)^2 = \lambda_1\lambda_3.$$

Finally, from $\langle R(e_3, e_1)e_2, e_3\rangle$ we have

$$e_1\left(\frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}\right) = \frac{e_1(\lambda_3)e_2(\lambda_3)}{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3)}. \tag{3.8}$$

On the other hand, from Step 1 we obtain

(3.9)

$$-\mu_{1,1} e_1 e_1(H_2) + \left(\mu_{2,1}\frac{e_1(\lambda_2)}{\lambda_2 - \lambda_1} + \mu_{3,1}\frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1}\right) e_1(H_2) - 9H_2^2(H - \frac{3}{2}\lambda_1) = 0.$$

By covariant derivative of (3.9) along e_2 and e_3 respectively, and using (3.5), (3.6) we get

$$e_2\left(\frac{e_1(\lambda_2)}{\lambda_2 - \lambda_1}\right) = \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3} - \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}\right), \tag{3.10}$$

$$e_3\left(\frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1}\right) = \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2}\left(\frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2} - \frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3}\right). \tag{3.11}$$

Using (3.4), we find that

$$[e_1,\ e_2] = \frac{e_1(\lambda_2)}{\lambda_2 - \lambda_1} e_2. \tag{3.12}$$

Applying both sides of the equality (3.12) on $\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}$, using (3.10), (3.7), and (3.8), we deduce that

$$\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)=0. \tag{3.13}$$

(3.13) shows that $e_2(\lambda_3)=0$ or

$$\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}=\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}. \tag{3.14}$$

From equation (3.14), by differentiating on its both sides along e_1 and applying (3.7), we get $\lambda_2=\lambda_3$, which is a contradiction, so we have to confirm the result $e_2(\lambda_3)=0$.

Analogously, using (3.4), we find that $[e_1,\ e_3]=\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}e_3$. By a similar manner, we deduce that

$$\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\left(\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}+\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right)=0, \tag{3.15}$$

and one can show that $e_3(\lambda_2)$ necessarily has to be vanished.

Hence, we have obtained $e_2(\lambda_3)=e_3(\lambda_2)=0$ which, by applying the Gauss equation for $<R(e_2,e_3)e_1,e_3>$, gives the following equality

$$\frac{e_1(\lambda_3)e_1(\lambda_2)}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)}=\lambda_2\lambda_3. \tag{3.16}$$

Finally, using (3.7), differentiating (3.16) along e_1 gives

$$\lambda_2\lambda_3\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)=0, \tag{3.17}$$

which implies $\lambda_2\lambda_3=0$ (since we have seen above that $\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)\neq 0$). Therefore, we obtain $H_2=0$ on U, which is a contradiction. Hence H_2 is constant on M_1^3. □

Theorem 3.4. *Every $\boldsymbol{H}_2$-proper $\mathcal{D}_1$-hypersurface of $\mathbb{M}_1^4(c)$ with constant mean curvature and three distinct principal curvatures is* 1*-minimal.*

Proof. By Proposition 3.3, the 2nd mean curvature H_2 is constant. We prove that $H_2\equiv 0$. Assume that $H_2\neq 0$ on a neighborhood around a point. The condition (2.3)(i) gives that H_3 is constant. Hence M_1^3 is isoparametric because H_1,H_2 and H_3 are constant. Using Corollary 2.7 in [12], we know that every isoparametric $\mathcal{D}_1$-hypersurface may not have more than one nonzero principal curvature. Therefore, we have a contradiction with the assumption that, M has three distinct principal curvatures. Hence $H_2\equiv 0$. □

Proposition 3.5. *Each $\boldsymbol{H}_2$-proper $\mathcal{D}_1$-hypersurface of $\mathbb{M}_1^4(c)$ with exactly two distinct principal curvatures has constant scalar curvature.*

Proof. Let $\mathbf{x} : M_1^3 \to \mathbb{M}_1^4(c)$ be such a hypersurface. It is enough to prove that H_2 is constant. In the method of reasoning on reductio ad absurdum, we assume that $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$ is non-empty. The shape operator S is of type $\mathcal{D}_1$ with two distinct eigenfunctions η and λ of multiplicities 1 and 2, respectively. Hence, we have $\mathrm{S}e_1 = \lambda e_1$, $\mathrm{S}e_2 = \lambda e_2$, $\mathrm{S}e_3 = \eta e_3$ and $\mathrm{N}_2 e_i = \mu_{i,2} e_i$ for $i = 1, 2, 3$, where

$$\mu_{1,2} = \mu_{2,2} = \lambda\eta,\ \mu_{3,2} = \lambda^2,\ 3H_2 = \lambda^2 + 2\lambda\eta. \tag{3.18}$$

By (2.3)(ii), we get $\mathrm{N}_2(\nabla H_2) = \frac{9}{2} H_2 \nabla H_2$, which using

$$\nabla H_2 = \sum_{i=1}^{3} \epsilon_i \langle \nabla H_2, e_i \rangle e_i,$$

gives

$$\epsilon_i \langle \nabla H_2, e_i \rangle (\mu_{i,2} - \frac{9}{2} H_2) = 0$$

on $\mathcal{U}$ for $i = 1, 2, 3$. Hence, for each i if $\langle \nabla H_2, e_i \rangle \neq 0$ on $\mathcal{U}$, then we get

$$\mu_{i,2} = \frac{9}{2} H_2. \tag{3.19}$$

Since $\nabla H_2 \neq 0$ on U, one or both of the following cases hold.

Case 1. $\langle \nabla H_2, e_i \rangle \neq 0$, for $i = 1$ or $i = 2$. By equalities (3.18) and (3.19), we obtain

$$\lambda\eta = \frac{9}{2}(\frac{2}{3}\lambda\eta + \frac{1}{3}\lambda^2),$$

which gives

$$\lambda(2\eta + \frac{3}{2}\lambda) = 0. \tag{3.20}$$

If $\lambda = 0$ then $H_2 = 0$. Otherwise, we get $\eta = -\frac{3}{4}\lambda$, $H_2 = -\frac{1}{6}\lambda^2$.

Case 2. $\langle \nabla H_2, e_3 \rangle \neq 0$. By equalities (3.18) and (3.19), we obtain

$$\lambda^2 = \frac{9}{2}(\frac{2}{3}\lambda\eta + \frac{1}{3}\lambda^2),$$

which gives

$$\lambda(3\eta + \frac{1}{2}\lambda) = 0. \tag{3.21}$$

If $\lambda = 0$ then $H_2 = 0$. Otherwise, we have $\eta = -\frac{1}{6}\lambda$, $H_2 = \frac{2}{9}\lambda^2$.

Both cases require the same calculation, so we consider for instance Case 2. Let us denote the maximal integral submanifold through $x \in \mathrm{U}$, corresponding to λ by $\mathrm{U}_1^{n-1}(x)$. We write

$$d\lambda = \sum_{i=1}^{3} \lambda_i \omega_i \ d\eta = \sum_{j=1}^{3} \eta_j \omega_j. \tag{3.22}$$

Then, we have $\lambda_1 = \lambda_2 = 0$. We can assume that $\lambda > 0$ on U, then we have

$$\eta = \frac{-1}{6}\lambda < 0. \tag{3.23}$$

From the formula of dh_{ij} in Section 2, we obtain

$$\sum_{k=1}^{3} h_{ijk}\omega_k = \delta_{ij} d\lambda_j + (\lambda_i - \lambda_j)\omega_{ij}, \tag{3.24}$$

for $i, j, k = 1, 2, 3$. Here, we take $a, b, c = 1, 2$.
From (3.22) and (3.24), we have

$$\begin{aligned} &h_{12k} = h_{21k} = 0, \\ &h_{aab} = 0,\ h_{aa3} = \lambda_{,3}, \\ &h_{33a} = 0,\ h_{333} = \mu_{,3}. \end{aligned} \tag{3.25}$$

From

$$\sum_{i=1}^{3} h_{a3i}\omega_i = dh_{a3} + \sum_{i=1}^{3} h_{i3}\omega_{ia} + \sum_{i=1}^{3} h_{ai}\omega_{i3} = (\lambda - \eta)\omega_{a3},$$

and equality (3.23) we obtain

$$\omega_{a3} = \frac{\lambda_{,3}}{\lambda - \eta}\omega_a = \frac{6\lambda_{,3}}{7\lambda}\omega_a. \tag{3.26}$$

Therefore we have

$$d\omega_3 = \sum_{a=1}^{2} \omega_{3a} \wedge \omega_a = 0.$$

Notice that we may consider λ to be locally a function of the parameter s, where s is the arc length of an orthogonal trajectory of the family of the integral submanifolds corresponding to λ. We may put $\omega_3 = ds$.
Thus, for $\lambda = \lambda(s)$, we have

$$d\lambda = \lambda_3 ds,\ \lambda_3 = \lambda'(s),$$

so from (3.26), we get

$$\omega_{a3} = \frac{\lambda_3}{\lambda - \eta}\omega_a = \frac{6\lambda'(s)}{7\lambda}\omega_a. \tag{3.27}$$

According to the structure equations of $\mathbb{E}^4_1$ and (3.27), we may compute
(3.28)

$$\text{(i) } \mathrm{d}\omega_{\mathrm{a3}} = \sum_{\mathrm{b}=1}^{2} \omega_{\mathrm{ab}} \wedge \omega_{\mathrm{b3}} + \omega_{\mathrm{a4}} \wedge \omega_{43} = \left(\frac{6\lambda'}{7\lambda}\right) \sum_{\mathrm{b}=1}^{2} \omega_{\mathrm{ab}} \wedge \omega_{\mathrm{b}} - \lambda\eta\omega_{\mathrm{a}} \wedge \mathrm{ds},$$

$$\begin{aligned} \text{(ii) } \mathrm{d}\omega_{\mathrm{a3}} &= \mathrm{d}\left\{\frac{6\lambda'}{7\lambda}\omega_{\mathrm{a}}\right\} = \left(\frac{6\lambda'}{7\lambda}\right)' \mathrm{ds} \wedge \omega_{\mathrm{a}} + \left(\frac{6\lambda'}{7\lambda}\right) \mathrm{d}\omega_{\mathrm{a}} \\ &= \left\{-\left(\frac{6\lambda'}{7\lambda}\right)' + \left(\frac{6\lambda'}{7\lambda}\right)^2\right\} \omega_a \wedge ds + \left(\frac{6\lambda'}{7\lambda}\right) \sum_{b=1}^{2} \omega_{ab} \wedge \omega_b. \end{aligned}$$

Comparing equalities (3.28)(i) and (3.28)(ii), we get $\left(\frac{6\lambda'}{7\lambda}\right)' - \left(\frac{6\lambda'}{7\lambda}\right)^2 - \lambda\eta = 0$, which, by combining with (3.23), gives

$$\left(\frac{6\lambda'}{7\lambda}\right)' - \left(\frac{6\lambda'}{7\lambda}\right)^2 - \left(\frac{-1}{6}\right)\lambda^2 = 0. \tag{3.29}$$

Defining function $\beta(s) := \left(\frac{1}{\lambda(s)}\right)^{\frac{6}{7}}$ for $s \in (-\infty, +\infty)$, from (3.29) we get $\beta'' = \left(\frac{1}{6}\right)\beta^{\frac{-8}{6}}$, which by integrating, gives $(\beta')^2 = -\beta^{\frac{-2}{6}} + c$, where c is the constant of integration. The last equation is equivalent to

$$(\lambda')^2 = -\left(\frac{7}{6}\right)^2 \lambda^4 + c\left(\frac{7}{6}\right)^2 \lambda^{\frac{26}{7}}. \tag{3.30}$$

Now, in order to compare two sides of condition (2.3)(i), we need to compute $\nabla_{e_i}\nabla H_2$ and $P_1(e_i)$ for $i = 1, 2, 3$. From (3.20) we have $\nabla H_2 = \frac{4}{9}\lambda\lambda' e_3$, which by using (3.27), gives

$$\begin{aligned} \nabla_{e_a}\nabla H_2 &= \frac{4}{9}\lambda\lambda'\nabla_{e_a} e_3 = \frac{4}{9}\lambda^r\lambda'\sum_b \omega_{3b}(e_a)e_b = -\frac{8}{21}\lambda'^2 e_a, \\ \nabla_{e_3}\nabla H_2 &= \frac{4}{9}\nabla_{e_3}(\lambda\lambda' e_3) = \frac{4}{9}\lambda'^2 e_3 + \frac{4}{9}\lambda\lambda'' e_3. \end{aligned} \tag{3.31}$$

By using (3.18) and (3.23), we compute $P_1(e_a)$ and $P_1(e_3)$.

$$P_1(e_1) = \frac{5}{6}\lambda e_1,\ P_1(e_2) = \frac{5}{6}\lambda e_2\ P_1(e_3) = 2\lambda e_3. \tag{3.32}$$

From (3.31) and (3.32), we get

$$\Box(H_2) = 6H_2\left(\frac{-10(\lambda')^2}{21\lambda} + \frac{2(\lambda')^2}{3\lambda} + \frac{2}{3}\lambda''\right). \tag{3.33}$$

From 2.3(i), we have $\Box(H_2) = H_2 tr(S^2 \circ P_1) = 2H_2\frac{11}{6}\lambda^3$, which combining with (3.33), gives

$$\lambda\lambda'' + \left(1 + \frac{-5}{7}\right)\lambda'^2 - 2\frac{33}{12}\lambda^4 = 0. \tag{3.34}$$

On the other hand, the equality (3.29) is equivalent to

$$\lambda\lambda'' = \frac{13}{7}\lambda'^2 + \frac{-7}{36}\lambda^4. \tag{3.35}$$

Now, substituting (3.35) and (3.34), we obtain

$$\frac{15}{7}\lambda'^2 + \frac{191}{36}\lambda^4 = 0. \tag{3.36}$$

From equations (3.30), (3.36) and (3.20), we get that H_2 is locally constant on U, which is a contradiction with the definition of U. Hence H_2 is constant on M. By a similar discussion, one can get the same result in Case 1. □

Theorem 3.6. *Each $\boldsymbol{H}_2$-proper $\mathcal{D}_1$-hypersurface of $\mathbb{M}_1^4(c)$ with constant mean curvature and at most two distinct principal curvatures is 1-minimal.*

Proof. The proof is similar to the proof of Theorem 3.4. □

Proposition 3.7. *Each $\boldsymbol{H}_2$-proper $\mathcal{D}_2$-hypersurface of $\mathbb{M}_1^4(c)$ with constant ordinary mean curvature and a constant real principal curvature has constant second and third mean curvatures.*

Proof. In the first stage, we show that the open subset $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$ is empty. By the method of reasoning on reductio ad absurdum, we assume that $\mathcal{U}$ is non-empty. Since S is of type $\mathcal{D}_2$, we have $\mathrm{S}e_1 = \kappa e_1 - \lambda e_2$, $\mathrm{S}e_2 = \lambda e_1 + \kappa e_2$, $\mathrm{S}e_3 = \eta e_3$ and then, $\mathrm{N}_2 e_1 = \kappa\eta e_1 + \lambda\eta e_2$, $\mathrm{N}_2 e_2 = -\lambda\eta e_1 + \kappa\eta e_2$ and $\mathrm{N}_2 e_3 = (\kappa^2 + \lambda^2) e_3$.

The condition (2.3)(ii) by using $\nabla H_2 = \sum_{i=1}^3 \epsilon_i e_i(H_2) e_i$ gives

$$\text{(3.37)}\qquad \begin{aligned} &\text{(i) } \epsilon_1 e_1(H_2)(\kappa\eta - \frac{9}{2}H_2) = \epsilon_2 e_2(H_2)\lambda\eta,\\ &\text{(ii) } \epsilon_2 e_2(H_2)(\kappa\eta - \frac{9}{2}H_2) = -\epsilon_1 e_1(H_2)\lambda\eta,\\ &\text{(iii) } \epsilon_3 e_3(H_2)(\kappa^2 + \lambda^2 - \frac{9}{2}H_2) = 0. \end{aligned}$$

The continuation of the proof is the confirmation of several equalities, which is done in two steps.

Step 1: $e_1(H_2) = e_2(H_2) = 0$.
If $e_1(H_2) \neq 0$, then we divide both sides of equations (3.37)(i), (ii) by $\epsilon_1 e_1(H_2)$, so we get

$$\text{(3.38)}\qquad \begin{aligned} &\text{(i) } \kappa\eta - \frac{9}{2}H_2 = \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}\lambda\eta,\\ &\text{(ii) } \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}(\kappa\eta - \frac{9}{2}H_2) = -\lambda\eta. \end{aligned}$$

Substituting (3.38)(i) in (3.38)(ii), we get $\lambda\eta(1 + (\frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)})^2) = 0$, which gives $\lambda\eta = 0$. Since $\lambda \neq 0$ is assumed, we have $\eta = 0$. So, by (3.38)(i), we get $H_2 = 0$.

In a Similar way, if $e_2(H_2) \neq 0$, then by dividing both sides of equations (3.37)(i), (ii) by $\epsilon_2 e_2(H_2)$ we get

$$\text{(3.39)}\qquad \begin{aligned} &\text{(i) } \frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}(\kappa\eta - \frac{9}{2}H_2) = \lambda\eta,\\ &\text{(ii) } \kappa\eta - \frac{9}{2}H_2 = -\frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}\lambda\eta, \end{aligned}$$

which, by substituting (3.39)(i) in (3.39)(ii), we have $\lambda\eta(1 + (\frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)})^2) = 0$, then $\lambda\eta = 0$. Since by assumption $\lambda \neq 0$, we get $\eta = 0$. So, by (3.39)(ii), we have $H_2 = 0$.

Step 2: $e_3(H_2) = 0$.
If $e_3(H_2) \neq 0$, then from equality (3.37)(iii) we have $\kappa^2+\lambda^2 = \frac{9}{2}H_2$, which gives $\kappa^2+\lambda^2 = -6\kappa\eta$, where $\eta = 3H_1 - 2\kappa$ and η and H_1 are assumed to be constant on U. So, κ is also constant on U, and then, we obtain $H_2 = \frac{-4}{3}\kappa\eta = \frac{8}{3}\kappa^2 - 4H_1\kappa$ and $H_3 = -6\kappa\eta^2 = -6\kappa(3H_1 - 2\kappa)^2$ are constant on U. □

Theorem 3.8. *Every $\boldsymbol{H}_2$-proper $\mathcal{D}_2$-hypersurface of $\mathbb{M}_1^4(c)$ with constant ordinary mean curvature and a constant real principal curvature is 1-minimal and 2-minimal.*

Proof. By Proposition 3.7, the second mean curvature of M_1^3 is constant, which gives $\mathrm{C}(H_2) = 0$. Then, by (2.3)(i), we have $9H_1H_2^2 - 3H_2H_3 = 0$, which gives $(7\eta - 4\kappa)\kappa^2\eta^2 = 0$.

Now, if $7\eta = 4\kappa$, then from $\kappa^2 + \lambda^2 = -6\kappa\eta$ we get $\frac{31}{7}\kappa^2 + \lambda^2 = 0$, and then $\kappa = \lambda = 0$, which gives $H_2 = H_3 = 0$. Also, if $\kappa^2\eta^2 = 0$, then we have $H_2 = H_3 = 0$. □

Proposition 3.9. *Every $\boldsymbol{H}_2$-proper $\mathcal{D}_3$-hypersurface of $\mathbb{M}_1^4(c)$ with constant ordinary mean curvature has constant second mean curvature.*

Proof. Suppose that, H_2 is non-constant. Considering the open subset $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$, we try to show $\mathcal{U} = \emptyset$. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1, e_2, e_3\}$ on M, the shape operator S has the matrix form $\mathcal{D}_3$, such that $\mathrm{S}e_1 = (\kappa + \frac{1}{2})e_1 - \frac{1}{2}e_2$, $\mathrm{S}e_2 = \frac{1}{2}e_1 + (\kappa - \frac{1}{2})e_2$, $\mathrm{S}e_3 = \lambda e_3$ and then, we have $\mathrm{N}_2e_1 = (\kappa - \frac{1}{2})\lambda e_1 + \frac{1}{2}\lambda e_2$, $\mathrm{N}_2e_2 = -\frac{1}{2}\lambda e_1 + (\kappa + \frac{1}{2})\lambda e_2$ and $\mathrm{N}_2e_3 = \kappa^2 e_3$.

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^3 \epsilon_i e_i(H_2)e_i$, from condition (2.3)(ii) we get

$$
\begin{aligned}
&\text{(i) } \epsilon_1 e_1(H_2)[(\kappa - \frac{1}{2})\lambda - \frac{9}{2}H_2] = \epsilon_2 e_2(H_2)\frac{\lambda}{2},\\
&\text{(ii) } \epsilon_2 e_2(H_2)[(\kappa + \frac{1}{2})\lambda - \frac{9}{2}H_2] = -\epsilon_1 e_1(H_2)\frac{\lambda}{2},\\
&\text{(iii) } \epsilon_3 e_3(H_2)(\kappa^2 - \frac{9}{2}H_2) = 0.
\end{aligned} \tag{3.40}
$$

Now, we prove a simple claim.

Claim : $e_1(H_2) = e_2(H_2) = e_3(H_2) = 0$.
If $e_1(H_2) \neq 0$, then by dividing both sides of equalities (3.40)(i), (ii) by $\epsilon_1 e_1(H_2)$ we get

$$
\begin{aligned}
&\text{(i) } (\kappa - \frac{1}{2})\lambda - \frac{9}{2}H_2 = \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}\frac{\lambda}{2},\\
&\text{(ii) } \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}[(\kappa + \frac{1}{2})\lambda - \frac{9}{2}H_2] = -\frac{\lambda}{2},
\end{aligned} \tag{3.41}
$$

which, by substituting (i) in (ii), gives $\frac{\lambda}{2}(1 + u)^2 = 0$, where $u := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$. Then $\lambda = 0$ or $u = -1$. If $\lambda = 0$, then we get $H_2 = 0$ from (3.41)(i). Also, by assumption $\lambda \neq 0$ we get $u = -1$ which gives $\kappa\lambda = \frac{9}{2}H_2$. Then $\kappa(3\kappa + 4\lambda) = 0$ and finally $\kappa = -\frac{4}{3}\lambda$ (since $\kappa = 0$ gives $H_2 = 0$ again). Hence, we have $H_2 = \frac{2}{9}\kappa\lambda = -\frac{8}{27}\lambda^2$ and $H_1 = -\frac{5}{9}\lambda$, and since H_1 is assumed to be constant, H_2 has to be constant and we have $e_1(H_2) = 0$, which is a contradiction. Therefore, the first claim is proved. The second claim (i.e. $e_2(H_2) = 0$) can be proven by a similar manner.

Now, if $e_3(H_2) \neq 0$, then by (3.40)(iii) we get $\kappa^2 = \frac{9}{2}H_2$, then $\kappa(\kappa+6\lambda) = 0$, which gives $\kappa = 0$ or $\kappa = -6\lambda$. If $\kappa = 0$, then $H_2 = 0$, and if $\kappa = -6\lambda$ then since $H_1 = -\frac{11}{3}\lambda$ is assumed to be constant, we get that H_2 is constant and then $e_3(H_2) = 0$. Which is a contradiction, so we have $e_3(H_2) = 0$. □

Theorem 3.10. *Let $\boldsymbol{x} : M_1^3 \to \mathbb{E}_1^4$ be a $\mathcal{D}_3$-hypersurface with proper second mean curvature vector field. If M_1^3 has constant ordinary mean curvature, then it is* 1*-minimal.*

Proof. By assumption H_1 is assumed to be constant and then, by Proposition 3.9 it is proved that H_2 has to be constant. We claim that H_2 is null. Since the shape operator is of type $\mathcal{D}_3$, there exist two possible cases as:

Case 1: M_1^3 has a principal curvature κ of multiplicity 3;

Case 2: M_1^3 has two principal curvatures κ and λ of multiplicities 2 and 1, respectively.

In Case 1, we have $H_1 = \kappa$, $H_2 = \kappa^2$ and $H_3 = \kappa^3$. By (2.3)(i), we have $3H_1H_2^2 = H_2H_3$, which gives $\kappa^5 = 0$, and then $H_2 = 0$.

In Case 2, we have $H_1 = \frac{1}{3}(2\kappa + \lambda)$, $H_2 = \frac{1}{3}(\kappa^2 + 2\kappa\lambda)$ and $H_3 = \kappa^2\lambda$. We assume that $H_2 \neq 0$ and continue in two subcases as follow. Since $H_2 \neq 0$, then $\kappa \neq 0$ and by using (2.3)(i) we obtain that H_3 is constant. Therefore, all of mean curvatures H_i (for $i = 1, 2, 3$) are constant, which means that M_1^3 is isoparametric. By Corollary 2.7 in [12], an isoparametric Lorentzian hypersurface of Case $\mathcal{D}_3$ in the Einstein space has at most one nonzero principal curvature, so we get $\lambda = 0$. Then $H_1 = \frac{2}{3}\kappa$, $H_2 = \frac{1}{3}\kappa^2$ and $H_3 = 0$, hence, by (2.3)(i), we get $\kappa = 0$, which contradicts with the assumption of this case. Therefore $H_2 = 0$. □

Proposition 3.11. *Let $\boldsymbol{x} : M_1^3 \to \mathbb{M}_1^4(c)$ be a $\mathcal{D}_4$-hypersurface with proper second mean curvature vector field. Then its second mean curvature is constant.*

Proof. Suppose that, H_2 be non-constant. Considering the open subset $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$, we try to show $\mathcal{U} = \emptyset$. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1, e_2, e_3\}$ on M, the shape operator S has the matrix form $\tilde{B}_3$, such that $Se_1 = \kappa e_1 + \frac{\sqrt{2}}{2}e_3$, $Se_2 = \kappa e_2 - \frac{\sqrt{2}}{2}e_3$, $Se_3 = -\frac{\sqrt{2}}{2}e_1 - \frac{\sqrt{2}}{2}e_2 + \kappa e_3$ and then, we have $P_2e_1 = (\kappa^2 - \frac{1}{2})e_1 - \frac{1}{2}e_2 - \frac{\sqrt{2}}{2}\kappa e_3$, $P_2e_2 = \frac{1}{2}e_1 + (\kappa^2 + \frac{1}{2})e_2 + \frac{\sqrt{2}}{2}\kappa e_3$ and $P_2e_3 = \frac{\sqrt{2}}{2}\kappa e_1 + \frac{\sqrt{2}}{2}\kappa e_2 + \kappa^2 e_3$.

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{3} \epsilon_i e_i(H_2) e_i$, from condition (2.3)(ii) we get

$$(3.42)\quad \begin{aligned} &\text{(i) } \epsilon_1 e_1(H_2)[(\kappa^2 - \frac{1}{2}) - \frac{9}{2}H_2] + \frac{1}{2}\epsilon_2 e_2(H_2) + \frac{\sqrt{2}}{2}\epsilon_3 e_3(H_2)\kappa = 0\\ &\text{(ii) } \frac{-1}{2}\epsilon_1 e_1(H_2) + \epsilon_2 e_2(H_2)[(\kappa^2 + \frac{1}{2}) - \frac{9}{2}H_2] + \frac{\sqrt{2}}{2}\epsilon_3 e_3(H_2)\kappa = 0\\ &\text{(iii) } \epsilon_1 e_1(H_2)\frac{-\sqrt{2}}{2}\kappa + \epsilon_2 e_2(H_2)\frac{\sqrt{2}}{2}\kappa + \epsilon_3 e_3(H_2)(\kappa^2 - \frac{9}{2}H_2) = 0. \end{aligned}$$

Now, we prove some simple claims.

Claim: $e_1(H_2) = e_2(H_2) = e_3(H_2) = 0$.

If $e_1(H_2) \neq 0$, then by dividing both sides of equalities (3.40)(i), (ii), (iii) by $\epsilon_1 e_1(H_2)$, and using the identity $H_2 = \kappa^2$, we get

$$(3.43)\quad \begin{aligned} &\text{(i) } -\frac{1}{2} - \frac{7}{2}\kappa^2 + \frac{1}{2}u_1 + \frac{\sqrt{2}}{2}u_2\kappa = 0\\ &\text{(ii) } \frac{-1}{2} + u_1(\frac{1}{2} - \frac{7}{2}\kappa^2) + \frac{\sqrt{2}}{2}u_2\kappa = 0\\ &\text{(iii) } \frac{-\sqrt{2}}{2}\kappa + \frac{\sqrt{2}}{2}u_1\kappa - \frac{7}{2}\kappa^2)u_2 = 0, \end{aligned}$$

where $u_1 := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$ and $u_2 := \frac{\epsilon_3 e_3(H_2)}{\epsilon_1 e_1(H_2)}$, which, by comparing (i) and (ii), gives $\kappa^2(u_1 - 1) = 0$. If $\kappa = 0$, then $H_2 = 0$. Assuming $\kappa \neq 0$, we get $u_1 = 1$, which, using (3.43)(iii), gives $u_2 = 0$. Substituting $u_1 = 1$ and $u_2 = 0$ in (3.43)(i), we obtain again $\kappa = 0$, which is a contradiction. Hence $e_1(H_2) \equiv 0$.

Therefore, using the result $e_1(H_2) \equiv 0$, the system of equations (3.42) gives

$$(3.44)\quad \begin{aligned} &\text{(i) } \frac{1}{2}\epsilon_2 e_2(H_2) + \frac{\sqrt{2}}{2}\epsilon_3 e_3(H_2)\kappa = 0\\ &\text{(ii) } \epsilon_2 e_2(H_2)(\frac{1}{2} - \frac{7}{2}\kappa^2) + \frac{\sqrt{2}}{2}\epsilon_3 e_3(H_2)\kappa = 0\\ &\text{(iii) } \epsilon_2 e_2(H_2)\frac{\sqrt{2}}{2}\kappa - \epsilon_3 e_3(H_2)\frac{7}{2}\kappa^2 = 0. \end{aligned}$$

Comparing (i) and (ii), we get $\kappa e_2(H_2) = 0$, which using (iii) gives $\kappa e_3(H_2) = 0$, and then, using (i), gives $e_2(H_2) = 0$. Then, the second claim (i.e. $e_2(H_2) = 0$) is proved.

Now, using the results $e_1(H_2) = e_2(H_2) = 0$, we get $\kappa e_3(H_2) = 0$, which, using $H_2 = \kappa^2$, implies $\kappa e_3(\kappa^2) = 0$ and then $e_3(\kappa^3) = 0$, and finally $e_3(H_2) = 0$. □

Theorem 3.12. *Let $\boldsymbol{x} : M_1^3 \to \mathbb{M}_1^4(c)$ be a $\mathcal{D}_4$-hypersurface with proper second mean curvature vector field. If the ordinary mean curvature of M_1^3 is constant, then it is 1-minimal. Furthermore, all of mean curvatures of M_1^3 are null.*

Proof. By Proposition 3.11, the 2th mean curvature of M_1^3 is constant, which, by (2.3)(i), gives $L_1 H_2 = 9H_1 H_2^2 - 3H_2 H_3 = 0$. Then $3H_1 H_2^2 = H_2 H_3$, which, using $H_1 = \kappa$, $H_2 = \kappa^2$ and $H_3 = \kappa^3$, gives $\kappa^5 = 0$, and then $H_1 = H_2 = H_3 = 0$. □

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